Chapter 2

No-Arbitrage Asset Pricing

This chapter explores no-arbitrage as a useful restriction on the prices of all tradable securities. In the next section, we provide an exhaustive list of the implications of no-arbitrage on asset pricing, together with a proof of the fundamental theorem of finance. Several applications of the fundamental theorem on option pricing, forward and futures pricing, the term structure of interest rates and the pricing of interest-rate contingent claims will be carried out in the remainder of this chapter. This chapter serves also as an introduction to financial derivatives and the term structure of interest rates.

2.1 Fundamental Theorem

In the Introduction of this book, we saw that the no-arbitrage condition is equivalent to the existence of a positive linear pricing rule, which constitutes a necessary and sufficient condition for the existence of an optimal solution to the agent's optimal choice problem. Consequently, no-arbitrage will necessarily constitute part of the restrictions on equilibrium security price. Precisely speaking, if an equilibrium price p exists, then there must exist a state price vector $\psi \gg 0$ such that $p_0 \cdot \phi = \psi \circ d^{\phi}$ for any trading strategy $\phi \in \Phi$ (here, d^{ϕ} is the future cash stream generated from ϕ). However, we must keep in mind that the existence of a positive pricing rule ψ on market span $\mathbb{D} = \mathbb{D}(p, \delta)$, generated from tradable securities (p, δ), does not necessarily imply that the underlying security price p can be supported as the equilibrium price. Therefore, the no-arbitrage approach to asset pricing is understood to be a *partial equilibrium approach*.

We now list the useful implications on security price of no-arbitrage.

- **(a)** *One-price principle*: Two trading strategies resulting in the same cash stream must cost the same in the current marketplace. For all $d \in \mathbb{D}$ that is spannable in the marketplace, from condition S1 (see Chapter 1), we see that the initial costs underlying all trading strategies ϕ that finance d must be the same. This is given by $\Psi_0(d) = \psi \circ d$.
- **(b)** *Positivity*: $\Psi_0(d) \geq 0$ for all $d \geq \emptyset$ in $\mathbb{D}$. $\Psi_0(d)$ is strictly positive if $d \neq \emptyset$; and it equals zero if $d = \emptyset$.
- **(c)** *Additivity*: $\Psi_0(d + d') = \Psi_0(d) + \Psi_0(d')$ for all d and $d' \in \mathbb{D}$.
- **(d)** *Homogeneity*: $\Psi_0(\alpha d) = \alpha \Psi_0(d)$ for all $d \in \mathbb{D}$ and for all real numbers α.
- **(e)** *Recursivity*: For all $d \in \mathbb{D}$ and $t \geq 0$, the time t market price $\Psi_t(d)$ is $\mathcal{F}_t$-measurable and is given by

$$\Psi_t(d, \omega) = \frac{\sum_{s \geq t+1} \psi_s \bullet d_s 1_{\{\mathcal{F}_t(\omega)\}}}{\sum_{\omega' \in \mathcal{F}_t(\omega)} \psi_t(\omega')} \tag{2.1}$$

where, for all x and y, $x \bullet y 1_{\{\mathcal{F}_t(\omega)\}}$ stands for $\sum_{\omega' \in \mathcal{F}_t(\omega)} x(\omega') y(\omega')$. Moreover, $\Psi_t(d, \omega)$ can be expressed as the time t present value of the immediate future cash flow d_{t+1} plus its market capitalization $\Psi_{t+1}(d)$. That is, at all (t, ω) it must hold true that

$$\Psi_t(d, \omega) = \frac{\psi_{t+1} \bullet [d_{t+1} + \Psi_{t+1}(d)] 1_{\{\mathcal{F}_t(\omega)\}}}{\sum_{\omega' \in \mathcal{F}_t(\omega)} \psi_t(\omega')}. \tag{2.2}$$

To illustrate the validity of the recursive pricing relationship, we consider a trading strategy ϕ that finances d. We compare the cash flows in each of the following three scenarios:

  1. Where the investor holds portfolio ϕ_0 and trades according to ϕ afterwards.
  2. Where the investor holds the same initial portfolio ϕ_0 at time 0 and sells the portfolio at time t (at the market price) when (and only when) event $\mathcal{F}_t(\omega)$ is observed at time t. The resulting cash stream, denoted as d', is the same as d except that at $\omega' \in \mathcal{F}_t(\omega)$ the cash flows after time t become

$$d'_t(\omega') = d_t(\omega) + \Psi_t(d; \omega)$$
$$d'_s(\omega') = 0 \text{ for all } s \geq t+1.$$

  3. Where everything is the same as in the second scenario except that the investor chooses to close the account at time $t+1$ after observing event $\mathcal{F}_t(\omega)$ at time t. Let d'' be the corresponding cash stream. We have:

$$d''_{t+1}(\omega') = d_{t+1}(\omega') + \Psi_{t+1}(d; \omega') \text{ for all } \omega' \in \mathcal{F}_t(\omega).$$

The initial costs associated with each of these three trading strategies are the same. Setting $\Psi_0(d) = \Psi_0(d') = \Psi_0(d'')$ obtains

$$\begin{aligned}\sum_{s\geq 1}\psi_s \bullet d_s = & \sum_{1\leq s\leq t}\psi_s \bullet d_s + \sum_{s\geq t+1}\psi_s \bullet d_s 1_{\{\Omega-\mathcal{F}_t(\omega)\}} \\ & + \left(\sum_{\omega'\in\mathcal{F}_t(\omega)}\psi_t(\omega')\right)\Psi_t(d,\omega) \\ = & \sum_{1\leq s\leq t}\psi_s \bullet d_s + \sum_{s>t}\psi_s \bullet d_s 1_{\{\Omega-\mathcal{F}_t(\omega)\}} \\ & + \psi_{t+1} \bullet [d_{t+1} + \Psi_{t+1}(d)]\, 1_{\{\mathcal{F}_t(\omega)\}}.\end{aligned}$$

The first equality yields expression (2.1), while comparing both sides of the second equality leads to the recursive relationship (2.2).

(f) *Risk-neutral pricing rule*: Define probability measure $\mathbb{Q}$ on the event space $(\Omega, \mathcal{F})$ such that

$$\mathbb{Q}(A \mid \mathcal{F}_t(\omega)) = \frac{\sum_{\omega'\in A}\psi_{t+1}(\omega')}{\sum_{\omega'\in\mathcal{F}_t(\omega)}\psi_{t+1}(\omega')} \tag{2.3}$$

for all (t, ω) and $A \subset \mathcal{F}_t(\omega)$. The probability measure $\mathbb{Q}$ so defined is referred to as a *risk-neutral probability measure*. Consider a one-period discount bond at time t with unit payoff at $t+1$. Let $R_t^f(\omega) = 1 + r_t(\omega)$ be the risk-free interest rate for the bond contingent on ω. By the recursiveness principle, we have

$$\frac{1}{1+r_t(\omega)} = \frac{\sum_{\omega'\in\mathcal{F}_t(\omega)}\psi_{t+1}(\omega')}{\sum_{\omega'\in\mathcal{F}_t(\omega)}\psi_t(\omega')} > 0. \tag{2.4}$$

With the prescribed expressions respectively for the risk-free interest rate and the risk-neutral probability measure $\mathbb{Q}$, the recursiveness principle for pricing future cash flows can be re-expressed into the following risk-neutral pricing rule. For all $d \in \mathbb{D}$, and $t \geq 0$, we have

$$\Psi_t(d) = \frac{1}{1+r_t}\mathbb{E}_{\mathbb{Q}}[d_{t+1} + \Psi_{t+1}(d) \mid \mathcal{F}_t]. \tag{2.5}$$

That is, the time t price equals the discounted value of the time $t+1$ dividend payment plus its market capitalization at $t+1$.

(g) *Risk-free return rule:* Under the risk-neutral probability measure $\mathbb{Q}$, the expected returns for all cash streams $d \in \mathbb{D}$ are the same, and are given by the risk-free interest rate; that is,

$$\mathbb{E}_{\mathbb{Q}}\left[R_{t+1}^d \mid \mathcal{F}_t\right] = 1 + r_t \tag{2.6}$$

in which $R_{t+1}^d = \frac{d_{t+1}+\Psi_{t+1}(d)}{\Psi_t(d)}$.

The following *fundamental theorem of asset pricing* is a summary of these listed observations.

Theorem 2.1. *For finite economies, the statements M3, S1, (e), (f) and (g) are equivalent to each other. Each of these statements implies, and is implied by, the statements (a), (b), (c) and (d) combined.*

Proof. In following this construction, we have S1 ⇒ (a), (b), (c) and (d) ⇒ M3. We also have S1 ⇒ (e) ⇒ (f) ⇔ (g) ⇒ M3. The step '(g) ⇒ M3' is valid because the risk-free return rule implies the validity of the following present value pricing rule

$$\Psi_0(d) = \mathbb{E}_{\mathbb{Q}}\left[\sum_{t\geq 1} \frac{d_t}{(1+r_0)\cdots(1+r_{t-1})}\right], \tag{2.7}$$

keeping in mind that, for finite economies, $\Psi_T(d) = d_T = 0$ when T is sufficiently large. The statement of the theorem follows from Theorem 1.2 on the equivalence between S1 and M3 for finite economies. ☐

We must differentiate the risk-neutral probability measure $\mathbb{Q}$ from the original probability measure $\mathbb{P}$. The risk-neutral probability is a hypothetical probability measure fully derived from the state price ψ, while the latter is, in general, affected by the measure $\mathbb{P}$ in equilibrium. Moreover, when the no-arbitrage condition is satisfied, the corresponding risk-neutral probability measure $\mathbb{Q}$ is, in general, not unique. It becomes unique when the underlying state price ψ satisfying condition S1 is unique, as is the case when the market is complete. Here is an illustrative example.

Example 2.1. Consider a one-period economy with three states $\{a, b, c\}$, each occurring with equal probability $\frac{1}{3}$. It has two tradable securities with payoffs respectively given by

$$\delta_1 = [110, 110, 110] \text{ and } \delta_2 = [50, 100, 150].$$

The initial market prices are $p_1 = p_2 = 100$. It is easy to verify that the market admits no arbitrage; in particular, condition S1 is satisfied with a continuum range of positive state price vectors ψ given by

$$\begin{bmatrix} \psi_a \\ \psi_b \\ \psi_c \end{bmatrix} = \begin{bmatrix} -\frac{2}{11} \\ \frac{12}{11} \\ 0 \end{bmatrix} + \psi_c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

for all $\psi_c \in \left(\frac{2}{11}, \frac{6}{11}\right)$. With a risk-free interest rate of $r = 10\%$ we obtain a continuum range of risk-neutral probability measures that is given by

$$q_a = q_c - 0.2 \text{ and } q_b = 1.2 - 2q_c$$

with q_c taking an arbitrary value in the open interval $(0.2, 0.6)$. So, in this case, the no-arbitrage condition fails to generate a unique risk-neutral probability measure. It is also noted that the objective probability measure with equal probability mass $\frac{1}{3}$ does not fall into the set of risk-neutral probability measures.

Further to this observation, the no-arbitrage condition would, in general, fail to determine a unique price for contingent claims. To illustrate this, we continue with this example and introduce a new security into the marketplace. The payoffs of the new security are given by $\delta_3 = [0, 0, 50]$. We wonder: what is the sensible price for the security? A sensible price for a security is understood to be a price at which the market with the newly introduced security remains to fulfill the no-arbitrage condition. Under the risk-neutral pricing rule, for arbitrary $q_c \in (0.2, 0.6)$, $p_3 = 50/1.1q_c$ must constitute arbitrage-free prices for the newly introduced security. This results in a continuum range of prices $p_3 \in (100/11, 300/11)$ which are all consistent with the no-arbitrage condition. So, for this particular example, we see that the no-arbitrage condition fails to provide a precise and unique assessment of the newly introduced security.

Generally speaking, a unique price for a contingent claim is obtained whenever the payoff of the contingent claim is redundant in the sense that it is spanned by the existing tradable securities. By definition, all contingent claims will be uniquely priced if, and only if, the marketplace is complete so that all contingent claims are redundant to the existing tradable securities. To illustrate this, we continue with the example and add a third security with payoffs $\delta_3 = [0, 0, 50]$. Suppose this security is traded at $p_3 = 200/11$. In this case, the market with three tradable securities becomes complete. In fact, it is straightforward to check that the 3×3 payoff matrix formed by the three tradable securities has a full rank. Consequentially, all arbitrary future payoffs $[x_a, x_b, x_c]$ can be obtained through a portfolio of these three tradable securities.

To determine the price of the contingent claim, we may compute the risk-neutral probability measure by applying the no-arbitrage condition. This yields

$$q_a = 0.2 \text{ and } q_b = q_c = 0.4.$$

In this case, the random payoffs $[x_a, x_b, x_c]$ correspond to a unique arbitrage-free price p_x that is given by

$$p_x = \frac{2x_a + 4x_b + 4x_c}{11}.$$

2.2 Primitive vs Derivative Securities

2.2.1 *Derivative Securities: An Introduction*

Securities tradable in the marketplace can be roughly classified into two broad categories: *primitive securities* and *derivative securities*. These securities differ from each other according to the specifications of the payoffs/future cash flows of the underlying securities.

