Chapter 1

Introduction

'May the best ye've ever seen, Be the warst ye'll ever see.' (Scottish blessing)

'It is not enough to do your best; you must know what to do, and then do your best.' (W. Edwards Deming)

'It's only the man who can look at the same problem from many different aspects that will make a true leader.' (Takao Fujisawa)

In this chapter, we explain why it is preferable to deal with set-theoretical rather than probabilistic or fuzzy uncertainty. It is demonstrated that probabilistic analysis with unbounded support is prone to high sensitivity of the results to the behavior of the tail of the distribution. This bad news is balanced by good news: a probability distribution with bounded support yields the same results as the simple and transparent anti-optimization technique. We argue that since there are apparently no physical parameters with unbounded support, one has to resort to probability densities with bounded ones or directly to the anti-optimization technique. This provides, in a nutshell, the main justification for this book to deal with bounded uncertainty and anti-optimization.

1.1 Probabilistic Analysis: Bad News

Apparently, the first study in which probabilistic methods were applied to structures was conducted by Mayer (1926). Since then, probabilistic methods have developed beyond the age of adolescence (Cornell, 1981).

Usually, probabilistic reliability studies involve assumptions on the probability densities, whose knowledge regarding the relevant input quantities is central. Given this, the probabilistic properties of the output quantities can be determined. As Wentzel (1980) stresses, probabilistic methods are

'frequently regarded as a kind of magic wand which produces in-

> formation out of a void. This is a fallacy; the theory of probability only enables information to be transformed, and conclusions on inaccessible phenomena to be drawn from data on observable ones.'

Probabilistic methods must yield a central quantity–reliability of the structure, namely, the probability that the structure will perform its mission satisfactorily. Modern society rightfully expects extremely high reliabilities and, consequently, extremely small probabilities of failure.

The deterministic mechanical theories represent a cornerstone of probabilistic mechanics. First, the phenomenon should be understood qualitatively; next, it should be understood sufficiently well quantitatively. Then the output quantities in their intricate dependence on their input counterparts are developed as a set of equations, algebraic expressions, or numerical codes of varying complexity. At this stage the parameters to be treated as uncertain variables or functions are identified. The deterministic relation or numerical code then serves as a transfer function determining the probabilistic characteristics of the output and the reliability of the structure.

Our goal being extremely high reliabilities, it is immediately understood that the deterministic relations or numerical codes must be of supreme accuracy, in order to avoid a GIGO ('garbage in–garbage out') situation. However, since each deterministic relation is based on simplifying assumptions with a specified degree of accuracy of their own, legitimate questions arise: Are less accurate tools suitable for determining the extremely accurate desired probabilities of failure? What happens to the reliability of structures designed on the basis of increasingly accurate theories?

No less central is the input data. In most cases, accurate information is unavoidable. Probabilistic analysts maintain that their theoretical analyses, presently devoid of experimental inputs, still are of values; and once information becomes available, it can be incorporated into a ready-made theoretical framework.

But in the meantime these analyses have taken on lives of their own: to make up for the lacking information, researchers resort to such assumptions as specific patterns of normal distribution, time-wise stationarity of a random process, space-wise homogeneity of a random field, ergodicity of the process, or the Markov property.

Thus the questions 'What can go wrong with probabilistic methods?' and 'To use or not to use them?' have become extremely relevant. Their various aspects have been addressed in monographs by Ben-Haim and Elishakoff (1990) and Ben-Haim (1996), and in papers, e.g., Elishakoff and Hasofer (1996). The need for a major revision of our understanding of the role of probabilistic methods is stressed in the monograph by Ben-Haim and Elishakoff (1990) and in the articles by Kalman (1994) and Elishakoff (1995b).

In this section, we elucidate difficulties inherent in structural reliability due to imperfect information, in conjunction with the ever increasing desire to allow only extremely small probability of failure.

First of all, we address this cardinal question: Why does one need probabilistic methods? The reply to this question is best given in terms of the design of a structure. In a deterministic design process, one requires

$$\sigma \le \sigma_{\mathrm{y}} \tag{1.1}$$

where σ is an actual stress that is assumed to be positive in tensile state, and σ_{y} is an yield stress. Here, uncertainty enters the picture in the sense that we may not know the loads precisely, or there may be imprecision in measuring the geometric parameters of the cross-section, or we may have built an imperfect mechanical model to describe the behavior of the structure. Accordingly, a *required* safety factor k_{req} is introduced, and Eq. (1.1) is replaced by

$$\sigma \le \frac{\sigma_{\mathrm{y}}}{k_{\mathrm{req}}} \tag{1.2}$$

Once the structure is designed, one can introduce the actual safety factor

$$k_{\mathrm{act}} = \frac{\sigma_{\mathrm{y}}}{\sigma_{\max}} \tag{1.3}$$

where $\sigma_{\max}$ is the maximum actual stress occurring in the structure. Thus the design requirement can be formulated as $k_{\mathrm{act}} \ge k_{\mathrm{req}}$, or, in other words, the actual safety factor should not be less than the required one.

