Chapter 1

The Generalized Moment Problem

We describe the abstract basic Generalized Moment Problem (GMP) and its dual with a few examples. We also provide some general results concerning the GMP with polynomial data and its dual and show why the theory of moments and its dual theory of positive polynomials can be useful to solve the GMP.

Problems involving moments of measures arise naturally in many areas of applied mathematics, statistics and probability, economics, engineering, physics and operations research. For instance, how do we obtain optimal bounds on the probability that a random variable belongs to a given set, given some of its moments? How do we price derivative securities in a financial economics framework without assuming any model for the underlying price dynamics, given only moments of the price of the underlying asset? But as we will see throughout the book, some (and even many) other problems seemingly different and which *a priori* do not involve any moment of some measure, have equivalent reformulations which involve moments or generalized moments.

In fact, these problems can be seen as particular instances of a linear infinite dimensional optimization problem, called the **Generalized Moment Problem** (in short, GMP).

Let $\mathbb{K}$ be a Borel subset of $\mathbb{R}^n$ and let $\mathscr{M}(\mathbb{K})$ be the space of finite *signed* Borel measures on $\mathbb{K}$, whose positive cone $\mathscr{M}(\mathbb{K})_+$ is the space of finite Borel measures μ on $\mathbb{K}$. Given a set of indices Γ, a set of reals $\{\gamma_j : j \in \Gamma\}$, and functions $f, h_j : \mathbb{K} \to \mathbb{R}$, $j \in \Gamma$, that are integrable with respect to every measure $\mu \in \mathscr{M}(\mathbb{K})_+$, the GMP is defined as follows:

$$\boxed{\text{GMP}: \quad \begin{array}{rl} \rho_{\text{mom}} = \displaystyle\sup_{\mu \in \mathscr{M}(\mathbb{K})_+} & \displaystyle\int_{\mathbb{K}} f\, d\mu \\ \text{s.t.} & \displaystyle\int_{\mathbb{K}} h_j\, d\mu \leqq \gamma_j, \qquad j \in \Gamma. \end{array} \qquad (1.1)}$$

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(Recall that the symbol "$\leqq$" stands for either an inequality "$\leq$" or an equality "$=$".) Note that we write "sup" instead of "max" to indicate that an optimal solution might not be attained. In this chapter, we present several examples that illustrate the modeling power of problem (1.1), develop a duality theory that forms the basis of future developments and briefly discuss the complexity of problem (1.1).

1.1 Formulations

In this section, our goal is to illustrate the modeling flexibility of formulation (1.1). Towards this goal, we present three examples from rather diverse areas such as probability theory, financial economics and optimization.

Moment problems in probability

Given vectors $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)' \in \mathbb{N}^n$ and $\mathbf{x} = (x_1, \ldots, x_n)' \in \mathbb{R}^n$, let

$$\mathbf{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}.$$

Let $\mathbf{S}$, $\mathbb{K} \subset \mathbb{R}^n$ be given Borel sets with $\mathbf{S} \subset \mathbb{K}$. Then, the problem of finding an optimal bound on the probability that a $\mathbb{K}$-valued random variable $\mathbf{X}$ belongs to $\mathbf{S}$, given some of its moments $\gamma_{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha} \in \Gamma \subset \mathbb{N}^n$, can be formulated as solving the problem:

$$\begin{aligned} \rho_{\text{mom}} = \sup_{\mu \in \mathscr{M}(\mathbb{K})_+} & \ \mu(\mathbf{S}) \\ \text{s.t.} & \int_{\mathbb{K}} \mathbf{x}^{\boldsymbol{\alpha}} d\mu = \gamma_{\boldsymbol{\alpha}}, \qquad \forall \, \boldsymbol{\alpha} \in \Gamma, \end{aligned} \tag{1.2}$$

a special case of formulation (1.1) with $f(\mathbf{x}) = 1_{\mathbf{S}}(\mathbf{x})$ and $h_{\boldsymbol{\alpha}}(\mathbf{x}) = \mathbf{x}^{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \in \Gamma$.

Moment problems in financial economics

A central question in financial economics is to find the price of a so-called derivative security given information on the underlying asset. Let us take the example of the so-called European call option on an underlying security (this is why an option is called derivative security as its value is derived from another) with strike k and maturity T. It gives its holder the option (but not the obligation) to buy the underlying security at time T at price k. Clearly, if the price S_T is more than k, then the holder will exercise the option and make a profit of $S_T - k$, while if it is less than k, he will not exercise and does not make a profit. Thus, the payoff of this option is $\max(S_T - k, 0)$. Clearly, as the payoff of this option is nonnegative, it has some value. A key problem in financial economics is to determine the price of such an option. This is exactly the area of the 1997 Nobel prize in economics awarded to Robert Merton and Myron Scholes (Fisher Black has passed away in 1995). Under the assumption that the price of the underlying asset follows a geometric Brownian motion and using the no-arbitrage assumption, the Black-Scholes formula provides an explicit and insightful answer to this question.