The payoffs of a primitive security are directly linked to the production performance of the issuing company. Typical examples of primitive securities are stocks issued by some trading companies, indexes such as the S&P 500 and the FTSE 300, and commodities. The cash flows associated with a stock are measured by the dividend payments made by the issuing company plus the capital gains at the closing date. The dividend payment is determined by many economic factors that directly affect the company's financial status. These factors include, for example, profitability, cash position, financial structure (debt-equity ratio), current market conditions, potential earnings, other future growth perspectives of the company and so on. The value of the stock reflects the real performance of the company, which is, of course, also affected by the overall well-being of the economy. Similarly, the index, the value of which reflects the real performance of the economy, belongs to the primitive securities category.

Bonds are another class of primitive securities. Included in this class are Treasury bills issued by the United States government and corporate bonds issued by individual companies. The bonds are also known as fixed income securities because their cash flows, called *coupons*, are fixed. The bonds have fixed terms on the delivery of the coupons and have a pre-specified time-to-maturity. The value of a government bond is particularly affected by several macroeconomic variables such as the growth rate and employment rate that summarize the overall fundamentals of the economy. The value of a government bond is also affected by monetary and fiscal policy variables such as the inflation rate and taxes. A corporate bond's value is particularly sensitive to the financial situation of the issuing company and is linked to their risk of default.

A derivative security is a security with its payoffs, or cash flows, derived from the price of other (primitive or derivative) securities. Typical examples of derivative securities are stock options, index options, financial/commodity forward and futures contracts. For example, a *European call option* on company A's stock is a right to buy the stock at a future time T at a pre-fixed price X. The investor who owns the option can decide whether he is to exercise his option or not. Of course, he will exercise the option if and only if the time T price, say S_T, of the stock is above the pre-fixed contract price X. Here, X is called the *exercise price* of the option, and T is called the *time-at-maturity* or the *maturity date* of the option. The cash flows associated with the option are thus derived from the price of the underlying stock at the maturity date, which is $C_T = (S_T - X)^+$. Here, $(x)^+ = x$ if $x \geq 0$ and equals 0 otherwise. Similarly, for a *European put option* on the stock, the maturity payoff is given by $P_T = (X - S_T)^+$. Therefore, the put option is the right to sell the stock at maturity date T at the exercise price X. *American options* are similar to European options except that the investor has the right to exercise the options at any time before the maturity dates. There are other types of options available for trade in the marketplace such as exotic options. Readers may wish to consult textbooks such as Hull (2000) that specialize in options and derivative trading.

The futures and forward contracts are not options, but 'obligations' to buy or sell an asset at some pre-specified price and at a pre-fixed maturity date. The payoff for a long position at maturity T with futures price $F_{0,T}$, which is fixed at $t = 0$, is thus given by $S_T - F_{0,T}$. A short position has a maturity payoff $F_{0,T} - S_T$. Unlike options, the payoff of a futures/forward contract at the maturity date could be positive or negative depending on whether the future spot price S_T of the underlying security is above or below the contract price $F_{0,T}$. The futures and forward contracts are treated as being the same here although, in practice, forward and futures contracts differ in their settlement procedures and are subjected to different regulatory restrictions (for instance, margin deposits).

2.2.2 *Trading with Derivatives*

There are various motivations for investors to trade in derivative securities. Roughly speaking, investors trading in derivatives can be divided into three categories: speculators, *arbitrageurs* and *hedgers*.

Speculators hold derivatives because they aim to profit from short-term price changes. For example, an investor who expects the stock price to drop in the near future will probably buy a put option, aiming to profit from the downside move in the stock prices. Similarly, if he expects the market prices to go up, he may want to hold long in calls. In either scenario, buying put or call options represents a much more effective way of trading in comparison with the traditional strategy of selling short or buying long the underlying securities. This effectiveness can be best traced back to the respective liquidity implications of both groups of strategies. Let us illustrate this with a numerical example. Suppose an investor has \$1000 to invest. He sees a stock, ABC, that is currently trading at \$100 per unit. Suppose he believes that the share price of the stock ABC will surpass \$110 within a one-month period. To profit from the situation, the investor can afford to buy 10 shares of the stock. By doing so, he expects to make a profit of about $10 \times (\$110 - \$100) = \$100$ if he bets correctly. Now, suppose the stock ABC is option eligible so that investors can trade options written on the stock. Assume a one-month call option with an exercise price of \$100 is traded at \$5 per unit. The investor with \$1000 in his pocket can afford to buy 200 option units. If he bets correctly, he will make as much as $200 \times (\$110 - \$100) = \$2000$ in profit. The profit resulting from the options trading is 20 times the profit made by following the prescribed trading strategy of holding 10 shares of stock ABC. However, with options trading the speculator might lose all his money (\$1000) if he bets incorrectly. This is the risk that the investor has to bear if speculating with options trading.

Some investors, known as hedgers, use derivative securities to hedge against the risk, particularly the downside risk, associated with a stock in the underlying primitive securities. Consider an investor who has a long position, say 10 units, in ABC stock that is traded at \$100 as described above. Suppose he expects a press release from the issuing company within a period of one month. Upon reading the press release, the stock price will either go up (to \$120) or drop (to \$80) depending on whether the news turns out to be good or bad. The investor faces a dilemma. On the one hand, he does not want to close his position and miss an 'up' opportunity if the news turns out to be good. On the other hand, he does not wish to bear the risk of losing big in the case of bad news. In the presence of options trading, he can simply hold long 10 units of the one-month put option on the stock with an exercise price of \$110. By doing so, he will not miss the growth opportunity if the stock price goes up. If there is bad news, the gain in the put option will largely offset the capital loss in the stock. Therefore, with the help of options trading he is able to secure himself a 'win and not-much-to-lose' situation. To illustrate, suppose the unit cost of the put option is \$12. Under the prescribed options trading, the payoff by the end of the month will be either \$1200 or \$1100 with the downside risk largely eliminated. The cost of the hedging is \$120.

The third group of investors trading in derivatives are arbitrageurs. The goal of an arbitrageur is to identify arbitrage opportunities in the marketplace. Arbitrageurs make a living by speculating on the imperfections of the marketplace. Indeed, the real market is not perfect compared to an ideal, arbitrage-free marketplace. Since payoffs of a derivative security are derived from the prices of those primitive securities, in an ideal, arbitrage-free world, the price of the derivative security along with the prices of the primitive securities must be constrained in a coherent fashion to rule out arbitrage opportunities. When such coherency breaks down, arbitrage opportunities will emerge.

To illustrate how arbitrageurs make profit, we consider again Example 2.1, in which security 1 is a risk-free bond with an interest rate of 10%, and security 2 is treated as a stock. Consider a call option written on the stock with an exercise price $X = \$100$. The payoffs of the option are thus given by $\delta_3 = [0, 0, \$50]$. Suppose the option is traded at $p_3 = \$9$, which is out of the no-arbitrage range $(10, 30)$. This creates an arbitrage opportunity for the arbitrageurs. The following trading strategy illustrates how arbitrageurs profit from situations like this:

- buy 10 bond units at \$100;
- sell short 11 stock units at \$100;
- buy 11 option units at \$9.

This results in future payoffs $[\$550, 0, 0]$ with an initial cost of capital given by $10 \times \$100 - 11 \times (\$100 - \$9) = -\$1$.

In the following section, we establish several useful parity relationships between the price of derivative securities (options and futures) and the price of primitive securities (bonds and stocks) in an arbitrage-free marketplace. Violations of any of these parity relationships would represent profit opportunities for arbitrageurs.

2.2.3 *European Put-Call Parity*

Consider a stock with the current market price S_0. Assume that the stock is associated with non-zero dividend payments $d = \{d_t\}_{t=1}^{T}$. Let $B_{0,T}$ be the price of a discount bond with maturity date T. Let $C_0(X, T)$ be the price of a European call option written on the stock with exercise price X and with maturity date T. Let $P_0(X, T)$ be the price of the corresponding European put option. We have:

Proposition 2.1. *In an arbitrage-free marketplace, the option price must satisfy the following put-call parity:*

$$S_0 + P_0(X,T) = \Psi_0(d) + XB_{0,T} + C_0(X,T) \tag{2.8}$$

for all T and X.

Proof. First, we assume that the stock pays no dividend before time T. Consider a buy-and-hold strategy with the initial portfolio consisting of a long position in one unit of stock and a long position in one unit of a put option written on the stock with maturity date T and exercise price X. The resulting maturity date payoff is given by $S_T + (X - S_T)^+$, where S_T is the time T market price of the stock. This maturity payoff $S_T + (X - S_T)^+$ can be re-written as $X + (S_T - X)^+$. The latter is the payoff resulting from a buy-and-hold trading strategy of an initial portfolio that consists of X units of a discount bond of maturity T and one unit of a European call option written on the stock with exercise price X and with the same maturity. Since both strategies involve zero cash flows before the maturity date, by the one-price principle, they must have the same initial market price; that is,

$$S_0 + P_0(X,T) = XB_{0,T} + C_0(X,T)$$

as desired.

In general, with non-zero dividend payments, the cash flow generated from the buy-and-hold trading strategy (I) involved in a portfolio with a long position in the stock (one unit) and one unit of a put option written on the stock (with maturity date T and exercise price X) is not the same as the cash flows resulting from a buy-and-hold trading strategy (II) with an initial portfolio consisting of X units of a discount bond of maturity T and one unit of a European call option written on the stock with exercise price X and with the same maturity. Comparing the difference between the two sources of cash flows gives the dividend stream d. We have

$$\begin{aligned} S_0 + P_0(X,T) &= \Psi_0(d) + \Psi_0\left(S_T + (X - S_T)^+\right) \\ &= \Psi_0(d) + \Psi_0\left(X + (S_T - X)^+\right) \\ &= \Psi_0(d) + XB_{0,T} + C_0(X,T). \end{aligned}$$

□

In this derivation, it is noted that the dividend stream $\{d_t\}_{t=1}^T$ belongs to the market span. So, $\Psi_0(d)$ is well-defined. In fact, the dividend stream $\{d_t\}_{t=1}^T$ can be generated by holding long one unit of a stock and one unit of a European put option on the stock, and by writing a covered European call of the same maturity and the same exercise price (as the put), and by selling short X units of a discount bond of maturity T. From the put-call parity, we see that if the dividends are positive, the initial cost associated with trading strategy (I) described in the proof must be greater than the cost resulting from trading strategy (II). The first trading strategy will generate higher cash flows than the second. So, we obtain

$$S_0 + P_0(X,T) > XB_{0,T} + C_0(X,T) \tag{2.9}$$

whenever $d \geq \emptyset$ and $d \neq \emptyset$.

As another observation, from the put-call parity we may assess the market value of the dividend stream with

$$\Psi_0(d) = S_0 + P_0(X,T) - C_0(X,T) - XB_{0,T}. \tag{2.10}$$

The assessment is invariant at the exercise price X. Consequently, if options with different exercise prices lead to different assessments with respect to the market price of the dividend stream (before the maturity date), the market must admit arbitrage opportunities. As a result of the put-call parity, we obtain the following parity relation for options of different exercise prices.

Corollary 2.1. *Given the prescribed S_0 and $B_{0,T}$, if the market admits no arbitrage, it must hold true that*

$$P_0(X_1,T) - C_0(X_1,T) = P_0(X_2,T) - C_0(X_2,T) + (X_1 - X_2)B_{0,T}$$

for all T and X_1 and X_2; or, in differential form,

$$\frac{\partial P_0(X,T)}{\partial X} = \frac{\partial C_0(X,T)}{\partial X} + B_{0,T}$$

for all T and X.

The put-call parity, in general, does not hold for American options. This is because investors may exercise their options at any time before the maturity date. Moreover, the exercising of the call and put options might not take place at the same point in time. A detailed study of American options will be carried out later on in Section 2.3.7.

2.2.4 *Forward and Futures Prices*

We start with a simple example of a futures contract on a stock index.

Example 2.2. Suppose the spot price of the index is \$30, which is expected to pay no dividend over a period of at least one year. Suppose also the bond price has an annual risk-free interest rate of 5%. Now imagine that the one-year futures contract on the index is quoted at \$33 on the futures exchange. That is, investors who hold the futures contract can buy or sell the stock index at a unit price of \$33 with delivery in a one-year period. Does this quoted price constitute a 'fair' price for the contract? By 'fair' we mean that all parties, no matter whether they were in the long or short sides of the market, would agree upon the settled price at the future time.

Given the quoted futures price of \$33, we consider the following trading strategy:

- borrow \$3000 dollars at an annual interest rate of 5% for one year;
- buy 100 units of the stock index at the spot price \$30;
- enter into a one-year futures contract to sell 100 units of the stock index at the quoted futures price of \$33.