Can one quantify that the actual safety factor of the uncertainty is not hidden, but is directly introduced into the scene? Let us attempt to answer this question. Let the random force p, acting on a tension-compression element (bar) with cross-sectional area a, have a Weibull (1951) distribution with the following probability distribution function

$$F_P(p) = 1 - \exp\left[-\left(\frac{p - p_0}{w - p_0}\right)^k\right], \quad k > 0, \quad w > p_0, \quad p \ge p_0 \tag{1.4}$$

For $p < p_0$, $F_P(p) \equiv 0$. Equation (1.4) is called the 3-parameter Weibull distribution, which is a generalization of the exponential distribution. It has been developed originally to describe failure strength of metals. More recently it has been utilized in connection with fracture, as well as life distribution of mechanical compounds. For the special case $p_0 = 0$, the distribution is called 2-parameter Weibull distribution; in this case $k = 1$ yields the exponential distribution, whereas $k = 2$ is associated with Rayleigh distribution. The shape parameter $k < 1$ is typical of wearing phenomena, whereas $k > 1$ is typical of aging effects. The 3-parameter Weibull distribution is useful in describing phenomena for which some minimum value p_0 exists for the random variable P, so that P takes on values greater than or equal to p_0. In the following, we denote random variables by upper-case, and lower-case notation is reserved for their possible values.

We are interested in the reliability of the structure, i.e., the probability of the structure performing its intended mission satisfactorily. Such a performance is identified with relationship holding Eq. (1.1) true. Thus, the reliability becomes:

$$\begin{aligned} R &= \mathrm{Prob}(\Sigma \le \sigma_{\mathrm{y}}) \\ &= \mathrm{Prob}(P/a \le \sigma_{\mathrm{y}}) \\ &= \mathrm{Prob}(P \le \sigma_{\mathrm{y}} a) \\ &= F_P(\sigma_{\mathrm{y}} a) \end{aligned} \tag{1.5}$$

Thus, we have

$$R = 1 - \exp\left[-\left(\frac{\sigma_y a - p_0}{w - p_0}\right)^k\right] \tag{1.6}$$

How can we define the safety factor in the context of probabilistic design? One natural way is to relate it to some characteristic load, say the average load. The latter equals

$$E(P) = p_0 + (w - p_0)\Gamma(1 + 1/k) \tag{1.7}$$

where $\Gamma(\cdot)$ is the Gamma function. The variance of the load is

$$V(P) = (w - p_0)[\Gamma(1 + 2/k) - \Gamma^2(1 + 1/k)] \tag{1.8}$$

The central safety factor n is defined as the ratio of the yield stress to its average counterpart $E(P)/a$:

$$n = \frac{\sigma_y a}{E(P)} = \frac{\sigma_y a}{p_0 + (w - p_0)\Gamma(1 + 1/k)} \tag{1.9}$$

Let us design the structure probabilistically. Probabilistic design requires that the reliability be not less than a codified value r:

$$R \geq r \tag{1.10}$$

Thus, in view of Eq. (1.6), we obtain

$$1 - \exp\left[-\left(\frac{\sigma_y a - p_0}{w - p_0}\right)^k\right] \geq r \tag{1.11}$$

The design value of the cross-sectional area a_{design} is found from the equality $R = r$, and reads

$$a_{\text{design}} = \frac{p_0 + (w - p_0)[\ln 1/(1 - r)]^{1/k}}{\sigma_y} \tag{1.12}$$

Substitution of a_{design} for a in Eq. (1.9) enables us to write the central safety factor explicitly in terms of the codified required reliability r:

$$n = \frac{p_0 + (w - p_0)[\ln 1/(1 - r)]^{1/k}}{p_0 + (w - p_0)\Gamma(1 + 1/k)} \tag{1.13}$$

For the set of parameters $w = 3p_0$, $k = 4$, we have

$$n = \frac{1 + 2[\ln 1/(1 - r)]^{1/4}}{1 + 2\Gamma(1.25)} \tag{1.14}$$

Therefore, from Eq. (1.14), $n = 1.2314$ for $r = 0.9$; $n = 1.3971$ for $r = 0.99$; $n = 1.5082$ for $r = 0.999$; $n = 1.5942$ for $r = 0.9999$; $n = 1.6653$ for $r = 0.99999$; $n = 1.7263$ for $r = 0.999999$, etc. As we see, a probabilistic model allows us to associate the required reliability directly with the safety factor.

Let us see now how a small error can affect the reliability calculations. Say that the actual values are w_1, p_1 and k_1, while those used in the analysis are w, p_0 and

k. The actual reliability $R_{\rm act}$ and the actual probability of failure $P_{\rm f,act}$ are related as

$$P_{\rm f,act} = 1 - R_{\rm act} \tag{1.15}$$

The actual reliability is given by Eq. (1.6) with the actual values substituted:

$$R_{\rm act} = 1 - \exp\left[-\left(\frac{\sigma_{\rm y} a - p_1}{w_1 - p_1}\right)^{k_1}\right] \tag{1.16}$$

and the actual probability by

$$P_{\rm f,act} = \exp\left[-\left(\frac{\sigma_{\rm y} a - p_1}{w_1 - p_1}\right)^{k_1}\right] \tag{1.17}$$

