Preface

The representation theory of Lie algebras is an important and intensively studied area of modern mathematics with applications in practically all major areas of mathematics and physics. There are several textbooks which specialize in different aspects of the representation theory of Lie algebras and its applications, but the usual topics covered in such books are finite-dimensional, highest weight or Harish-Chandra modules.

The smallest simple Lie algebra $\mathfrak{sl}_2$ differs in many aspects from all other semi-simple Lie algebras. One could, for example, mention that $\mathfrak{sl}_2$ is the only semi-simple Lie algebra for which *all* simple (not necessarily finite-dimensional) modules are in some sense understood. The algebra $\mathfrak{sl}_2$ is generated by only two elements and hence is an invaluable source of computable examples. Moreover, in many cases the ideas which one gets from working with $\mathfrak{sl}_2$ generalize relatively easily to other Lie algebras with a minimum of extra knowledge required.

The aim of these lecture notes is to give a relatively short introduction to the representation theory of Lie algebras, based on the Lie algebra $\mathfrak{sl}_2$, with a special emphasis on explicit examples. Using this Lie algebra, we can examine and describe many more aspects of the representation theory of Lie algebras than are covered in standard textbooks.

The notes start with two conventional introductory chapters on finite-dimensional modules and the universal enveloping algebra. The third chapter moves on to the study of weight modules, including a complete classification and explicit construction of all weight modules and a description of the category of all weight modules with finite-dimensional weight spaces, via quiver algebras. This is followed by a description and study of the primitive spectrum of the universal enveloping algebra and its primitive quotients. The next step is a relatively complete description of the Bernstein–Gelfand–

Gelfand category $\mathcal{O}$ and its properties. The two last chapters contain a description of all simple $\mathfrak{sl}_2$-modules and various categorifications of simple finite-dimensional modules. The material presented in the last chapter is based on papers which were published in the last two years.

The notes are primarily directed towards postgraduate students interested in learning the basics of the representation theory of Lie algebras. I hope that these notes could serve as a textbook for both lecture courses and reading courses on this subject. Originally, they were written and used for reading courses which I gave in Uppsala in 2008.

The prerequisites for understanding these notes depend on the chapter. For the first two chapters, one needs only some basic knowledge in linear algebra and rings and modules. For the next two chapters, it is assumed that the reader is familiar with the basics of the representation theory of finite-dimensional associative algebras and basic homological algebra. The last three chapters also require some basic experience with category theory.

At the end of each chapter are comments including some historical background, brief descriptions of more advanced results, and references to some original papers. I tried to present these comments to the best of my knowledge and I would like to apologize in advance for any unforeseen errors or omissions.

There are numerous exercises in the main text and at the end of each chapter. The exercises in the main text are usually relatively straightforward and required to understand the material. It is strongly recommended that the reader at least looks through them. Answers and hints are supplied at the end of the notes.

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Uppsala, August 2009 *Volodymyr Mazorchuk*