This trading strategy results in a zero cash flow in the current spot market, with the future cash flow at the delivery date to be given by $\$33 \times 100 - \$3000 \times 1.05 = \$150$. This constitutes an arbitrage. It is easy to see that at any futures price quoted above \$31.50, there are arbitrage opportunities. Similarly, any futures price quoted below \$31.50 will also violate the no-arbitrage condition. For example, an investor can short the \$3000 index at the current market price, invest \$3000 in the one-year discount bond, and enter into a futures contract to buy 100 units of the stock index at the quoted price. This will result in a risk-free positive cash flow at the delivery date with zero cash injection into the current marketplace. Therefore, the only price for the futures contract that is sustainable in a no-arbitrage marketplace is $\$31.50 = \$30 \times 1.05$.

In a general sense we consider an asset S_T with the current market price S_0. Let $\{B_{0,t}\}_{t\geq 1}^{T}$ be the discount bond prices of all maturities before T.

Proposition 2.2. *For a zero-cash payout asset S_T, the fair price $F_{0,T}$ for the futures contract with delivery date T is given by $F_{0,T} = S_0 B_{0,T}^{-1}$. Moreover, if the asset is associated with non-zero cash flows $d = \{d_t\}_{t=1}^{T}$ at and before T, the fair price of the futures contract is given by*

$$F_{0,T} = (S_0 - \Psi_0(d))\, B_{0,T}^{-1}. \tag{2.11}$$

Proof. The cash flow to an investor who enters into the futures contract in the long side is $S_T - F_{0,T}$ at the maturity date T and with zero cash flows for $t \neq T$. Since investors access the contract at a zero initial cost, the fair futures price $F_{0,T}$, by the fundamental theorem, must be such that the cash flow $S_T - F_{0,T}$ has zero initial market value. We see that $\Psi_0 (S_T - F_{0,T}) = 0$; or, by the additivity principle, $\Psi_0 (S_T) - \Psi_0 (F_{0,T}) = 0$. Notice further that the futures price $F_{0,T}$ is deterministic, its initial market price $\Psi_0 (F_{0,T})$ is given by a discount bond of face value $F_{0,T}$ at maturity date T; that is, $\Psi_0 (F_{0,T}) = F_{0,T} B_{0,T}$. Therefore, we have

$$F_{0,T} = \Psi_0 (S_T) B_{0,T}^{-1}.$$

Furthermore, if the asset has zero cash flows $d = \emptyset$ before the maturity date T, we have $S_0 = \Psi_0 (S_T)$. Otherwise, if the asset has non-zero cash flows $d = \{d_t\}_{t=1}^T$, we have, by the additivity principle, $S_0 = \Psi_0 (d) + \Psi_0 (S_T)$. We may write this equivalently as $\Psi_0 (S_T) = S_0 - \Psi_0 (d)$. In either case, we have $F_{0,T} = (S_0 - \Psi_0 (d)) B_{0,T}^{-1}$ as desired. □

When $d_t > 0$, the cash flow can be regarded as dividend payments for stocks and stock indexes, and coupons for bonds. Similarly, when $d_t < 0$, the cash flow can be interpreted as the carrying costs, storage costs of the asset until the delivery dates. Such costs apply to many commodity futures.

From the derived formula for the futures price, we see that to determine the futures price we need to estimate the market value of the underlying cash flows which will, in general, require knowledge of the risk-neutral probability measure. Nevertheless, for the simple case of deterministic cash flows, we have the simple expression $\Psi_0 (d) = \sum_{t=1}^T d_t B_{0,t}$. In particular, if the cash flows are proportional to the initial price of the asset so that $d_t = k_t S_0, t = 1, \cdots, T$. The futures price is given by

$$F_{0,T} = S_0 \left(1 - \sum_{t=1}^{T} k_t B_{0,t} \right) B_{0,T}^{-1}. \tag{2.12}$$

It is also possible to obtain an analytic expression for the futures price when the dividend rates are stochastic. The following two examples show that when the stochastic dividend rates have either a constant dividend yield (i.e. $d_{t+1} = \kappa S_t$ for all $t \geq 0$) or a constant dividend-price ratio (i.e. $d_{t+1} = \kappa S_{t+1}$ for all $t \geq 0$), we can express the corresponding futures price in terms of the current price of the underlying asset.

Example 2.3. We consider a futures contract written on an asset with a constant dividend yield written as $d_{t+1} = \kappa S_t$ for all $t \geq 0$. We assume the interest rates are constant. With $S_t = \frac{\kappa}{1+r} S_t + \Psi_t (S_{t+1})$ we obtain $\Psi_0 (S_{t+1}) = \left(1 - \frac{\kappa}{1+r}\right) \Psi_0 (S_t)$ with the solution $\Psi_0 (S_t) = \left(1 - \frac{\kappa}{1+r}\right)^t S_0$.

Moreover, with $\Psi_0(S_T) = \frac{1}{(1+r)^T} F_{0,T}$ we obtain the futures price $F_{0,T} = (1+r-\kappa)^T S_0$.

Example 2.4. As in the previous example, except with the dividend rates having a constant dividend-price ratio, we write $d_{t+1} = kS_{t+1}$ for all $t \geq 0$. We now have $S_t = (1+\kappa)\Psi_t(S_{t+1})$ and $\Psi_0(S_t) = (1+\kappa)\Psi_0(S_{t+1})$. The solution to the first-order difference equation is given by $\Psi_0(S_t) = \frac{1}{(1+\kappa)^t} S_0$. This, together with $\Psi_0(S_T) = \frac{1}{(1+r)^T} F_{0,T}$, results in the futures price $F_{0,T} = \left(\frac{1+r}{1+\kappa}\right)^T S_0$.

2.2.5 *Put-Call-Futures Parity*

With this expression for futures price, we may re-express the put-call parity in terms of the futures price by substituting $S_0 - \Psi_0(d)$ with $F_{0,T}B_{0,T}$. This yields the following *put-call-futures parity.*

Proposition 2.3. *Consider European options and futures contracts written on a primitive security S_T. No-arbitrage implies the validity of the following put-call-futures parity:*

$$P_0(X,T) - C_0(X,T) = (X - F_{0,T})B_{0,T}. \tag{2.13}$$

Here, we must emphasize that the put-call-futures parity holds true for all securities with arbitrary future cash flows.

To enhance our understanding of the logic underlying the put-call-futures parity, we consider an investor entering into a futures contract and wishing to hedge against his risk exposure (resulting from the contract) with the help of options trading. We will use European options on the underlying asset on which the futures contract is written. As it turns out, a perfect hedge can be obtained no matter whether the investor enters into the long or the short side of the futures contract.

For instance, suppose the investor is long in the futures contract with the maturity payoff $S_T - F_{0,T}$. The investor can hedge fully against the risk by holding long in a put option on the same security with maturity payoff $(X - S_T)^+$, and by writing a call option on the security of the same exercise price and of the same maturity with maturity payoff $-(S_T - X)^+$. This trading strategy is associated with zero cash flows after the initial trading and before the maturity date. The portfolio payoff at the maturity date is constant and is given by $X - F_{0,T}$, which is the same as the payoff of a T-period bond of face value $X - F_{0,T}$. By the one-price principle of no-arbitrage, the initial cost of the trading strategy, which is $P_0(X,T) - C_0(X,T)$, must be the same as the market price of the bond of face value $X - F_{0,T}$. The latter is given by $(X - F_{0,T})\, B_{0,T}$. This leads, as desired, to the validity of put-call-futures parity.

This prescribed trading strategy is known as *long hedging* because it aims to hedge against the risk associated with a long position in a futures contract. A similar trading strategy can be created to hedge against the risk of entering into the short side of the futures contract. This involves a long position in the call and a short position in the put, and is known as *short hedging*.

2.2.6 *Options and Futures*

As a separate observation on pricing futures contracts, from the put-call-futures parity we can deduce the futures price in terms of the price of European options and bond price (of the same maturity). We have:

Corollary 2.2. *In an arbitrage-free marketplace, it must hold true that*

$$F_{0,T} = X + B_{0,T}^{-1}\left[C_0(X,T) - P_0(X,T)\right]. \tag{2.14}$$

The futures price is invariant in X; in particular, for at-the-money options we obtain the following expression for the futures price as a function of the spot price of the underlying asset:

$$F_{0,T} = S_0 + B_{0,T}^{-1}\left[C_0(S_0,T) - P_0(S_0,T)\right]. \tag{2.15}$$

It is interesting to note that the futures price given by Eq. (2.15) is obtained without explicitly referring to the market value of future cash flows $\{d_t\}_{t=1}^{T}$ associated with the underlying asset. The presence of futures dividends or cash flows affects the futures price only through the current price S_0 of the underlying asset and through the price of the at-the-money European options written on the asset, taking as given the bond price. Moreover, with the validity of Eq. (2.14), we see that the futures price can be obtained from the price of the European options of any arbitrary exercise price. This is without explicitly referring to the quotation of the price of the underlying asset and the assessment of the market value of future cash flows! A violation of Eq. (2.14) for any given exercise price implies the existence of arbitrage opportunities.

2.2.7 *Option Pricing: A Single-Period Model*

Here, we are concerned with how the price of derivative securities, say options, are determined, taking as given the price of the underlying primitive securities. We ask the following question:

Problem 2.1. *Suppose we know the price and payoffs for primitive securities such as stocks and bonds. What can we then say about the value of a derivative security written on the primitive securities?*

Generally speaking, if the primitive securities failed to span the entire consumption space, no-arbitrage conditions may not be sufficient to determine a unique price for a derivative security. This is because the primitive securities may not be able to reveal uniquely the state price; and different state prices may induce different prices for the derivative securities. Nevertheless, the no-arbitrage conditions do impose some useful joint restrictions on the price of the derivative securities and on those of the primitive securities. For example, we can determine a unique price for a 'redundant' derivative security according to the one-price principle (a) of no-arbitrage, since payoffs of a redundant security can be replicated by forming a portfolio among the primitive securities. We illustrate this with the following simple example.

Example 2.5. Assume that the current stock price is \$100. Suppose that there is an equal chance that after three months, the stock price will either go up by 20% to \$120 or drop by 10% to \$90. Suppose also that the three-month T-bill rate is 2%, which implies that a face value of \$100 invested in the T-bill will result in a final payoff of \$102 after three months. Now consider a European call option written on the stock with an exercise price of \$100 and with a time-to-maturity of three months. What is the sensible price for the option?

Notice that under the subjective belief of equal chance, the expected present value of the option's maturity payoff equals $\frac{0.5\times(120-100)+0.5\times 0}{1.02} = 9.80$. Is this a sensible price? The answer is NO. Suppose $C_0 = \$9.80$ is the market price of the option. Consider the following trading strategy: Borrow \$9700 in T-bills, then buy 113 units of the stock and sell 170 units of the option. This results in a net cash surplus at $t = 0$ of $170\times 9.8+9700-113\times 100 = \$66$. Three months later, if the stock price goes up, the cash balance is $-170\times 20-9700\times 1.02+113\times 120 = \$266$; otherwise, if the stock price goes down, it becomes $-170\times 0-9700\times 1.02+113\times 90 = \$276$. This is a gain-gain situation as you receive (not invest) \$66 in the current marketplace, and will also receive no less than \$266 three months later regardless of the stock price movement. This clearly constitutes an arbitrage opportunity. Therefore, the hypothetical option price of $C_0 = \$9.80$ cannot be a sensible market price for the option.

We apply the fundamental theorem to determine the sensible price for the option. The existence of no-arbitrage implies the existence of a risk-neutral probability measure $\mathbb{Q}$ such that the expected returns of all tradable securities under measure $\mathbb{Q}$ are the same, which is the risk-free rate of return (2%). Let $q_h > 0$ and $q_l = 1 - q_h > 0$ be the probabilities assigned respectively to the 'high' and 'low' return states under $\mathbb{Q}$. For the stock, the no-arbitrage condition (g) becomes $1.20q_h + 0.90q_l = 1.02$, which implies $q_h = 0.4$ and $q_l = 0.6$. The option price that satisfies the no-arbitrage condition must be given by the expected present value of its maturity payoff under the risk-neutral measure $\mathbb{Q}$, which equals $\frac{0.4\times(120-100)+0.6\times 0}{1.02} \doteq \$7.84$. Therefore, the only sensible market price for the option is $C_0 = \$7.84$.

As the option in this economy is a redundant security, hold long $\frac{2}{3}$ units of stock and borrow $\frac{60}{102} \times 100$ dollars of the bond. The resulting payoffs over a three-month period for the trading strategy can be easily computed: \$20 at the 'high' state and \$0 at the 'low' state. These coincide with the payoffs of the underlying option at the maturity date. By the one-price principle (a) of no-arbitrage, the price of the option must be given by the cost of the portfolio, which again yields $\frac{2}{3} \times 100 - \frac{60}{102} \times 100 = \$7.84$.