If the design of the structure has been performed using the values w, p_0 and k, the appropriate value of the cross-sectional area is given by Eq. (1.12). To calculate the actual probability of failure corresponding to the design value $a_{\rm design}$, we substitute the expression $a_{\rm design}$ into Eq. (1.17) and have

$$P_{\rm f,act} = \exp\left[-\left(\frac{\sigma_{\rm y} a_{\rm design} - p_1}{w_1 - p_1}\right)^{k_1}\right] \tag{1.18}$$

or

$$P_{\rm f,act} = \exp\left[-\left(\frac{p_0 - p_1 + (w - p_0)[\ln 1/(1-r)]^{1/k}}{w_1 - p_1}\right)^{k_1}\right] \tag{1.19}$$

Let us consider some particular cases. In the simplest case $p_1 = p_0$, $w = w_1$, but $k_1 \neq k$,

$$P_{\rm f,act} = \exp\left[-\left(\ln\frac{1}{1-r}\right)^{k_1/k}\right] \tag{1.20}$$

Since r is the required reliability, $1 - r$ is recognized in Eq. (1.20) as the allowed probability of failure $P_{\rm f,all}$. Thus, Eq. (1.20) can be rewritten as:

$$P_{\rm f,act} = \exp\left[-\left(\ln\frac{1}{P_{\rm f,all}}\right)^{k_1/k}\right] \tag{1.21}$$

Let $P_{\rm f,all} = 10^{-6}$. Then

$$P_{\rm f,act} = \exp[-13.81551056^{k_1/k}] \tag{1.22}$$

This function of the ratio k_1/k is shown in Fig. 1.1. As is seen, when $k_1/k = 1$,

$$P_{\rm f,act} = P_{\rm f,all} = 10^{-6} \tag{1.23}$$

is satisfied.

By contrast, when $k_1 \neq k$, the actual probability of failure may differ from the allowed one. Remarkably, there can be a serendipitous situation, namely, when an error in measurement of k_1 may be of a *favorable* nature: for $k_1/k > 1$, the actual

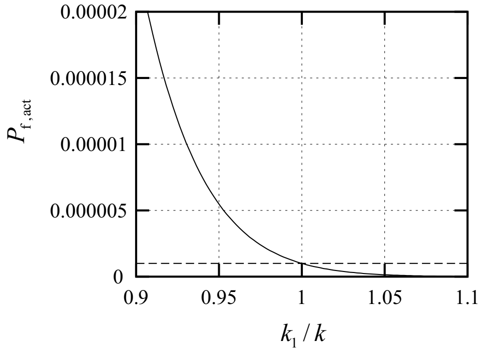

Fig. 1.1 Actual probability of failure as a function of the ratio k_1/k (solid line); for $k_1 = k$ it coincides with the required one (dotted line); for $k_1/k > 1$ it is less than the required one; for $k_1/k < 1$ it may well exceed the allowed value, resulting in a detrimental state.

probability of failure is less than the allowable one. However, when $k_1/k > 1$, the effect of a small error in evaluating k may be detrimental: If $k_1/k = 0.95$, the actual probability is about five times the allowed one; if $k_1/k = 0.93$, the actual probability is approximately ten times as large as the one which was permitted! We conclude that the probability of failure is too sensitive a parameter to make do with imprecise input characteristics, and that accurate determination of the probabilistic characteristics of input must be an integral part of a rigorous probabilistic analysis.

Regarding the effect of a small deviation in the probability density on the reliability estimate, we consider a bar with cross-sectional area a subjected to a load P which is a random variable with the probability density $f_P(p)$, p being a possible value of the load. The material of the bar is assumed to be perfectly elastic in compression and has a yield stress in tension σ_y. We also assume that the bar cannot lose its stability, or undergo any other form of failure. To recapitulate, its reliability is defined as the probability of the stress $\Sigma = P/a$ not exceeding the yield stress:

$$R = \mathrm{Prob}(\Sigma \le \sigma_\mathrm{y}) \tag{1.24}$$

Let us consider the situation in which the data is suggestive for the analyst to assume the probability density of the load in the form of the log-normal variable

$$f_P(p) = \begin{cases} \dfrac{1}{pc\sqrt{2\pi}} \exp\left[\dfrac{-(\ln p - b)^2}{2c^2}\right], & \text{for } p > 0 \\ 0, & \text{otherwise} \end{cases} \tag{1.25}$$

where b and c characterize the probability density. The mean value $E(P)$ and the variance $V(P)$ are expressed as

$$\begin{aligned} E(P) &= \exp\left(b + \frac{1}{2}c^2\right) \\ V(P) &= \exp(2b + c^2)[\exp(c^2) - 1] \end{aligned} \tag{1.26}$$