No arbitrage means that one cannot make money deterministically. For example, if a stock trades in two exchanges, it will trade at the same price, since otherwise there is an arbitrage opportunity. It turns out that the assumption of no-arbitrage is equivalent to the existence of a probability measure μ, such that the price of any European call option with strike k is given by $\mathrm{E}_\mu[\max(S_T - k, 0)]$. If we do not assume a particular stochastic process for the price dynamics S_t, but only moments of the price S_T at time $t = T$, and under the no-arbitrage assumption, the problem of finding an optimal upper bound on the price of a European call option with strike k given the first m moments γ_j, $j = 0, 1, \ldots, m$, $(\gamma_0 = 1)$ of the price of the underlying asset, is given by:

$$\begin{aligned} \rho_{\mathrm{mom}} = \sup_{\mu \in \mathscr{M}(\mathbb{R}_+)_+} & \int_{\mathbb{R}_+} \max(x-k, 0)\, d\mu \\ \text{s.t.} & \int_{\mathbb{R}_+} x^j\, d\mu = \gamma_j, \quad j = 0, 1, \ldots, m, \end{aligned} \tag{1.3}$$

a special case of formulation (1.1) with $\mathbb{K} = \mathbb{R}_+$, $f(x) = \max(x - k, 0)$ and $h_j(x) = x^j$. As another example, if the prices p_j of European call options of strike k_j, $j = 1, \ldots, m$ are given, the problem of finding an optimal upper bound on the price of a European call option with strike k is a special case of formulation (1.1) with $\mathbb{K} = \mathbb{R}_+$, $f(x) = \max(x - k, 0)$, $h_j(x) = \max(x - k_j, 0)$, and $\gamma_j = p_j$.

Global optimization over polynomials

With $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbb{K} \subset \mathbb{R}^n$, consider the constrained optimization problem:

$$\begin{aligned} f^* = \sup\ & f(\mathbf{x}) \\ \text{s.t. } & \mathbf{x} \in \mathbb{K}, \end{aligned} \tag{1.4}$$

which we rewrite as

$$\begin{aligned} \rho_{\text{mom}} = \sup_{\mu \in \mathscr{M}(\mathbb{K})_+} & \int_{\mathbb{K}} f d\mu \\ \text{s.t. } & \int_{\mathbb{K}} d\mu = 1. \end{aligned} \tag{1.5}$$

Theorem 1.1. *Problems (1.4) and (1.5) are equivalent, that is* $f^* = \rho_{\text{mom}}$.

Proof. If $f^* = +\infty$, let M be arbitrary large, and let $\mathbf{x} \in \mathbb{K}$ be such that $f(\mathbf{x}) \geq M$. Then, with $\mu := \delta_{\mathbf{x}} \in \mathscr{M}(\mathbb{K})_+$ (with $\delta_{\mathbf{x}}$ being the Dirac measure at $\mathbf{x}$), we have $\int f\, d\mu = f(\mathbf{x}) \geq M$, and so $\rho_{\text{mom}} = +\infty$.

We next consider the case $f^* < +\infty$. Since $f(\mathbf{x}) \leq f^*$ for all $\mathbf{x} \in \mathbb{K}$, then $\int_{\mathbb{K}} f d\mu \leq f^*$ and thus $\rho_{\text{mom}} \leq f^*$. Conversely, with every $\mathbf{x} \in \mathbb{K}$, we associate the Dirac measure $\delta_{\mathbf{x}} \in \mathscr{M}(\mathbb{K})_+$ which is a feasible solution of problem (1.5) with value $f(\mathbf{x})$, leading to $\rho_{\text{mom}} \geq f^*$. This combined with $f^* \geq \rho_{\text{mom}}$ leads to $f^* = \rho_{\text{mom}}$, the desired result. Notice also that if $\mathbf{x}^* \in \mathbb{K}$ is a global minimizer of problem (1.4), then the probability measure $\mu^* := \delta_{\mathbf{x}^*}$ is an optimal solution of problem (1.5). □

In other words, we can formulate the general nonlinear optimization problem as a special case of the generalized moment problem (1.1), which underscores the modeling flexibility of problem (1.1). In contrast to problem (1.4), problem (1.5) is **linear**, and thus **convex**. It is, however, infinite-dimensional.

Given the diversity and generality of these examples, it is evident that the generalized moment problem is a problem of remarkable modeling power. In later chapters and in the exercises we explore several other examples. The goal of the book is to understand the complexity of problem (1.1) and its variations, to explore applications in a variety of applied contexts and to develop algorithms for providing bounds, and sometimes solutions.