The argument used in this example also applies to the general case for a single-period economy that contains two primitive securities, say a stock and a bond, and two states of nature $\Omega = \{h, l\}$. Here, h and l are interpreted as realizations of a stock's (gross) return R_T at the high state and the low state, respectively. For the moment we assume that the stock has a zero dividend payout. Let $R^f = 1 + r$ be the risk-free interest rate for the bond. Assume that $h > 1 + r > l$. Let $q_h = Q\ (\{h\})$ and $q_l = Q\ (\{l\}) = 1 - q_h$ be the risk-neutral probabilities assigned to the states. By applying the risk-free return rule (g) of no-arbitrage to the stock return, one obtains $q_h h + (1 - q_h)\, l = 1 + r$. This results in this expression for the risk-neutral probabilities:

$$q_h = \frac{1 + r - l}{h - l} = 1 - q_l. \tag{2.16}$$

With this notion and by the risk-neutral pricing formula (f) of no-arbitrage, the initial market price of any contingent claim X_T must be given by

$$\Psi_0\,(X_T) = (1 + r)^{-1}\,[q_h X_h + q_l X_l]\,. \tag{2.17}$$

If we now look at the European call option written on the stock with an exercise price of X, the initial price is uniquely determined and is given by

$$C_0 = (1+r)^{-1}\left[q_h\left(hS_0 - X\right)^{+} + q_l\left(lS_0 - X\right)^{+}\right] \tag{2.18}$$

where S_0 is the current stock price.

Similar exercises can be carried out for dividend-paying stocks. As it turns out, the option pricing formula (2.18) and pricing rule (2.17) for general contingent claims remain valid for dividend-paying stocks. The corresponding risk-neutral probabilities, however, must be modified by taking into account the additional source of cash flows when defining the total returns of the stock. We must keep in mind that the total returns of the stock equals 'dividend yield plus growth rate'. Different risk-neutral probabilities are obtained depending on the specifications of the dividend payouts d_T. Now we consider two special cases:

(a) Stocks with a constant dividend yield $d_T = \kappa S_0$ and $h + \kappa > 1 + r > l + \kappa$. The risk-free return rule for the stock yields the following linear equation for the risk-neutral probability:

$$q_h\left(h+\kappa\right) + q_l\left(l+\kappa\right) = 1 + r$$

with $q_h = \frac{1+r-l-\kappa}{h-l} = 1 - q_l$.

(b) Stocks with a constant dividend-price ratio $d_T = \kappa S_T$ and $h > \frac{1+r}{1+\kappa} > l$. In this case the risk-free return rule results in

$$\left(1+\kappa\right)\left(q_h h + q_l l\right) = 1 + r$$

with $q_h = \frac{(1+r)/(1+\kappa)-l}{h-l} = 1 - q_l$.

With these expressions for the risk-neutral probabilities, one can readily carry out several static analysis exercises for the option price. We can now establish the following relationships, verifications of which are left as exercises.

(1) The option price increases in the risk-free interest rate r.
(2) The option price decreases in the dividend yield.
(3) The option price decreases in the dividend price ratio.
If we let $h = \mu + \sigma$ and $l = \mu - \sigma$ with $\sigma > 0$, we will have two additional relationships.
(4) The option price increases in σ.
(5) The sign $\frac{\partial C_0}{\partial \mu}$ is, in general, indefinite.

Property 1 is a result of the trade-off between the negative effect of the discount factor and the positive effect of the upward risk-neutral probability with respect to a raise in the risk-free interest rate. Properties 2 and 3 are self-evident because a raise in dividend payout reduces the upward risk-neutral probability, and thus reduces the price of the option (keeping other things unchanged). When the stock return has equal probability of either going upward or downward, the parameter μ gives the expected capital growth rate for the stock, while σ measures stock price volatility. Property 4 suggests that the option price increases in stock volatility. Property 5 suggests that a change in the expected growth rate has an ambiguous impact on the price of the option. This does not come as a surprise because an increase in μ reduces the upward risk-neutral probability but increases the payoffs of the option across states. These two effects offset each other with an uncertain net impact on the option price.

2.3 CRR Binomial Model

The Cox, Ross and Rubinstein (1979) model, known as the CRR model of option pricing, is a successful application of the no-arbitrage condition to asset pricing in a dynamic framework.

2.3.1 *Setup of the Model*

The economy considered by Cox, Ross and Rubinstein (1979) consists of two types of primitive securities: zero-coupon bonds of all maturities and a zero-dividend paying stock. The randomness of the economy is assumed to be driven by a *Bernoulli random process* $\{\xi_t\}_{t=1}^{\infty}$ on Ω. Here, ξ_t takes two possible values of $\{0, 1\}$. The state space Ω can be viewed as consisting of all possible sample paths in the realization of $\{\xi_t\}_{t=1}^{\infty}$. For each $\omega \in \Omega$, $\mathcal{F}_t(\omega)$ contains all paths that are consistent with the historic observations up to time t along path ω. $\{\xi_t\}_{t=1}^{\infty}$ is referred to as a state process, and ξ_t is the time t state variable.

We impose the following conditions on the bond price and stock returns:

- A flat term structure of interest rates written as $R^f = 1 + r$.
- Binomial stock returns. As illustrated in Fig. 2.1, the stock returns $\{R_t\}_{t\geq 1}$ are adapted to the state process $\{\xi_t\}_{t=1}^{\infty}$ so that

$$R_t = l + (h - l)\,\xi_t \text{ for all } t = 1, 2, \cdots . \tag{2.19}$$

Here, h and l correspond to the upward and the downward returns for the stock. It is assumed that

$$h > 1 + r > l.$$

These conditions ensure the no-arbitrage conditions are fulfilled in this context.

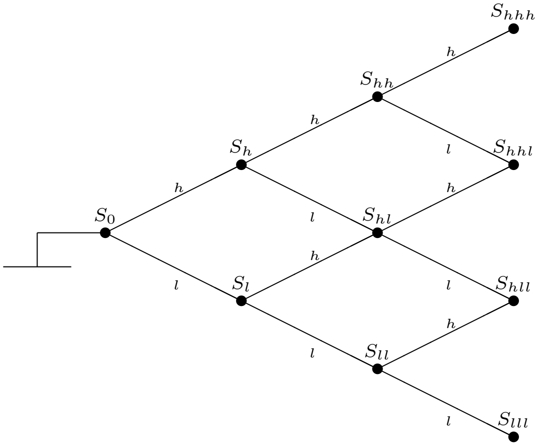

Fig. 2.1 CRR binomial tree

- The stock has a zero-dividend payment up to time T. Its time 0 price is S_0.

The assumption on the constant interest rate along with the zero-dividend assumption on the stock are for simplicity. These assumptions will be relaxed later on. The objective is to determine the price of a contingent claim written on S_T with maturity payoff $X = X(S_T)$.

2.3.2 *CRR Binomial Option Pricing Formula*

By the risk-free return rule (g), there exists a risk-neutral probability measure $\mathbb{Q}$ such that $\mathbb{E}_{\mathbb{Q}}[R_{t+1} \mid \mathcal{F}_t] = 1 + r$, which is equivalent to $\mathbb{E}_{\mathbb{Q}}\left[\xi_{t+1} \mid \mathcal{F}_t\right] = \frac{1+r-l}{h-l}$. Let

$$q_h(t,\omega) = \mathbb{Q}\left(\xi_{t+1} = 1 \mid \mathcal{F}_t(\omega)\right) = 1 - q_l(t,\omega)$$

be respectively the upward and downward risk-neutral probability at (t, ω). We obtain a unique constant risk-neutral probability given by

$$q_h = \frac{1+r-l}{h-l} = 1 - q_l. \tag{2.20}$$

With these we can derive the following pricing formula for any contingent claim $X(S_T)$ written on the stock.

Proposition 2.4. *The current market price for contingent claim* $X(S_T)$ *at maturity time* T *is given by*

$$\Psi_0(X) = \frac{1}{(1+r)^T} \sum_{m=0}^{T} \binom{T}{m} q_h^m q_l^{T-m} X\left(h^m l^{T-m} S_0\right) \tag{2.21}$$

where $\binom{T}{m} = \frac{T!}{m!(T-m)!}$ *with* $\binom{T}{0} = 1$.

Proof. Following the risk-neutral pricing rule (f), we have

$$\Psi_0(X) = (1+r)^{-T} \mathbb{E}_{\mathbb{Q}}[X(S_T)].$$

For any given $\omega \in \Omega$, the stock price at ω is determined by the number of h's and l's along the path ω. For example, if ω contains m highs and $T-m$ lows, the stock price at ω is given by $S_T(\omega) = h^m l^{T-m} S_0$, which is invariant with the order of highs and lows along the path. Let $Q_{T,m}$ be the probability of a path of length T that contains m highs. We may write

$$\Psi_0(X) = (1+r)^{-T} \sum_{m=0}^{T} Q_{T,m} X\left(h^m l^{T-m} S_0\right).$$

It remains to show $Q_{T,m} = \binom{T}{m} q_h^m q_l^{T-m}$ for all $0 \leq m \leq T$. The proof follows by applying the induction argument. First, the statement holds true for $T = 1$ with $m = 0$ and 1. Suppose it holds true for $T \geq 1$ with all $m \leq T$. Now, consider a path of length $T+1$. The statement holds with $m = 0$ because $Q_{T+1,0} = q_l^{T+1}$ is the probability of $T+1$ consecutive lows. Similarly, it holds for $m = T+1$. For $0 < m \leq T$, we may write

$$Q_{T+1,m} = Q_{T,m-1} q_h + Q_{T,m} q_l$$

where the first term is the probability that there is a high return in the final period following $m-1$ high returns in the past T periods, and the second term is the probability that it has a low return in the final period following m high returns in the past T periods. These exhaust all the possibilities of m high returns along the path of length $T+1$. With the assumed expressions for $Q_{T,m-1}$ and $Q_{T,m}$, we see that

$$Q_{T+1,m} = \binom{T+1}{m} q_h^m q_l^{T+1-m}$$

following from the fact that $\binom{T}{m-1} + \binom{T}{m} = \binom{T+1}{m}$. □

2.3.3 *The Laplace Inverse Representation of Option Price*

If we refer to Appendix B, we see that the characteristic function of a random variable X is defined as the bilateral Laplace transform of the p.d.f. $f_X(\cdot)$ that is induced by X. Let $\mathfrak{m} : \mathbb{Z} \to \mathbb{Z}$ be the characteristic function for the logarithmic stock return $\ln R_t$ that is calculated under the risk-neutral probability measure $\mathbb{Q}$. The $\mathfrak{m}$-function is also known as the *risk-neutral moment generating function*, or simply the *risk-neutral m.g.f.*, for the stock return R_t.

Under this binomial specification on stock returns, we obtain

$$\mathfrak{m}(s) = q_h h^{-s} + q_l l^{-s}, s \in \mathbb{Z}. \tag{2.22}$$

Let $z = \ln \frac{X}{S_0}$. The variable z is referred to as the *moneyness ratio* of the option. The option is said to be *in-the-money,* at-the-money or *out-of-the-money,* depending on the sign $\{-, 0, +\}$ of the moneyness ratio z.

With these notions, the price of the option can be expressed in terms of the risk-neutral m.g.f. and the moneyness ratio.

Proposition 2.5. *The time* 0 *price of a European call option with exercise price* X *and time-to-maturity* T *is given by*

$$C_0 = \frac{X}{(1+r)^T} \mathcal{L}^{-1} \left\{ \frac{\mathfrak{m}^T(s)}{s(s+1)} \right\} \left(\ln \frac{X}{S_0} \right) \tag{2.23}$$

with $\operatorname{Re}(s) < -1$.

Proof. We need to verify that the expression on the right-hand side of (2.23) coincides with the option price given by the CRR binomial model (2.21) with maturity payoff $X(S_T) = (S_T - X)^+$. First, we have

$$\begin{aligned} \mathfrak{m}^T(s) &= \left(q_h h^{-s} + q_l l^{-s}\right)^T \\ &= \sum_{m=0}^{T} \binom{T}{m} \left(q_h h^{-s}\right)^m \left(q_l l^{-s}\right)^{T-m} \\ &= \sum_{m=0}^{T} \binom{T}{m} q_h^m q_l^{T-m} e^{-(m \ln h + (T-m)\ln l)s}. \end{aligned}$$

Substituting this expression into the right-hand side of Eq. (2.23), we obtain

$$
\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{\mathfrak{m}^{T}(s)}{s(s+1)}\right\}\left(\ln\frac{X}{S_0}\right)\\
&=\sum_{m=0}^{T}\binom{T}{m}q_h^m q_l^{T-m}\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}\left(\ln\frac{X}{S_0}-m\ln h-(T-m)\ln l\right)\\
&=\sum_{m=0}^{T}\binom{T}{m}q_h^m q_l^{T-m}\left(e^{\ln\frac{S_0}{X}+m\ln h+(T-m)\ln l}-1\right)^{+}\\
&=\sum_{m=0}^{T}\binom{T}{m}q_h^m q_l^{T-m}\left(h^m l^{T-m}S_0-X\right)^{+}/X
\end{aligned}
$$

in which the first equality follows from the definition of the Laplace inverse transform and the second equality follows from the fact that $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}(x)=(e^{-x}-1)^{+}$ for $\mathrm{Re}(s)<-1$. This ends the proof. □

The Laplace inverse transform representation of the CRR binomial option pricing model can be regarded as a special case of a class of option pricing models with the underlying asset to follow arbitrary distributions, both in discrete- and continuous-time settings. The option pricing problem will be thoroughly studied later on in Chapters 10, 13, 14 and 19. As an application of the CRR binomial option pricing model, in the following section, we derive the Black–Scholes option pricing model as a limit of the CRR binomial model.