The reliability is thus given by

$$\begin{aligned} R &= \text{Prob}(P \leq \sigma_{\text{y}} a) \\ &= \text{Prob}(\ln P \leq \ln \sigma_{\text{y}} a) \\ &= \frac{1}{2} + \text{erf}\left(\frac{\ln \sigma_{\text{y}} a - b}{c}\right) \end{aligned} \tag{1.27}$$

where

$$\text{erf}(x) = \frac{1}{2\pi} \int_0^x \exp(-t^2/2)\text{d}t \tag{1.28}$$

since $\ln P$ is a normal variable with mean b and variance c^2. Let us assume now that the actual probability density differs slightly from the true one and reads, for $p > 0$:

$$f_P^{(\varepsilon)}(p) = \frac{1}{pc\sqrt{2\pi}} \exp\left[-\frac{(\ln p - b)^2}{2c^2}\right] \{1 + \varepsilon \sin[2\pi(\ln p - b)]\} \tag{1.29}$$

where ε is a constant belonging to an interval $[-1, 1]$. For $p < 0$, the probability density vanishes. It can be shown that $f_P^{(\varepsilon)}(p)$ is indeed a probability density, non-negative and satisfies the equality

$$\int_{-\infty}^{\infty} f_P^{(\varepsilon)}(p)\text{d}p = 1 \tag{1.30}$$

To prove this property, we have to demonstrate that

$$\int_{-\infty}^{\infty} f_P(p) \sin[2\pi(\ln p - b)]\text{d}p = 0 \tag{1.31}$$

To do this, we make a substitution $\ln p = t$. Thus, the integral to be calculated reads

$$I = \int_{-\infty}^{\infty} \frac{1}{c\sqrt{2\pi}} \exp\left[-\frac{(t-b)^2}{2c^2}\right] \sin[2\pi(t - b)]\text{d}t \tag{1.32}$$

and further substitution $t - b = u$ leads to

$$I = \int_{-\infty}^{\infty} \frac{1}{c\sqrt{2\pi}} \exp\left(-\frac{u^2}{2c^2}\right) \sin(2\pi u)\text{d}u \tag{1.33}$$

which vanishes since the integrand is an odd function of u. Thus, the function defined in Eq. (1.29) represents a probability density of some random variable, denoted by $P^{(\varepsilon)}$. Obviously, if $\varepsilon = 0$, $f_P^{(\varepsilon)}$ is equal to f_P as per Eqs. (1.25) and (1.29). Stoyanov (1987, pp. 89–91) demonstrates that for any $k = 1, 2, \ldots$, we have

$$E(P^{(\varepsilon)})^k = E(P^k) \tag{1.34}$$

i.e., the *perturbed* random variable $P^{(\varepsilon)}$ has the same moments as those of an *unperturbed* variable P.

Let us calculate now the true reliability associated with $f^{(\varepsilon)}$:

$$R = \text{Prob}(P \leq \sigma_{\text{y}} a) = \int_0^{\sigma_{\text{y}} a} f_P^{(\varepsilon)}(p)\text{d}p \tag{1.35}$$

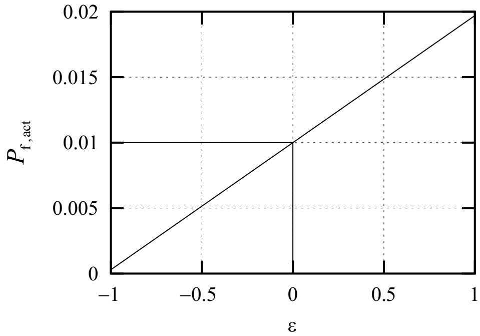

Fig. 1.2 Relation between ε and $P_{\mathrm{f,act}}$ for $c = 0.1$, $P_{\mathrm{f,all}} = 0.01$ $(r = 0.99)$.

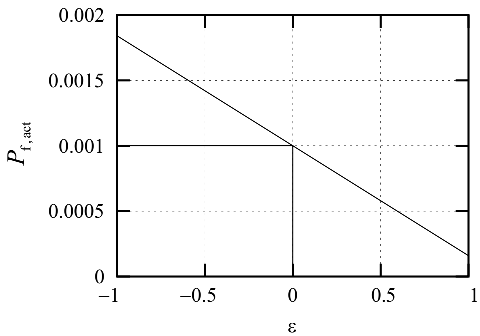

Fig. 1.3 Relation between ε and $P_{\mathrm{f,act}}$ for $c = 0.2$, $P_{\mathrm{f,all}} = 0.001$ $(r = 0.999)$.

or

$$R = \int_0^{\sigma_{\mathrm{y}} a} \frac{1}{pc\sqrt{2\pi}} \exp\left[-\frac{(\ln p - b)^2}{2c^2}\right] \{1 + \varepsilon \sin[2\pi(\ln p - a)]\} \mathrm{d}p \tag{1.36}$$

Introducing again the new variable of integration $\ln p = t$, we reduce reliability to

$$R = \int_0^{\ln \sigma_{\mathrm{y}} a} \frac{1}{c\sqrt{2\pi}} \exp\left[-\frac{(t - b)^2}{2c^2}\right] \{1 + \varepsilon \sin[2\pi(t - b)]\} \mathrm{d}t \tag{1.37}$$

and with $t - b = u$, we have

$$R = \frac{1}{2} + \mathrm{erf}\left(\frac{\ln \sigma_{\mathrm{y}} a - b}{c}\right) + \frac{\varepsilon}{c\sqrt{2\pi}} \int_0^{\ln \sigma_{\mathrm{y}} a - b} \frac{1}{c\sqrt{2\pi}} \exp\left(-\frac{u^2}{2c^2}\right) \sin(2\pi u) \mathrm{d}u \tag{1.38}$$