For the case $f = 0$, $h_{\boldsymbol{\alpha}} = \mathbf{x}^{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \in \Gamma \subset \mathbb{N}^n$, problem (1.1) becomes a *feasibility* problem known as the $\mathbb{K}$-moment problem.

Definition 1.1. Given $\gamma_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \in \Gamma \subset \mathbb{N}^n$ and a set $\mathbb{K} \subseteq \mathbb{R}^n$, **the $\mathbb{K}$-moment problem** asks whether there exists a finite Borel measure $\mu \in \mathscr{M}(\mathbb{K})_+$ such that

$$\int_{\mathbb{K}} \mathbf{x}^{\boldsymbol{\alpha}} d\mu = \gamma_{\boldsymbol{\alpha}}, \quad \boldsymbol{\alpha} \in \Gamma.$$

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In the next section, we develop a duality theory that forms the basis of algorithms and relaxations that we will utilize in later chapters.

1.2 Duality Theory

Problem (1.1) is linear and its dual problem is given by:

$$\begin{aligned}\rho_{\text{pop}} = \inf_{\boldsymbol{\lambda}} &\sum_{j\in\Gamma} \gamma_j \lambda_j \\ \text{s.t.} &\sum_{j\in\Gamma} \lambda_j h_j(\mathbf{x}) \geq f(\mathbf{x}), \quad \forall\, \mathbf{x} \in \mathbb{K}. \\ &\lambda_j \,\geq\, 0, \quad j \in \Gamma_+\end{aligned} \tag{1.6}$$

where $\Gamma_+ \subseteq \Gamma$ stands for the set of indices j for which the generalized moment constraint is the inequality $\int h_j\, d\mu \leq \gamma_j$.

As we will apply general results from conic duality in convex optimization, we write problems (1.1) and (1.6) as conic optimization problems. So we introduce the convex cones:

$$\begin{aligned} C(\mathbb{K}) &= \{(\boldsymbol{\gamma}, \gamma_0)' : \exists\, \mu \in \mathscr{M}(\mathbb{K})_+ \text{ s.t. } \gamma_0 = \textstyle\int_{\mathbb{K}} f\, d\mu,\ \gamma_j \geqq \int_{\mathbb{K}} h_j\, d\mu,\ \forall\, j \in \Gamma\}, \\ P(\mathbb{K}) &= \left\{(\boldsymbol{\lambda}, \lambda_0)' : \lambda_j \geq 0,\ j \in \Gamma_+;\ \sum_{j\in\Gamma} \lambda_j h_j(\mathbf{x}) + \lambda_0 f(\mathbf{x}) \geq 0, \quad \forall\, \mathbf{x} \in \mathbb{K}\right\}, \end{aligned}$$

and rewrite problems (1.1) and (1.6) as

$$\rho_{\text{mom}} = \sup_{\gamma_0} \{\gamma_0 \ :\ (\boldsymbol{\gamma}, \gamma_0)' \in \overline{C(\mathbb{K})}\,\}, \tag{1.7}$$

$$\rho_{\text{pop}} = \inf_{\boldsymbol{\lambda}} \{\sum_{j\in\Gamma} \gamma_j \lambda_j \ :\ (\boldsymbol{\lambda}, -1)' \in \overline{P(\mathbb{K})}\,\}, \tag{1.8}$$

where $\overline{R}$ denotes the closure of the set R (whereas $\operatorname{int} R$ denotes its interior).

The *weak duality* property holds for problems (1.7) and (1.8) if for any two feasible solutions γ_0 of (1.7) and $\boldsymbol{\lambda}$ of (1.8), one has $\gamma_0 \leq \sum_{j\in\Gamma} \lambda_j \gamma_j$ (and so $\rho_{\text{mom}} \leq \rho_{\text{pop}}$). If $\rho_{\text{mom}} < \rho_{\text{pop}}$ then one says that there is a *duality gap* for problems (1.7) and (1.8). Finally *strong duality* holds if there is no duality gap, i.e., $\rho_{\text{pop}} = \rho_{\text{mom}}$.

Using general results of conic duality in convex optimization, we obtain the following weak and strong duality.

Theorem 1.2.
(a) (Weak duality) *The optimal values of (1.7) and (1.8) satisfy* $\rho_{\mathrm{mom}} \leq \rho_{\mathrm{pop}}$.