2.3.4 *Continuous-Time Limit: Black–Scholes Formula*

Here, we compute the limit of the CRR option pricing model. The limit is given by the Black–Scholes option pricing model, which was originally derived by Black and Scholes (1973) in continuous time.

Let $T>0$ be the maturity date for a European call option. Divide the time interval $[0,T]$ into n equal sub-intervals of length $\frac{T}{n}$. Assume that

- the risk-free interest rate of each sub-interval is deterministic and is given by $R_n^f=1+\frac{rT}{n}+o\left(\frac{1}{n}\right)$;
- the stock returns $\left\{R_k^{(n)}\right\}_{k=1}^{n}$ of each sub-interval is an independent

binomial process with upward and downward returns given by

$$\begin{aligned} h_n &= e^{\frac{\mu T}{n}+a\sqrt{\frac{T}{n}}+o\left(\frac{1}{n}\right)} \\ &= 1+\left(\mu+0.5a^2\right)\frac{T}{n}+a\sqrt{\frac{T}{n}}+o\left(\frac{1}{n}\right); \qquad (2.24) \\ l_n &= e^{\frac{\mu T}{n}-a\sqrt{\frac{T}{n}}+o\left(\frac{1}{n}\right)} \\ &= 1+\left(\mu+0.5a^2\right)\frac{T}{n}-a\sqrt{\frac{T}{n}}+o\left(\frac{1}{n}\right) \qquad (2.25) \end{aligned}$$

where μ and a are positive constants. Notice that if there is an equal chance of receiving a high return or a low return, then a measures the volatility of the return. Keeping a unchanged, we see that the higher the μ, the higher the expected return.

Now, for n sufficiently large, we have $h_n > R_n^f > l_n$ so that the no-arbitrage condition is satisfied for the corresponding binomial economies. The resulting risk-neutral probabilities are well-defined and are given by

$$q_h^{(n)} = \frac{R_n^f - l_n}{h_n - l_n} = \frac{1}{2} - \frac{\mu - r + 0.5a^2}{2a}\sqrt{\frac{T}{n}} + o\left(\frac{1}{\sqrt{n}}\right) \qquad (2.26)$$

with $q_l^{(n)} = 1 - q_h^{(n)}$. Moreover, the corresponding m.g.f. for $\ln R_k^{(n)}$ under the risk-neutral measure $\mathbb{Q}_n$ is given by

$$\mathfrak{m}_n(s) = q_h^{(n)} h_n^{-s} + q_l^{(n)} l_n^{-s}, s \in \mathbb{Z}. \qquad (2.27)$$

Accordingly, $\mathfrak{m}_n^n(\cdot)$ is the m.g.f. for $\sum_{k=1}^n \ln R_k^{(n)}$, the sum of n identical independent random variables under measure $\mathbb{Q}_n$.

To compute the limit of the resulting option price $\left\{C_0^{(n)}\right\}$ as $n \to \infty$, one may follow a two-step procedure. First, compute the limit of the probability distribution function under $\mathbb{Q}_n$, for the maturity stock price S_n at time T $(= n \times \frac{T}{n})$, by applying the central limit theorem. Second, verify that the limiting distribution function for the stock price is log-normal under the corresponding risk-neutral probability, and is the same as that assumed in the original continuous-time Black–Scholes model.

Here, we provide a different approach. Consider the sequence of m.g.f.s $\{\mathfrak{m}_n^n\}_{n=0}^\infty$ for the log-total returns up to maturity date T. First, we show that it has a limit $\mathfrak{m}_{\mathrm{BS}}^T$, where $\mathfrak{m}_{\mathrm{BS}}$ corresponds to an m.g.f. for a Gaussian normal random variable (as is assumed by Black and Scholes (1973) for the log-stock returns). Then, we can substitute the limit m.g.f. into the right-hand side of (2.23), with the limit discount factor given by e^{-rT}, to obtain the Laplace inverse transform expression of the Black–Scholes formula.

Theorem 2.2. *The option price* $\left\{C_0^{(n)}\right\}_{n=1}^{\infty}$ *corresponding to the CRR binomial option price model (2.23) has a limit which is given by the Black–Scholes option price*

$$C_0^{BS} = S_0 N\left(d_1\left[z\right]\right) - Xe^{-rT} N\left(d_2\left[z\right]\right) \tag{2.28}$$

in which $N\left(x\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{x^2}{2}}dx$ *is the standard Gaussian normal c.d.f., and* $d_{1,2}\left[z\right] = \frac{\left[r \pm 0.5a^2\right]T - z}{a\sqrt{T}}$ *and* $z = \ln\frac{X}{S_0}$*. Furthermore, the Black–Scholes option price admits this alternative expression*

$$C_0^{BS} = Xe^{-rT}\mathcal{L}^{-1}\left\{\frac{\mathfrak{m}_{BS}^{T}\left(s\right)}{s\left(s+1\right)}\right\}\left(z\right) \tag{2.29}$$

with $\mathfrak{m}_{BS}\left(s\right) = e^{-\left(r-0.5a^2\right)s + 0.5a^2s^2}$ *and* $\mathrm{Re}\left(s\right) < -1$*.*

Proof. We start with some order assessments respectively for h_n^{-s} and l_n^{-s}:

$$\begin{aligned}
h_n^{-s} &= e^{-\mu s\frac{T}{n} - as\sqrt{\frac{T}{n}} + o\left(\frac{1}{n}\right)} \\
&= 1 - \left(\mu s - 0.5a^2s^2\right)\frac{T}{n} - as\sqrt{\frac{T}{n}} + o\left(\frac{1}{n}\right) \\
l_n^{-s} &= e^{-\mu s\frac{T}{n} + as\sqrt{\frac{T}{n}} + o\left(\frac{1}{n}\right)} \\
&= 1 - \left(\mu s - 0.5a^2s^2\right)\frac{T}{n} + as\sqrt{\frac{T}{n}} + o\left(\frac{1}{n}\right).
\end{aligned}$$

These, together with $q_h^{(n)}$ given above, enable us to obtain the risk-neutral m.g.f. for the stock return on each sub-interval as described by

$$\begin{aligned}
&q_h^{(n)}h_n^{-s} + q_l^{(n)}l_n^{-s} \\
&= 1 - \left(\mu s - 0.5a^2s^2\right)\frac{T}{n} + as\left(1 - 2q_h^{(n)}\right)\sqrt{\frac{T}{n}} + o\left(\frac{1}{n}\right) \\
&= 1 - \left(\mu s - 0.5a^2s^2\right)\frac{T}{n} + s\left(\mu - r + 0.5a^2\right)\frac{T}{n} + o\left(\frac{1}{n}\right) \\
&= 1 - \left[\left(r - 0.5a^2\right)s - 0.5a^2s^2\right]\frac{T}{n} + o\left(\frac{1}{n}\right).
\end{aligned}$$

Therefore the risk-neutral m.g.f. at the maturity date is given by

$$\begin{aligned}\mathfrak{m}_n^n(s) &= \left[q_h^{(n)} h_n^{-s} + q_l^{(n)} l_n^{-s}\right]^n \\ &= e^{n \ln\left(1-\left[\left(r-0.5a^2\right)s-0.5a^2s^2\right]\frac{T}{n}+o\left(\frac{1}{n}\right)\right)}\end{aligned}$$

We set $n \to \infty$ to obtain $\lim_{n\to\infty} \mathfrak{m}_n^n(s) = e^{-\left(r-0.5a^2\right)Ts+0.5Ta^2s^2} = \mathfrak{m}_{\mathrm{BS}}^T(s)$.

Now, consider the CRR binomial option pricing model (2.23) with $\mathfrak{m}^T(s)$ replaced by $\mathfrak{m}_n^n(s)$. Again, setting $n \to \infty$ we produce expression (2.29) as the limit of the CRR binomial option prices. The expression (2.28) for the Black–Scholes option price is then obtained from expression (2.29) by computing the Laplace inverse transform of $\frac{\mathfrak{m}_{\mathrm{BS}}^T(s)}{s(s+1)}$ with $\mathrm{Re}(s) < -1$. □

Remark 2.1. For a dividend-paying stock with a constant dividend yield κ, by repeating the same procedure as for the zero-dividend paying stock, we see the validity of the option pricing formula (2.29) with a revised $\mathfrak{m}_{\mathrm{BS}}(\cdot)$-function

$$\mathfrak{m}_{\mathrm{BS}}(s) = e^{\left(\kappa-r+0.5a^2\right)s+0.5a^2s^2}.$$

The corresponding Black–Scholes option price with a constant dividend yield κ is thus given by

$$C_0^{\mathrm{BS}} = S_0 e^{-\kappa T} N\left(d_1[z]\right) - X e^{-rT} N\left(d_2[z]\right)$$

with $d_{1,2}[z] = \frac{\left[r-\kappa\pm 0.5a^2\right]T-z}{a\sqrt{T}}$ and $z = \ln\frac{X}{S_0}$.

2.3.5 *Option Pricing and State Prices*

From the fundamental theorem, if we know the state prices we can price all contingent claims, particularly those European options written on the underlying stock. This section considers the inversion problem.

Problem 2.2. *Suppose we know how options are priced with the knowledge of the risk-free interest rate and the initial stock price, what can we say about the state prices and the corresponding risk-neutral probability measure? Or can we find out the state prices and the risk-neutral probability measure from the option pricing rule?*

The CRR option pricing formula (2.23) in the binomial framework constitutes an option pricing rule which is summarized by the Laplace inverse transform of an analytic function $\frac{\mathfrak{m}^T(s)}{s(s+1)}$. Let

$$g(x) = \mathcal{L}^{-1}\left\{\frac{\mathfrak{m}(s)}{s(s+1)}\right\}(x), x \in \mathbb{R}.$$

Knowing the g-function, we can reveal uniquely the complex function

$$\mathfrak{m}(s) = s(s+1)\mathcal{L}\{g(x)\}(s), s \in \mathbb{Z}.$$

Therefore, the option pricing rule is fully summarized by the g-function: one can price European call options of all maturities T and of all exercise prices X with knowledge of the risk-free interest rate r and the initial stock price S_0. The g-function reveals the $\mathfrak{m}$-function, which in turn determines the risk-neutral probabilities. Therefore, with the knowledge of the option pricing rule summarized by g, one can price all contingent claims and all derivative securities written on the stock.

Moreover, we have the following alternative expression for the price $\Psi_0(X)$ of any contingent claim $X(S_T)$ written on the stock with maturity date T.

Proposition 2.6. *The current market price for contingent claim $X(S_T)$ at maturity time T is given by*

$$\Psi_0(X) = (1+r)^{-T} \int_{\mathbb{R}} \mathcal{L}^{-1}\left\{\mathfrak{m}^T(s)\right\}(x - \ln S_0) X(e^x)\, dx. \tag{2.30}$$

Proof. Its price $\Psi_0(X)$ is given by Eq. (2.21). This price admits an alternative expression in terms of the risk-neutral p.d.f.

$$\begin{aligned}\Psi_0(X) &= (1+r)^{-T}\mathbb{E}_{\mathbb{Q}}\left[X\left(S_0\prod_{t=1}^{T} R_t\right)\right]\\ &= (1+r)^{-T}\int_{\mathbb{R}} X\left(e^{\ln S_0 + x}\right) f(x)\, dx\\ &= (1+r)^{-T}\int_{\mathbb{R}} f(x - \ln S_0) X(e^x)\, dx\end{aligned}$$

where f is the p.d.f. for $\sum_{t=1}^{T}\ln R_t$ under the risk-neutral measure $\mathbb{Q}$. Since $\ln R_t$ has its risk-neutral m.g.f. given by $\mathfrak{m}(s)$, the resulting risk-neutral m.g.f. for $\sum_{t=1}^{T}\ln R_t$ is given by $\mathfrak{m}^T(s)$. Therefore the risk-neutral p.d.f. f can be expressed as $f(x) = \mathcal{L}^{-1}\left\{\mathfrak{m}^T(s)\right\}(x)$. This gives the desired expression (2.30) for the price of the contingent claim. □

Remark 2.2. We have established the relationship

$$g \Leftrightarrow \mathfrak{m} \Leftrightarrow \psi \Leftrightarrow \Psi_0(\cdot),$$

taking the interest rate r as given. This equivalence between the state prices and the option pricing rule is for the CRR binomial model. Later, we shall show that such equivalence holds true for a fairly general class of models.

2.3.6 *Backward Procedure*

The CRR binomial model can be analyzed alternatively by following a backward recursive procedure or, simply, *backward procedure*. To illustrate the backward procedure, let $\zeta_t(\omega) = \sum_{s=1}^{t} \xi_s(\omega)$, which corresponds to the number of upward moves up to t along path ω. We have $\zeta_t \in \{0, 1, \cdots, t\}$. The time t price of the contingent claim, denoted by Ψ_t, depends on the number of upward moves realized at and before t, and is thus adapted to $\{\zeta_t\}$. We may write $\Psi_t = \Psi(t, \zeta_t)$.