Evaluation of this integral yields

$$\begin{aligned} R = \frac{1}{2} + \mathrm{erf}\left(\frac{\ln \sigma_{\mathrm{y}} a - b}{c}\right) + \frac{\varepsilon}{c\sqrt{2\pi}} \Bigg\{ - e^{-2\pi^2 c^2} \sqrt{\frac{\pi}{2}} \\ \times \left(\mathrm{erfi}\left[\frac{2\pi c^2 - i(b + \ln[\sigma_{\mathrm{y}} a])}{\sqrt{2c}}\right] + \mathrm{erfi}\left[\frac{2\pi c^2 + i(b + \ln[\sigma_{\mathrm{y}} a])}{\sqrt{2c}}\right]\right) \Bigg\} \end{aligned} \tag{1.39}$$

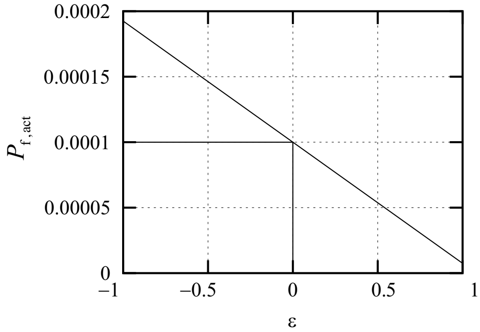

Fig. 1.4 Relation between ε and $P_{\mathrm{f,act}}$ for $c = 0.2$, $P_{\mathrm{f,all}} = 0.0001$ ($r = 0.9999$).

where $\mathrm{erfi}(z)$ is an imaginary error function, defined as (Wolfram, 1996, p. 745)

$$\mathrm{erfi}(z) = \mathrm{erf}(iz)/i \tag{1.40}$$

where $i = \sqrt{-1}$. One can visualize that the reliability calculations have been performed with the probability density of the load set at $f_P^{(0)}(p)$.

Let required reliability be r, so that the design is performed using the expression in Eq. (1.27). Figures 1.2–1.4, associated with $r = 0.99$, $r = 0.999$ and $r = 0.9999$, respectively, depict the actual situation in which the control parameter ε describing the deviation from reality does not vanish identically. Thus, the actual probability of failure $P_{\mathrm{f,act}}$ varies with ε. Naturally, when $\varepsilon = 0$, $P_{\mathrm{f,act}}$ takes the value $1 - r$, or 0.01, 0.001 and 0.0001, respectively, in Figs. 1.2–1.4. The integrand in Eq. (1.36) is an oscillatory function. Therefore, depending on the value of c, the value of the integral is either positive or negative; hence, different behaviors are exhibited in Figs. 1.2–1.4. The diagrams demonstrate the possibility of *safe* errors; i.e., deviation of the model from the reality may turn out to be on the safe side. Indeed, if ε in Fig. 1.2 tends to -1 from above, the actual probability of failure tends to zero. Likewise, in Fig. 1.3, associated with $c = 0.2$ and $P_{\mathrm{f,all}} = 10^{-3}$, if ε tends to unity from below, the actual reliability too approaches unity from below. Analogously, in Fig. 1.4, with $c = 0.2$, $P_{\mathrm{f,all}} = 10^{-4}$, and ε tending to unity from below, the actual probability of failure tends to zero from above. It is prudent, however, to consider what can go wrong when the theoretical model differs from reality. As Fig. 1.2 demonstrates, when ε tends to unity, the actual probability of failure becomes double the allowable one. This implies, through the frequency interpretation of the probability notion, that the number of unsuccessful performances may be double the allowed. Nearly analogous situations are depicted in Figs. 1.3 and 1.4 when ε tends to -1.

For a critical appraisal of methods to determine the possibility of failure, the reader may like to consult, for example, the paper by Schuëller and Stix (1987) and the monograph by Elishakoff (2004).

1.2 Probabilistic Analysis: Good News

The somewhat discouraging picture painted above by no means justifies boycotting the probabilistic approach. In fact, the latter makes it possible to substantiate the mysterious *fudge* factor, or factor of safety. Taking an example from a monograph (Elishakoff, 2004) on safety factors, consider an element subjected to a stress modeled as a random variable.

In a similar manner as in previous section, random variables are denoted with capital letters. Hence the random stress is denoted by Σ with its realization being denoted by lower-case notation σ. We also assume that the yield stress σ_{y} is a deterministic quantity. Both the actual and yield stresses can be treated as random variables, but for greater convenience only one therein is so treated. We also adopt the simplest possible assumption of a continuously distributed Σ, i.e., a uniform probability density

$$f_{\Sigma}(\sigma) = \begin{cases} 0, & \text{for } \sigma < \sigma_{\mathrm{L}},\ \sigma > \sigma_{\mathrm{U}} \\ \dfrac{1}{\sigma_{\mathrm{U}} - \sigma_{\mathrm{L}}}, & \text{for } \sigma_{\mathrm{L}} \le \sigma \le \sigma_{\mathrm{U}} \end{cases} \tag{1.41}$$

where σ_{L} and σ_{U} are the lower and upper bounds of the stress.