With these, together with the prescribed risk-neutral probabilities, the risk-neutral pricing rule

$$\begin{aligned}\Psi_t &= \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}\left[\Psi_{t+1} \mid \mathcal{F}_t\right] \\ &= \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}\left[\Psi\left(t+1, \zeta_t + \xi_{t+1}\right) \mid \zeta_t\right]\end{aligned}$$

yields the following recursive equation for the price process $\{\Psi_t\}$ of the contingent claim:

$$\Psi(t, \zeta_t) = (1+r)^{-1}\left[q_h \Psi(t+1, \zeta_t + 1) + q_l \Psi(t+1, \zeta_t)\right] \tag{2.31}$$

for all $t = 0, 1, \cdots, T$.

Consequently, the option price given by Eq. (2.21) can be regarded as a solution to the following set of recursive equations:

- At maturity date T, setting $\Psi_T(m) = X\left(h^m l^{T-m} S_0\right)$ for all $0 \leq m \leq T$;
- For $0 \leq t < T$, setting $\Psi_t(m) = (1+r)^{-1}[q_h \Psi_{t+1}(m+1) + q_l \Psi_{t+1}(m)]$ for all $0 \leq m \leq t$; and
- The initial price $\Psi_0(X)$ of the contingent claim is given by $\Psi_0(0)$.

The procedure used to determine the initial price of the contingent claim by solving the recursive system is referred to as 'backward procedure'. It is so named because the system can be solved by working backwards starting from the calculations of the payoffs of the contingent claim at the maturity date to determine the spot price of the claim in the previous period, so on and so forth, and ending up with a unique time 0 price for the contingent claim.

The usefulness of the backward procedure lies in the fact that it applies to a general binomial framework with stochastic interest rates along with dividend-paying stocks with stochastic dividend rates. As will be illustrated below, the backward procedure provides a framework to compute the price of a contingent claim in the face of stochastic interest rates and stochastic dividend payments when an analytic expression for the price of a contingent claim is not readily available. We shall also illustrate in Section 2.3.7. how the backward procedure is useful for pricing American options for which investors may choose to exercise their options at any time before a pre-specified maturity date. It is well-known in the literature that analytic expressions for American options are generally not available.

2.3.6.1 *Backward Procedure with Stochastic Interest Rates*

We analyze contingent claims written on zero-dividend paying stocks with constant upward and downward returns $\{h, l\}$. We consider the case when the time t interest rate r_t is adapted to the number of historic upward moves so that we may write $r_t = r(t, \zeta_t)$. Assume that

$$h > 1 + r(t, m) > l \tag{2.32}$$

for all t and for all $m \in \{0, 1, \cdots, t\}$. Under this assumption, the time t spot market satisfies the no-arbitrage condition with risk-neutral probabilities well-defined and adapted to the number of historic upward moves ζ_t. We have

$$q_h(t, \zeta_t) = \frac{1 + r(t, \zeta_t) - l}{h - l} = 1 - q_l(t, \zeta_t). \tag{2.33}$$

According to the risk-neutral pricing rule (f) of no-arbitrage, the price process $\{\Psi_t\}$ for a contingent claim must be adapted to the number of historic upward moves $\{\zeta_t\}$ with $\Psi_t = \Psi(t, \zeta_t)$ to satisfy

$$\Psi(t, \zeta_t) = \frac{q_h(t, \zeta_t)\,\Psi(t+1, \zeta_t+1) + q_l(t, \zeta_t)\,\Psi(t+1, \zeta_t)}{1 + r(t, \zeta_t)}. \tag{2.34}$$

With the derived risk-neutral probabilities, the backward procedure for computing the price of a contingent claim $X(S_T)$ can be revised accordingly. In this case, the backward procedure can be stated precisely as follows:

- At maturity date T, for $m = 0, 1, \cdots, T$, setting $\Psi(T, m) = X\left(h^m l^{T-m} S_0\right)$;
- For $0 \le t < T$ and $m = 0, 1, \cdots, t$, setting

$$\Psi(t, m) = \frac{q_h(t, m)\,\Psi(t+1, m+1) + q_l(t, m)\,\Psi(t+1, m)}{1 + r(t, m)}; \tag{2.35}$$

- Setting $\Psi_0(X) = \Psi(0, 0)$.

2.3.6.2 *Backward Procedure for Dividend-Paying Stocks*

We consider a contingent claim written on a dividend-paying stock. The setup is the same as that described in the original CRR model with constant upward and downward returns $\{h, l\}$ except (i) we allow for the possibility of stochastic interest rates and (ii) the stock has non-zero and stochastic dividend yields, or has non-zero stochastic dividend-price ratios. We assume that both the interest rate $\{r_t\}$ and the dividend yields or dividend price ratios $\{\kappa_t\}$ are adapted to the number of historic upward moves $\{\zeta_t\}$ so that we may write $r_t = r(t, \zeta_t)$ and $\kappa_t = \kappa(t, \zeta_t)$.

For the case of stochastic dividend yields, we need to assume

$$h > 1 + r(t, m) - \kappa(t, m) > l$$

for all t and $m \in \{0, 1, \cdots, t\}$ so that the market admits no-arbitrage opportunities. The resulting time t risk-neutral upward probabilities are given by

$$q_h(t, \zeta_t) = \frac{1 + r(t, \zeta_t) - \kappa(t, \zeta_t) - l}{h - l} = 1 - q_l(t, \zeta_t). \tag{2.36}$$

Similarly, for the case of a stochastic dividend-price ratio, we assume

$$h > \frac{1 + r(t, m)}{1 + \kappa(t, m)} > l$$

for all t and $m \in \{0, 1, \cdots, t\}$. In this case, the no-arbitrage condition is satisfied with the risk-neutral probability to be given by

$$q_h(t, \zeta_t) = \frac{\frac{1 + r(t, \zeta_t)}{1 + \kappa(t, \zeta_t)} - l}{h - l} = 1 - q_l(t, \zeta_t). \tag{2.37}$$

For either case, the corresponding price process $\{\Psi_t\}$ for the contingent claim must satisfy the recursive Eq. (2.34) with its current price determined by following the backward procedure:

- At maturity date T, for $m = 0, 1, \cdots, T$, setting
$$\Psi(T, m) = X\left(h^m l^{T-m} S_0\right);$$
- For $0 \le t < T$ and $m = 0, 1, \cdots, t$, setting
$$\Psi(t, m) = \frac{q_h(t, m)\Psi(t + 1, m + 1) + q_l(t, m)\Psi(t + 1, m)}{1 + r(t, m)}; \text{ and}$$
- Setting $\Psi_0(X) = \Psi(0, 0)$.

Notice that, for European call options written on a dividend-paying stock, the presence of positive dividend payout reduces the upward risk-neutral probability, and hence reduces the option price.

2.3.7 *American Options*

As before, we take as given the stock price process S and constant risk-free interest rate $R^f = 1 + r$. The payoffs associated with an American option are summarized by $\{X_t\}_{t=0}^T$ such that, for all (t, ω), $X_t(\omega)$ is the payoff if the option is exercised at (t, ω). For an American call option written on a stock S with exercise price X, the payoff at (t, ω) is $X_t(\omega) = S_t(\omega) - X$ if $t < T$, and $X_T(\omega) = (S_T(\omega) - X)^+$ at the maturity date T. Similarly, for an American put option, the payoff becomes $X - S_t(\omega)$ if $t < T$, and $(X - S_T(\omega))^+$ at T.

American options are different from the corresponding European options since investors who have a long position in an American option can exercise the option at any time before the maturity date T. This, generally speaking, makes American options more valuable than the corresponding European options (given the same exercise price and the same maturity dates). Two related questions need to be answered to price American options:

- At each point of time t before the maturity date T and with time t information $\mathcal{F}_t(\omega)$ along each path ω, what is the sensible price for the option?
- At which times and at what states will the option holder choose to exercise the option?

Given the payoffs $\{X_t\}_{t=0}^T$, let $\Psi_t(X; \omega)$ be the corresponding market price for the option at (t, ω) if it is not exercised before t along the path ω. We take as given the underlying risk-neutral probability measure $\mathbb{Q}$. We have:

Theorem 2.3. *The American option price* $\{\Psi_t(X)\}_{t=0}^T$ *and the optimal exercise time* τ^* *are such that*

(1) At all (t, ω),

$$\Psi_t(X; \omega) = \max\left\{X_t(\omega), (1 + r)^{-1} \mathbb{E}_{\mathbb{Q}}\left[\Psi_{t+1}(X) \mid \mathcal{F}_t(\omega)\right]\right\}; \text{ and} \tag{2.38}$$

(2) For all (t, ω), $\tau^*(\omega) = t$ *if and only if* $\Psi_s(X; \omega) > X_s(\omega)$ *for all* $s < t$, *and* $\Psi_t(X; \omega) = X_t(\omega)$; *or, equivalently,*

$$\tau^*(\omega) = \inf\{t \in \{0, \cdots, T\} : \Psi_t(X; \omega) = X_t(\omega)\}. \tag{2.39}$$

Proof. Consider the following trading strategy induced by τ: for all ω, let $\tau(\omega) \in \{0, \cdots, T\}$ be the time to exercise the option along the path

$\omega \in \Omega$. Here, τ is $\mathcal{F}_t$-measurable, and is referred to as a *random stopping time*, or simply a *stopping time*. The resulting time t cash flow $d \in \mathbb{D}$ induced by the stopping time τ is such that $d_t(\omega) = X_t(\omega)$ if $\tau(\omega) = t$ and that $d_t(\omega) = 0$ if $\tau(\omega) \neq t$. By the no-arbitrage condition, the initial cost of the trading strategy corresponding to τ is $\mathbb{E}_{\mathbb{Q}}\left[(1+r)^{-\tau} X_\tau\right]$, which is the present value of d under the measure $\mathbb{Q}$.

First, we claim that the value of the American option is no less than the initial cost of a trading strategy resulting from any arbitrary stopping times; that is,

$$\Psi_0(X) \geq \sup_{0 \leq \tau \leq T} \mathbb{E}_{\mathbb{Q}}\left[(1+r)^{-\tau} X_\tau\right].$$

Otherwise, if investors can acquire the option at a lower price, they can achieve at least the same payoffs as those resulting from τ except with a lower initial cost. This violates the no-arbitrage condition.

Second, the value of the American option $\Psi_0(X)$ must be no more than $\sup_{0 \leq \tau \leq T} \mathbb{E}_{\mathbb{Q}}\left[(1+r)^{-\tau} X_\tau\right]$. This is because, for finite economies, there exists an optimal random time τ^* to exercise the option. Therefore we must have

$$\Psi_0(X) = \max_{0 \leq \tau \leq T} \mathbb{E}_{\mathbb{Q}}\left[(1+r)^{-\tau} X_\tau\right]. \tag{2.40}$$

Similarly, for all (t, ω), consider the price $\Psi_t(X; \omega)$ of the American option at $(t, \mathcal{F}_t(\omega))$, which admits the same mathematical expression by substituting T with $T - t$. The option price expressed in this way obviously satisfies the following recursiveness principle: if $\tau^*(\omega) > t$, then

$$\Psi_t(X; \omega) = (1+r)^{-1} \mathbb{E}_{\mathbb{Q}}\left[\Psi_{t+1}(X) \mid \mathcal{F}_t(\omega)\right]$$

and, for all $t \leq T$ and $\omega \in \Omega$, $\tau^*(\omega) = t$, if and only if $\Psi_t(X; \omega) = X_t(\omega)$ and $\Psi_s(X; \omega) > X_s(\omega)$ for all $s < t$. These are equivalent to statements 1 and 2 of the theorem. □

As an example, we consider an American call option written on a zero-dividend paying stock with exercise price X and maturity date T.

Proposition 2.7. *For American call options written on zero-dividend paying stocks, investors will not exercise their options before maturity date T; that is, $\tau = T$. In particular, the price of an American call option must be the same as that of the European call option.*

Proof. By definition, the payoff of the American call option is given by $S_t(\omega) - X$ if the investor chooses to exercise the option at (t, ω). Also, by definition, the value of the option at (t, ω) is no less than that of the European call option. That is,

$$\begin{aligned} C_t(\omega) &\geq (1+r)^{-(T-t)} \mathbb{E}_{\mathbb{Q}} \left[(S_T - X)^+ \mid \mathcal{F}_t(\omega) \right] \\ &\geq (1+r)^{-(T-t)} \mathbb{E}_{\mathbb{Q}} \left[S_T - X \mid \mathcal{F}_t(\omega) \right] \\ &= S_t(\omega) - X(1+r)^{-(T-t)}. \end{aligned}$$

With positive interest rate $r > 0$, we have $\Psi_t(X; \omega) > S_t(\omega) - X$ unless $T = t$. This implies that the investor will not exercise his option at $t < T$. At maturity date T, the option is exercised only when $S_T(\omega) > X$. Therefore, by the one-price principle (a), the value of the American call option must coincide with that of the European call option since they have the same exercise time, which is the maturity date, and the same payoffs. □

This observation about American call options, in general, does not hold true for options written on dividend-paying stocks. This is because, by exercising his option early, an investor not only receives a lump-sum payoff of $S_t(\omega) - X$, but also gains the claims towards all future dividend payments for the stock. But this is at a cost of losing the opportunity to exercise the option at a possibly much higher payoff $S_{t'}(\omega) - X$ at a future time t'. Therefore, there is a trade-off between early and late exercising, which makes the earlier exercising possible.