Now, the reliability R, i.e., the probability that the actual stress does not exceed the yield stress σ_{y}

$$R = \mathrm{Prob}(\Sigma \le \sigma_{\mathrm{y}}) \tag{1.42}$$

equals

$$R = \int_{-\infty}^{\sigma_{\mathrm{y}}} f_{\Sigma}(\sigma)\mathrm{d}\sigma = \begin{cases} 0, & \text{for } \sigma_{\mathrm{y}} < \sigma_{\mathrm{L}} \\ \dfrac{\sigma_{\mathrm{y}} - \sigma_{\mathrm{L}}}{\sigma_{\mathrm{U}} - \sigma_{\mathrm{L}}}, & \text{for } \sigma_{\mathrm{L}} \le \sigma_{\mathrm{y}} \le \sigma_{\mathrm{U}} \\ 1, & \text{for } \sigma_{\mathrm{y}} > \sigma_{\mathrm{U}} \end{cases} \tag{1.43}$$

Let us concentrate on the range $\sigma_{\mathrm{L}} \le \sigma_{\mathrm{y}} \le \sigma_{\mathrm{U}}$. We first note that the mean value $E(\Sigma)$ of the actual stress equals

$$E(\Sigma) = \frac{1}{2}(\sigma_{\mathrm{L}} + \sigma_{\mathrm{U}}) \tag{1.44}$$

whereas the standard deviation d_{Σ} is given by

$$d_{\Sigma} = \frac{\sigma_{\mathrm{U}} - \sigma_{\mathrm{L}}}{2\sqrt{3}} \tag{1.45}$$

By Eqs. (1.44) and (1.45), the lower and upper bounds of the stress are expressed, respectively, as

$$\begin{aligned} \sigma_{\mathrm{L}} &= E(\Sigma) - \sqrt{3}d_{\Sigma} \\ \sigma_{\mathrm{U}} &= E(\Sigma) + \sqrt{3}d_{\Sigma} \end{aligned} \tag{1.46}$$

Thus, the reliability is rewritten, in the range $\sigma_{\mathrm{L}} \le \sigma_{\mathrm{y}} \le \sigma_{\mathrm{U}}$ as

$$R = \frac{\sigma_{\mathrm{y}} - E(\Sigma) + \sqrt{3}d_{\Sigma}}{2\sqrt{3}d_{\Sigma}} \tag{1.47}$$

At this juncture it is instructive to introduce the central safety factor as the following ratio

$$n = \frac{\sigma_{\mathrm{y}}}{E(\Sigma)} \tag{1.48}$$

along with the coefficient of variation ν_Σ of the stress

$$\nu_\Sigma = \frac{d_\Sigma}{E(\Sigma)} \tag{1.49}$$

Dividing the numerator and denominator of Eq. (1.47) by $E(\Sigma)$, we obtain

$$R = \frac{n - 1 + \sqrt{3}\nu_\Sigma}{2\sqrt{3}\nu_\Sigma} \tag{1.50}$$

This formula relates reliability, safety factor, and the coefficient of variation of the stress. Expressing the safety factor from Eq. (1.50), we have

$$n = 1 + 2\sqrt{3}\nu_\Sigma \left(R - \frac{1}{2} \right) \tag{1.51}$$

Probabilistic design is based on the requirement

$$R \geq r \tag{1.52}$$

where r is the required, codified reliability, and the required safety factor n_{r} is obtained from Eq. (1.51) by setting $R = r$:

$$n_{\mathrm{r}} = 1 + 2\sqrt{3}\nu_\Sigma \left(r - \frac{1}{2} \right) \tag{1.53}$$

This is a simple formula connecting the required safety factor n_{r} with the required reliability r and coefficient of variation ν_Σ of the stress.

We can deduce several useful conclusions:

(i) The safety factor, so often criticized by practitioners and researchers alike, is actually a powerful concept, which can be given a probabilistic interpretation.
(ii) Probability theory strips the mystery from the safety factor, which instead of being assigned at the will of the designer, or *out of the sky* as it were, gains an analytical framework for its determination.
(iii) If one can quantify the required reliability, say through legislation, one can quantify the safety factor as well.
(iv) If the required reliability r is greater than 0.5, i.e., if we do not tolerate nearly half of our products being defective (and, hopefully, we don't!), the required safety factor exceeds unity. This is in agreement with all textbooks and designs, where the required safety factor is in excess of unity.
(v) The higher the coefficient of variation of the stress, the higher the required safety factor. This matches one's anticipation, as it should be. Moreover, Eq. (1.53) tells us exactly how much the safety factor must be, depending on the magnitude of uncertainty ν_Σ. Were the coefficient of variation negligibly small, i.e., $\nu_\Sigma = 0_+$, the safety factor would be $n_{\mathrm{r}} = 1_+$. This implies that the truly deterministic design is valid only when ν_Σ tends to zero. This fact clearly indicates that purely deterministic design is contained as a particular case in probabilistic design.

Consider now a closely related example from a textbook (Elishakoff 1983, 1999) on the probabilistic theory of structures. Again, our objective is a tension–compression element (bar), in this case a uniform one subjected to random load P which is uniformly distributed between the lower bound p_{L} and the upper bound p_{U}. The bar must be designed so that its reliability R equals the required value r or exceeds it. The design itself amounts to choosing the cross-sectional area a of the bar. The reliability equals

$$R = \mathrm{Prob}(P/a \leq \sigma_{\mathrm{y}}) = \mathrm{Prob}(P \leq \sigma_{\mathrm{y}} a) = F_P(\sigma_{\mathrm{y}} a) \tag{1.54}$$

where $F_P(p)$ is the probability distribution function of the load. Their reliability formula maintains that the reliability of the bar equals the value of the probability distribution of the load function computed at the product of the yield stress and the cross-sectional area. Calculation of the reliability yields

$$R = \begin{cases} 0, & \text{for } \sigma_{\mathrm{y}} a < p_{\mathrm{L}} \\ \dfrac{\sigma_{\mathrm{y}} a - p_{\mathrm{L}}}{p_{\mathrm{U}} - p_{\mathrm{L}}}, & \text{for } p_{\mathrm{L}} \leq \sigma_{\mathrm{y}} a \leq p_{\mathrm{U}} \\ 1, & \text{for } \sigma_{\mathrm{y}} a > p_{\mathrm{U}} \end{cases} \tag{1.55}$$

As noted, the range of P is $p_{\mathrm{L}} \leq \sigma_{\mathrm{y}} a \leq p_{\mathrm{U}}$. The design criterion stipulates

$$\frac{\sigma_{\mathrm{y}} a - p_{\mathrm{L}}}{p_{\mathrm{U}} - p_{\mathrm{L}}} \geq r \tag{1.56}$$

The minimum required cross-sectional area a_{r} corresponds to

$$r = \frac{\sigma_{\mathrm{y}} a_{\mathrm{r}} - p_{\mathrm{L}}}{p_{\mathrm{U}} - p_{\mathrm{L}}} \tag{1.57}$$

which leads to the needed cross-sectional area

$$a_{\mathrm{r}} = \frac{(p_{\mathrm{U}} - p_{\mathrm{L}})r + p_{\mathrm{L}}}{\sigma_{\mathrm{y}}} \tag{1.58}$$

We observe that when r tends to unity, the numerator tends to p_{U} and the design value a_{r} approaches the value

$$a_{\mathrm{r}} = \frac{p_{\mathrm{U}}}{\sigma_{\mathrm{y}}} \tag{1.59}$$

This expression is also obtainable from the third alternative in Eq. (1.55) with unity reliability being obtained for $\sigma_{\mathrm{y}} a \geq p_{\mathrm{U}}$.

It is remarkable that the expression Eq. (1.58) can *also* be obtained without any probabilistic analysis. Indeed, since the load varies in the interval $[p_{\mathrm{L}}, p_{\mathrm{U}}]$, the stress also is an interval variable $\sigma_{\mathrm{I}} = [p_{\mathrm{L}}/a, p_{\mathrm{U}}/a]$, which must be less than the yield stress σ_{y}. This requirement is satisfied if

$$\frac{p_{\mathrm{U}}}{a} \leq \sigma_{\mathrm{y}} \tag{1.60}$$

which leads to the minimum required value of the area as given in Eq. (1.58). We arrive at the elegant conclusion: *The probabilistic and anti-optimization approaches* *lead to the same result!* Not only is there no antagonism between them, but they both tell us the same thing!

This implies that we can interchangeably utilize either of the two approaches! Those who prefer the probabilistic sophistification, must be *allowed* (does anyone need permission, really?) to continue development and usage of probabilistic mechanics. Those who prefer simplicity will stick to the anti-optimization technique.

Naturally, this begs the question: *Is one of the approaches preferable to the other?* It appears that the anti-optimization approach, as the simpler of the two, has a certain advantage. Moreover, as one of the respondents stressed in the poll conducted by Elishakoff (2000b):

> 'Engineers, especially those at the top, do not trust statements and *predictions* of probabilistic character. They must make decisions on important projects, large structures, and big investments, and they prefer to be completely sure that their decisions are true. Of course, proper education in the theory of probability and mathematical statistics helps here, but there is a more profound, maybe subconscious cause of such an attitude.
>
> Most engineers are more happy to look at samples of the behavior of a system, at its time signatures, than at probability density functions, and cumulative distribution functions that all seem alike to practicing engineers. In addition, they become suspicious when a probabilist talks about probabilities of failure, say 3.14×10^{-6} and all that.
>
> In my own experience, it is expedient to show a set of samples of the system behavior, in particular the *worst* sample, the *best* one and an *average* or a typical one...'

At the same time, it should be borne in mind that in its pure form, without proper improvements in the structure via optimization, anti-optimization is likely to yield ultra-conservative results.

This is why this monograph opts for a combination of anti-optimization *and* optimization.