This observation about American call options would, in general, not hold true for American put options. Investors may choose to exercise American put options early even for stocks with zero dividend payments.

In general, to compute American option prices and to determine the optimal exercising times for American options, we follow the backward procedure illustrated below.

Let the stock price $\{S_t\}_{t=0}^{T}$ follow a binomial process with returns $\{h, l\}$. Suppose the risk-free interest rate and dividend yields or dividend-price ratio are state-dependent so that $r_t = r(t, \zeta_t)$ and $\kappa_t = \kappa(t, \zeta_t)$ with the risk-neutral probability measures $\mathbb{Q}$ respectively given by Eq. (2.36) and Eq. (2.37). Let $\Psi(t, m)$ be the time t price of the American option $\{X_t(S_t)\}_{t \geq 0}^{T}$ written on the stock (with zero or positive constant dividend yield) following m upward moves before time t. For all t and $m \leq t$, let $X_t(m) = X_t(h^m l^{t-m} S_0)$. Then, we see that the American option price $\{\Psi_t\}_{t=0}^{T}$ in this CRR binomial model can be computed recursively as follows:

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- At T, for all $m \in \{0, \cdots, T\}$, setting $\Psi(T, m) = X_T(m)$;
- For $0 \leq t < T$ and $m \in \{0, \cdots, t\}$, setting

$$\Psi(t, m) = \max\left\{X_t(m), \frac{q_h(t, m)\,\Psi(t+1, m+1) + q_l(t, m)\,\Psi(t+1, m)}{1 + r(t, m)}\right\};$$

- Setting $\Psi_0 = \Psi(0, 0)$.

Moreover, the exercising time τ^* for the American option can be determined accordingly with the help of the backward procedure. In fact, given

$$\{\Psi(t, m), m \in \{0, 1, \cdots, t\}\}_{t=0}^{T}$$

as a solution to the backward procedure, then, along each path $\omega \in \Omega$ it is optimal to exercise the option at t (i.e. $\tau^*(\omega) = t$) if and only if

(a) $\Psi(t, \zeta_t(\omega)) = X_t(\zeta_t(\omega))$; and
(b) $\Psi(s, \zeta_s(\omega)) > X_s(S_s(\omega))$ for all $s < t$.

2.4 Ho–Lee Model of the Term Structure of Interest Rates

Another successful application of the fundamental theorem for pricing securities in a dynamic framework is the Ho and Lee (1986) binomial model of the term structure of interest rates.

The primitive securities in the Ho–Lee model consist of discount bonds of all maturities, denoted by $\{B_{t,T}\}_{t,T=0}^{\infty}$, which constitute the entire term structure of interest rates at time t. Here, $B_{t,T}$ is the time t price of a *discount bond* with time-to-maturity T (or with time-at-maturity $T + t$). Recall that a discount bond is a zero-coupon bond with unit payoff at maturity date.

Similar to that described in the CRR model, the dynamics of the bond price is assumed to be driven by a binomial random process $\{\xi_t\}_{t=1}^{\infty}$, where, for all t, ξ_t will take the values $\{1, 0\}$. Here 1 and 0 stand for 'bull' and 'bear', or simply 'high' and 'low', market respectively. Therefore, the universal state space Ω consists of all distinct paths on the realization of $\{\xi_t\}_{t=1}^{\infty}$. For each $\omega \in \Omega$, $\mathcal{F}_t(\omega)$ contains all paths that are consistent with the historic observations up to time t along path ω. $\{\xi_t\}_{t=1}^{\infty}$ is referred to as a state process and ξ_t is the time t state variable.

The theory is built on the following assumptions:

- **Stationarity**: There exists a deviation function $u : \{1, 0\} \times \mathcal{T} \mapsto \mathbb{R}_+$, which is deterministic and time-independent, such that, for all t and T,
$$B_{t+1,T} = B_{t,1}^{-1} B_{t,T+1} u\left(\xi_{t+1}, T\right). \tag{2.41}$$
- **Path independence**: For all t, $\{B_{t,T}\}_{T=0}^{\infty}$ is $\mathcal{F}_t(\omega)$-measurable with $B_{t,0} = 1$. In particular, $B_{t,T}$ is assumed to be adapted to the number of 'bulls' at and before t.

It follows from this last assumption of path independence that we may write $B_{t,T} = B_{t,T}(\zeta_t)$ so that, for all t and $m = 0, \cdots, t$, $B_{t,T}(m)$ stands for the time t bond price following m highs along the historic path at and before t (see Fig. 2.2).

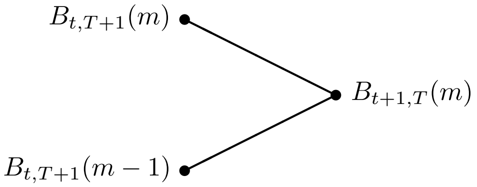

Fig. 2.2 Path independence

Under the stationarity assumption, the time t return $R_{t+1,T} = \frac{B_{t+1,T}}{B_{t,T+1}}$ of a $(T+1)$-period bond admits the following expression

$$R_{t+1,T} = R_t^f u\left(\xi_{t+1}, T\right)$$

with risk-free interest rate $R_t^f = B_{t,1}^{-1}$ corresponding to the return of the one-period bond at t. So, the function $u(\xi, T)$ measures the deviation of the long-term bond returns from the short-term risk-free return, and is thus known as the deviation function.

Alternatively, with the futures price for delivering a T-period discount bond in the following period at time $t+1$ given by $F_t(1, T) = B_{t,1}^{-1} B_{t,T+1}$, we may write $B_{t+1,T} = F_t(1,T) u\left(\xi_{t+1}, T\right)$. Hence, the deviation function $u(\xi, T)$ also measures the deviation of the actual spot price $B_{t+1,T}$ of the T-period bond from its futures price $F_t(1,T)$ settled in the previous period (at time t).

Let $u_i(T) = u(i, T), i \in \{0, 1\}$. The assumptions of stationarity and path independence, together with the no-arbitrage condition for the bond market, enable us to obtain the following characterization for the deviation function that is summarized by $\{u_0(T), u_1(T)\}$.

Proposition 2.8. *Under the assumptions of stationarity and path independence, no-arbitrage implies the existence of two positive constants* q^* *and* $\gamma \in (0,1)$ *such that the deviation function* u *is expressed as*

$$u_1(T) = \frac{1}{q^* + (1-q^*)\gamma^T} \text{ and } u_0(T) = \frac{\gamma^T}{q^* + (1-q^*)\gamma^T}. \tag{2.42}$$

Proof. For all (t,ω), let $q_1(t,\omega) = \mathbb{Q}\left(\xi_{t+1} = 1 \mid \mathcal{F}_t(\omega)\right)$ and $q_0(t,\omega) = 1 - q_1(t,\omega)$ be the risk-neutral probabilities. The risk-free return rule of no-arbitrage leads to

$$q_1(t,\omega)\,u_1(T) + (1 - q_1(t,\omega))\,u_0(T) = 1 \tag{2.43}$$

which holds true at all (t,ω) and for all T. Solving for $q_1(t,\omega)$ we obtain $q_1(t,\omega) = \frac{1-u_0(T)}{u_1(T)-u_0(T)}$. This implies that the risk-neutral probability $q_1(t,\omega)$ must be constant in (t,ω), which is thus denoted by q^*. This results in the following equation for the deviation function:

$$q^* u_1(T) + (1-q^*)\,u_0(T) = 1 \text{ for all } T. \tag{2.44}$$

By path independence, as illustrated by Fig. 2.2, we obtain

$$B_{t,1}^{-1}(m)\,B_{t,T+1}(m)\,u_0(T) = B_{t,1}^{-1}(m-1)\,B_{t,T+1}(m-1)\,u_1(T)$$

in which both sides are equal to $B_{t+1,T}(m)$. This gives

$$\frac{B_{t,T+1}(m)}{B_{t,T+1}(m-1)} = \frac{B_{t,1}(m)}{B_{t,1}(m-1)}\frac{u_1(T)}{u_0(T)}.$$

This, together with the stationarity assumption, yields

$$\frac{u_1(T+1)}{u_0(T+1)} = \frac{u_1(1)}{u_0(1)}\frac{u_1(T)}{u_0(T)}. \tag{2.45}$$

Let $\gamma = \frac{u_0(1)}{u_1(1)} < 1$. From Eq. (2.45) we obtain the second restriction on the deviation functions; that is,

$$u_1(T) = \gamma^{-T} u_0(T). \tag{2.46}$$

From Eqs. (2.44) and (2.46) we obtain the desired expression for the deviation function that is summarized by $\{u_1(T), u_0(T)\}$. □

Remark 2.3. When the state variable ξ is given by the stock return as in the CRR framework, q^* must coincide with that of Eq. (2.36). This leads to a constant interest rate R^f with $\gamma = 1$ and a flat initial term structure of interest rates given by $B_{0,t} = \left(R^f\right)^{-t}$. Therefore, in the following we assume that the state variable ξ captures a source of uncertainty that affects the bond prices only. We may interpret ξ as a state variable that captures the uncertainty associated with monetary policy controlled by the government. In this case, the CRR model remains valid with stochastic interest rates along with stochastic risk-neutral probabilities for the stock returns.

2.4.1 *A No-Arbitrage Bond Pricing Model*

We have seen that the no-arbitrage condition imposes some very specific restrictions on the dynamics of the term structure of interest rates through the deviation function u. The resulting Ho–Lee bond pricing model depends on two unknown parameters (q^*, γ). Hence we call the Ho–Lee model a model of a two-parameter class.

Proposition 2.9. *A stationary and path-independent term structure of interest rates* $\{B_{0,T}\}_{T=0}^{\infty}$ *satisfies the no-arbitrage condition if and only if there exists* $q^*, \gamma \in (0,1)$ *such that*

$$\ln B_{t,T} = \varphi\left(q^*, \gamma; t, T\right) + T \ln \gamma \times \left(t - \zeta_t\right), \tag{2.47}$$

where $\varphi\left(q^*, \gamma; t, T\right) = \ln\left(\frac{B_{0,T+t}}{B_{0,t}} \frac{u_1(T+t-1)\cdots u_1(T)}{u_1(t-1)\cdots u_1(0)}\right)$ *is a deterministic function of* (t, T).

Proof. For the given initial term structure $\{B_{0,T}\}_{T=0}^{\infty}$, with the deviation function u given by (2.42), we follow the standard induction arguments and show that the time t bond price is fully determined by the number of highs along its historic path and is given by

$$B_{t,T}\left(m\right) = \frac{B_{0,T+t}}{B_{0,t}} \frac{u_1\left(T+t-1\right)\cdots u_1\left(T\right)}{u_1\left(t-1\right)\cdots u_1\left(0\right)} \gamma^{(t-m)T} \tag{2.48}$$

for all $t \geq 1, T \geq 0, m = 0, \cdots, t$. The necessary part of the proposition follows from Eq. (2.48) as the number of highs at and before t is given by $\zeta_t = \sum_{s=1}^{t} \xi_s$. For sufficiency, suppose the dynamics of $\{B_{t,T}\}_{T=0}^{\infty}$ are governed by Eq. (2.47) with $u_1\left(T\right) = \left(q^* + \left(1-q^*\right)\gamma^T\right)^{-1}$ and $u_0\left(T\right) = u_1\left(T\right)\gamma^T$. We can easily verify that bond prices satisfy the stationarity and path independence conditions. Moreover, we may define the risk-neutral measure by setting $\mathbb{Q}\{\xi_t = 1\} = q^* = 1 - \mathbb{Q}\{\xi_t = 0\}$ and verify that the bond price of all maturities satisfies the risk-free return rule (g) of no-arbitrage. This concludes the proof. $\square$

Remark 2.4. The logarithmic bond price $\ln B_{t,T}$ is bounded within the interval

$$[\varphi(q^*, \gamma; t, T) + tT \ln \gamma, \varphi(q^*, \gamma; t, T)].$$

Accordingly, the curves $\left\{-\frac{\varphi(q^*,\gamma;t,T)}{T}\right\}_{T>0}$ and $\left\{-\frac{\varphi(q^*,\gamma;t,T)}{T} - t \ln \gamma\right\}_{T>0}$ constitute respectively the lower and the upper bounds for the time-t yield curve $\left\{-\frac{1}{T} \ln B_{t,T}\right\}_{T>t}$.

Remark 2.5. If we know the true probability distribution for the state process, we can derive the resulting distribution function for the bond price. For instance, if the Bernoulli process $\left\{\xi_t \overset{\text{dist}}{\sim} (q, 1; 1-q, 0), t \in \mathcal{T}\right\}$ is independent, then the mean and variance of the bond return can be readily determined.

$$\begin{aligned} \mathbb{E}_0[\ln B_{t,T}] &= \varphi(q^*, \gamma; t, T) + (1-q)\, tT \ln \gamma \\ \mathrm{Var}_0[\ln B_{t,T}] &= q(1-q)\, t\, (T \ln \gamma)^2 . \end{aligned}$$

2.4.2 *Risk-Free Interest Rate*

From expression (2.48) we see that short-term bonds ($T = 1$) prices are given by

$$B_{t,1}(m) = \frac{B_{0,t+1}}{B_{0,t}} u_1(t)\, \gamma^{t-m}. \tag{2.49}$$

The corresponding one-period risk-free return is

$$R_t^f(m) = \frac{B_{0,t}}{B_{0,t+1}} u_1^{-1}(t)\, \gamma^{m-t}. \tag{2.50}$$

In light of the term structure of interest rates model (2.47), the risk-free interest rate process becomes

$$\ln R_t^f = -\varphi(q^*, \gamma; t, 1) + \ln \gamma \times (\zeta_t - t), \forall t \geq 1 \tag{2.51}$$

with $\ln R_0^f = -\ln B_{0,1}$.

2.4.3 *Yield Curve*

The *yield-to-maturity* (at time t) for a T-period discount bond is defined by setting $Y_{t,T}^{-T} \triangleq B_{t,T}$. Since, by the risk-neutral pricing rule,

$$B_{t,T} = \mathbb{E}_{\mathbb{Q}}\left[\left(R_t^f \cdots R_{T+t-1}^f\right)^{-1}\right] = Y_{t,T}^{-T},$$

the yield-to-maturity $Y_{t,T}$ of the bond is thus interpreted as (the certainty equivalent of) the 'average' discount rate for the next T period, say, from t to $t+T$. From expressions (2.51) and (2.47), we may express the *yield curve* $\{Y_{t,T}\}$ in terms of the risk-free interest rates and write it as

$$\begin{aligned}\ln Y_{t,T} &= -\frac{1}{T}\varphi\left(q^*, \gamma; t, T\right) + \ln\gamma \times \left(\zeta_t - t\right) \\ &= \ln R_t^f + \varphi\left(q^*, \gamma; t, 1\right) - \frac{1}{T}\varphi\left(q^*, \gamma; t, T\right)\end{aligned}$$

with $\ln Y_{0,T} = -\frac{1}{T}\ln B_{0,T}$. Therefore, the entire yield curve is driven by and is perfectly correlated with the risk-free interest rate. Also, the slope of the yield curve, that is defined as $\ln Y_{t,T} - \ln R_t^f$, is deterministic and is given by

$$\ln Y_{t,T} - \ln R_t^f = \varphi\left(q^*, \gamma; t, 1\right) - \frac{1}{T}\varphi\left(q^*, \gamma; t, T\right).$$

2.4.4 *Forward Rates*

For all $t, \Delta \geq 0, T \geq 1$, let $F_t\left(\Delta, T\right)$ be the time t futures price that delivers a T-period bond at time $t+\Delta$. The *forward rate* $f_t\left(\Delta, T\right)$ is thus defined as the certainty equivalent of the 'average' interest rates for the future instant from $t+\Delta$ to $t+\Delta+T$. That is,

$$f_t^{-T}\left(\Delta, T\right) \triangleq F_t\left(\Delta, T\right). \tag{2.52}$$

By definition, $Y_{t,T} = f_t\left(0, T\right)$ and $R_t^f = f_t\left(0, 1\right)$. Therefore, the forward rates contain all the information about the risk-free interest rates and the yield curve.

As with the yield curve and the risk-free interest rates, the forward rates can be deduced from the term structure of interest rates $\{B_{t,T}\}$. In fact, with

$$B_{t,\Delta+T} = B_{t,\Delta}F_t\left(\Delta, T\right) = B_{t,\Delta}f_t^{-T}\left(\Delta, T\right)$$

we obtain $\ln f_t\left(\Delta, T\right) = \frac{\ln B_{t,\Delta} - \ln B_{t,\Delta+T}}{T}$. So, with bond prices given by Eq. (2.47) we can readily express the forward rates in terms of the risk-free interest rates as

$$\ln f_t\left(\Delta, T\right) = \frac{\varphi\left(q^*, \gamma; t, \Delta\right) - \varphi\left(q^*, \gamma; t, T+\Delta\right)}{T} + \varphi\left(q^*, \gamma; t, 1\right) + \ln R_t^f. \tag{2.53}$$

2.4.5 *Interest-Rate Contingent Claims*

Given the path-independent feature of the Ho–Lee model of the term structure of interest rates, we consider contingent claims of the American type with payoffs exercised at and before the maturity date T to be summarized by $\{X_t(m), m \in \{0, \cdots, t\}\}_{t=1}^{T}$. That is to say that an investor may wish to claim $X_t(m)$ at (t, m) or to continue to hold the option until a future time before the maturity date T. Here the contingent claims could be written on bonds, bond futures, the risk-free interest rates, yields and the forward rates. Following the same argument of pricing American options on stocks, under the no-arbitrage condition, the price of an *interest-rate contingent claim* can be expressed as

$$\Psi_0(X) = \max_{\tau \geq 0} \mathbb{E}_{\mathbb{Q}} \left[\frac{X_\tau}{R_0^f \cdots R_{\tau-1}^f} \right] \tag{2.54}$$

with $R_{-1}^f = 1$.

From Proposition 2.9 we see that the no-arbitrage condition on the bond market, in general, fails to generate a unique price for the contingent claim. This is because the risk-neutral probability measure $\mathbb{Q}$ and the dynamics of the risk-free interest rate process $\left\{R_t^f\right\}$ are expressed in terms of two unknown parameters q^* and γ, both taking values in $(0, 1)$. For all arbitrary $q^*, \gamma \in (0, 1)$, the corresponding price for the contingent claim, given by Eq. (2.54), will necessarily be consistent with the no-arbitrage condition. Hence, there is a continuum price range for the contingent claim consistent with the condition of no-arbitrage.

Given $q^*, \gamma \in (0, 1)$, similar to pricing American options on stocks in the CRR framework, the no-arbitrage condition for the bond market implies the following backward recursive procedure to compute the price of interest-rate contingent claims:

- At T, for $m \in \{0, \cdots, T\}$, setting $\Psi_T(m) = X_T(m)$;
- For $0 \leq t < T$ and $m \in \{0, \cdots, t\}$, setting

$$R_t^f(m) = \frac{B_{0,t}}{B_{0,t+1}} \frac{\gamma^{m-t}}{u_1(t)}$$

$$\Psi_t(m) = \max\left\{ X_t(m), \frac{q^* \Psi_{t+1}(m+1) + (1-q^*)\Psi_{t+1}(m)}{R_t^f(m)} \right\};$$

- Setting $\Psi_0(X) = \Psi_0(0)$ to obtain the current market price of the claim.

We illustrate this procedure with the following numerical example.

Example 2.6. Consider a one-year European put option on a three-month T-bill. The face value of the T-bill is \$1000. The exercise price of the option is \$980. Assume $q^* = 0.5$ and $\gamma = 0.997$. Let the initial term structure of interest rates be

$$[B_{0,1}, B_{0,2}, B_{0,3}, B_{0,4}, B_{0,5}] = [0.9826, 0.9650, 0.9474, 0.9296, 0.9119]$$

where $B_{0,t}$ is the price of a $(3 \times t)$-month discount bond. We apply the backward procedure to compute the initial price of the option.

First, note that $\$1000 \times B_{4,1}(m)$ is the price of a three-month T-bill at $t = 4$ following m highs along the path. At $T = 4$, the maturity payoffs of the option are

$$\Psi_4(m) = X_4(m) = (980 - 1000 \times B_{4,1}(m))^+,$$

where $B_{4,1}(m) = \frac{B_{0,5}}{B_{0,4}} u_1(4)\gamma^{4-m}, m = 0, \cdots, 4$. With $q^* = 0.5$, the backward recursive procedure becomes

$$\Psi_t(m) = \frac{\Psi_{t+1}(m+1) + \Psi_{t+1}(m)}{2} \frac{B_{0,t+1}}{B_{0,t}} u_1(t)\gamma^{t-m}$$

for $t = 0, 1, 2, 3$ and $m = 0, \cdots, t$. In light of (2.42), we obtain

$$[u_1(4), u_1(3), u_1(2), u_1(1)] = [1.006, 1.0045, 1.003, 1.0015].$$

These give us a data set for the option prices:

$$\begin{aligned}
[\Psi_4(0), \Psi_4(1), \Psi_4(2), \Psi_4(3), \Psi_4(4)] &= [4.84, 2.00, 0, 0, 0] \\
[\Psi_3(0), \Psi_3(1), \Psi_3(2), \Psi_3(3)] &= [3.39, 0.98, 0, 0] \\
[\Psi_2(0), \Psi_2(1), \Psi_2(2)] &= [2.14, 0.48, 0] \\
[\Psi_1(0), \Psi_1(1)] &= [1.28, 0.24] \\
\Psi_0(0) &= 0.75.
\end{aligned}$$

Therefore, the current market price of the option is \$0.75.

2.4.6 *Estimating Unknown Parameters*

To implement the Ho–Lee model of the term structure of interest rates, one requires an estimation of the unknown parameters q^*, γ and $q \in (0, 1)$, where $q = \mathbb{P}(\xi_t = h)$ is the probability that the state variable ξ_t takes a high value. These unknown parameters can be estimated by applying Hansen (1982) *generalized methods of moment (GMM)* with respect to the random variable

$$x_t = \ln R_t^f + \varphi(q^*, \gamma; t, 1) + t \ln \gamma.$$

The corresponding incremental process $\{\Delta x_t = x_{t+1} - x_t\}_{t \geq 0}$ is identical and is independent.

Let $\Theta = (0,1) \times (0,1) \times (0,1)$. The moments of the random variable Δx_t can be all expressed as functions of $\theta \triangleq (q^*, \gamma, q) \in \Theta$. Let $m_i(\theta) = \mathbb{E}, i = 1, 2, \cdots, I$, be the ith moment of the incremental process. Here, $I \geq 3$.

For any given sample of observations $\{\Delta x_{t_k}\}_{k=1}^K$, we denote the sample moments by $M_i = \frac{1}{K}\sum_{k=1}^K (\Delta x_{t_k})^i$, $i = 1, 2, \cdots, I$. The GMM estimator for θ is obtained by choosing $\theta \in \Theta$ to minimize the total square deviation of the moments from the corresponding sample moments; or,

$$\widehat{\theta} = \arg\min_{\theta \in \Theta} \sum_{i=1}^{I} (m_i(\theta) - M_i)^2 . \tag{2.55}$$

Readers are referred to Campbell, Lo and MacKinlay (1997, Appendix A.2) for detailed coverage of the GMM along with the asymptotic properties of the GMM estimator $\widehat{\theta}$.

2.5 Remarks

In this chapter we explored the usefulness of the no-arbitrage approach to pricing financial securities. Put-call parity, put-call-futures parity and several other parity relationships among options, futures, bonds and underlying security price were some of the useful applications of this no-arbitrage approach to pricing.

The Cox, Ross and Rubinstein (1979) binomial option pricing model and the Ho and Lee (1986) model of the term structure of interest rates are two typical examples of the asset pricing models developed purely based on the no-arbitrage principle. We showed that the CRR binomial option pricing model has a continuous-time limit that is given by the Black and Scholes (1973) option pricing model. The Heath, Jarrow and Morton (HJM, 1992) approach to the term structure of interest rates, which is regarded as another successful application of the no-arbitrage approach to asset pricing after Black–Scholes option pricing model, traces its merits back to the Ho–Lee binomial model covered in this chapter. The Black–Scholes and HJM models will be covered in detail in Part III of this book.

Musiela and Rutkowski (1997) is a good source for extensive coverage of applications of the no-arbitrage approach to pricing interest-rate contingent claims. Sercu and Uppal (1997) is an excellent source for applications in international finance. Other well-known applications of the no-arbitrage approach to pricing not covered in this book include the proof of the Modigliani and Miller (1958) proposition on capital structure irrelevance, and the arbitrage pricing theory (APT) of Ross (1976).

To implement the no-arbitrage approach to asset pricing, we require an estimation with respect to the risk-neutral probability measure $\mathbb{Q}$ and the underlying risk-free interest rate dynamics $\{r_t\}$. From the Ho and Lee (1986) model we see that even with observations of the entire initial term structure of interest rates which contains the prices of an infinite number of securities in a highly simplified binomial world, the no-arbitrage condition might fail to generate a unique risk-neutral probability and a unique risk-free interest rate process. In fact we have seen that, for all arbitrary fixed parameters q^* and γ in $(0, 1)$, the resulting motion of the term structure of interest rates is fully consistent with the no-arbitrage condition and fully conforms to the observed initial term structure of interest rates. For a general information structure, the no-arbitrage condition does not always result in a model that involves just a few identifiable unknown parameters as in Ho and Lee (1986). Interested readers are referred to Jarrow (2002) for detailed coverage of this.

Hence, modelers face an unavoidable task as they must estimate the risk-neutral probability measure in order to implement those asset pricing models based on no-arbitrage. The degree of technicality required for carrying out non-parametric estimation of this sort is well-recognized in the literature of empirical finance. We refer interested readers to Campbell, Lo and MacKinlay (1997) for the illustration of the non-parametric estimation of the risk-neutral probability measure or the pricing kernel.