1.3 Convergence of Probability and Anti-Optimization

Another natural question may pop up: if the probability approach is so good as to provide the same answer as its intellectual competitor, anti-optimization, why was it necessary to talk, in the previous section, about it being (sometimes) bad?

It is important to have a reply to this question, because the variously named non-probabilistic approaches are presented as sound *alternatives* to the probabilistic description. It appears that this happens because probability theory often uses random variables which stretch from minus infinity to plus infinity. Such is the case with the best-known discussion – the normal or Gaussian; do the exponential and Gamma distributed random variables take on values from zero to infinity? Are any of these random variables, as mental constructs, good for describing geometric or material characteristics?

Whereas the exponential and Gamma distributions with their only positive values can be utilized for describing positive quantities, it is doubtful that there exists any physical parameter that can take on values beyond physical limits. Likewise, use of the normal distribution to describe positive quantities appears to be doubtful, if not outright abnormal (pun intended). At best, these distributions are convenient analytical approximations.

The above implies that engineers ought to use truncated distributions, i.e., distributions with bounded support. It is gratifying that awareness of this fact grows amongst stochastic analysts. For example, Grigoriu (2006) mentions: 'All distributions with bounded support in $(0, \infty)$ are consistent with available information.' Ma, Leng, Meng and Fang (2004) write: 'Bounded random parameters ... are more reasonable for engineering structures than the unbounded Gaussian random ones.' In their study, Minciarelli, Gioffre, Grigoriu and Simiu (2001) write: 'Unless otherwise indicated, all uncertainty variables ... will be assumed to have truncated normal distribution.'

Cai and Wu (2004) note:

> 'When investigating a dynamical system under random excitation, it is important that each existing process should be modeled properly to resemble its measured or estimated statistical and probabilistic properties. In many cases, Gaussian distribution is assumed for convenience of analysis. However, the range of Gaussian distribution is unbounded; namely, there exists probability of having very large values. This violates the very nature of a real physical quantity, which is always bounded.'

Likewise, Cai and Lin (2006) write: 'Physically realistic random procedures are bounded, and they may deviate far from being Gaussian.' At the end of their paper these authors are even more forceful: 'Physically realistic stochastic processes must be bounded.' (see also Cai (2003)).

Ang and Tang (1975) realized this fact quite long time ago:

> 'In engineering, the information may often have to be expressed in terms of the lower and upper limits of the variable. For example, when judgment is necessary it is often convenient and perhaps more realistic to express judgmental information in the form of a range of possibilities given the range of possible values of a random variable and the underlying uncertainty may be evaluated by prescribing a suitable distribution within the range.'

In their study, Simiu and Heckert (1996) analyzed the wind speed model and fitted an approximate extreme value distribution. They concluded that the reverse Weibull distribution is more appropriate than those of Gumbel or Fréchet. Their conclusion is supported by the physical fact that non-tornadic extreme winds are expected to be bounded:

> 'In our opinion, the analysis provided persuasive evidence that extreme wind speeds are described predominantly by reverse Weibull distributions, which unlike the Gumbel distribution have finite upper tail and lead to reasonable estimates of wind load factors.'

They also stress

> 'It is a physical fact that extreme winds are bounded. Their probabilistic model should reflect this fact. To the extent that an extreme value distribution would be a reasonable model of extreme wind behavior, one would intuitively expect that the best fitting distribution to have finite tail...'

Kanda (1994) also showed that extreme winds are best fitted by distributions with limited tails; see also Walshan (1994). Holmes (2002) notes that extreme winds have a physical upper limit (which may differ for different storm types); hence, they have a bounded distribution with data from nearly all stations in Australia using various fitting methods. He reached a bounded generalized extreme value (GEV) distribution in over 80% of the cases. He stresses: 'The approach ... in substituting one third of the observation by "unbounded", "randomly generated" normal variables is not convincing.'

Thus, the pragmatic engineer cannot agree with Lindley (1987) who states: 'Probability is the only satisfactory description of uncertainty.' Neither can one agree with the notion implied by the title of Taleb's 2001 book that we are 'fooled by randomness.' (Readers are also advised to read insightful articles on the different sides of the issue by Klir (1989, 1994).)

Berleant and Goodman-Strauss (1998) used interval analysis to bound the results of arithmetic operations on random variables of unknown dependency. Thus, randomness and boundedness may pragmatically interact with each other. Kim, Ovseyevich and Reshetnyak (1993) and Elishakoff, Cai and Starnes (1994a) compared probabilistic and anti-optimization approaches. Ferson and Ginzburg (1996) advocate for application of different methods to describe ignorance and variability.

We opt in this book for the anti-optimization, or worst-scenario approach because it is simpler than the probabilistic one, not because we are against as Kosko (1994) calls it 'the probability monopoly'. Moreover, it yields the same results as a more complicated probabilistic methodology (according to Hans Hoffmann, 'the ability to simplify means to eliminate the unnecessary so that the necessary may speak'). We just add a *spice* to the approach: anti-optimized, worst-scenario results need to be optimized so as to yield cheaper, less voluminous designs. This is why this book is fully dedicated to marrying these two concepts: optimization and anti-optimization.