(b) (Strong duality) *If* $(\boldsymbol{\gamma}, \gamma_0) \in C(\mathbb{K})$ *for some* γ_0 *, and there exists* $\boldsymbol{\lambda} \in \mathbb{R}^{|\Gamma|}$ *(with* $\lambda_j \geq 0$ *for all* $j \in \Gamma_+$*) such that* $(\boldsymbol{\lambda}, -1)' \in \operatorname{Int} P(\mathbb{K})$*, then* $\rho_{\mathrm{mom}} = \rho_{\mathrm{pop}}$ *and problem (1.1) has an optimal solution, that is the* sup *is attained.*

Proof. **(a)** Let γ_0 and $\boldsymbol{\lambda}$ be arbitrary feasible solutions of (1.7) and (1.8) respectively. By definition of the cone $C(\mathbb{K})$ there exists a finite Borel measure $\mu \in \mathscr{M}(\mathbb{K})_+$ such that

$$\gamma_0 = \int_{\mathbb{K}} f d\mu \leq \int_{\mathbb{K}} \left(\sum_{j \in \Gamma} \lambda_j h_j \right) d\mu = \sum_{j \in \Gamma} \lambda_j \int_{\mathbb{K}} h_j d\mu \leq \sum_{j \in \Gamma} \lambda_j \gamma_j,$$

that is, weak duality holds (and so, $\rho_{\mathrm{mom}} \leq \rho_{\mathrm{pop}}$).
(b) Strong duality follows from general results of conic duality in convex optimization that requires, however, that there exists a vector $(\boldsymbol{\lambda}, -1)'$ in the interior of the cone $P(\mathbb{K})$, which is known as a Slater type condition. □

In important special cases we do not need to impose Slater type conditions for strong duality to hold, as the next theorem states.

Theorem 1.3. *Suppose that* $\mathbb{K}$ *is compact,* f *is bounded and upper-semicontinuous on* $\mathbb{K}$*,* h_j *is continuous on* $\mathbb{K}$ *for every* $j \in \Gamma$*, and there exists* $k \in \Gamma$ *such that* $h_k > 0$ *on* $\mathbb{K}$*. Then:*
(a) $\rho_{\mathrm{mom}} = \rho_{\mathrm{pop}}$ *and if problem (1.1) has a feasible solution then it has an optimal solution, that is, the* sup *is attained.*
(b) *If the* sup *is attained, problem (1.1) has an optimal solution* μ *supported on finitely many points of* $\mathbb{K}$*, i.e.,* μ *is a finite linear combination of Dirac measures.*

A proof which uses results from infinite-dimensional linear optimization, can be found in Appendix C.4 where an appropriate summarized background is also provided.

There are cases where one can relax the compactness of the set $\mathbb{K}$ and still maintain strong duality. One such case is when $-f$ is lower-semicontinuous, bounded from below and inf-compact (or a moment function); see Definition B.11. In addition, for each $\boldsymbol{\alpha} \in \Gamma$, the moment constraint is an inequality constraint $\int_{\mathbb{K}} h_{\boldsymbol{\alpha}} d\mu \leq \gamma_{\boldsymbol{\alpha}}$, and $h_{\boldsymbol{\alpha}}$ is a nonnegative lower-semicontinuous function.

Countably many constraints.

We next consider the moment problem (1.1) and extend Theorem 1.3 when there are *countably many* constraints present in problem (1.1).

Corollary 1.4. *Suppose that the assumptions of Theorem 1.3 hold with* Γ *a countable set. If (1.1) has a feasible solution, then it is solvable (i.e., the sup is attained), and there is no duality gap, i.e.,* $\rho_{\text{mom}} = \rho_{\text{pop}}$.

Proof. Let (Γ_m) be a sequence of finite sets such that $\Gamma_m \subset \Gamma$ for all m and $\Gamma_m \uparrow \Gamma$ as $m \to \infty$. We may assume without loss of generality that there exists an index, say $j = 0 \in \Gamma_1$ (hence $0 \in \Gamma_m$ for all m) such that $h_0 > 0$ on $\mathbb{K}$. Suppose that (1.1) has a feasible solution and consider the moment problem (1.1) with finitely many constraints indexed in Γ_m; let ρ^m_{mom} be its optimal value. Similarly let ρ^m_{pop} denote the optimal value of its dual. By Theorem 1.3, $\rho^m_{\text{mom}} = \rho^m_{\text{pop}}$, for all m. In addition, $\rho_{\text{mom}} \leq \rho^m_{\text{mom}}$ for each m and the sequence (ρ^m_{mom}) is monotone nonincreasing as more constraints are added as m increases. Moreover, let $\mu_m \in \mathscr{M}(\mathbb{K})_+$ be an optimal solution of the primal. As $h_0 > 0$ on $\mathbb{K}$ we have $h_0 \geq \delta$ on $\mathbb{K}$ for some $\delta > 0$, and so $\mu_m(\mathbb{K}) \leq \gamma_0/\delta$ for all m. Therefore, there is a subsequence $\{m_i\}$ and a finite Borel measure μ on $\mathbb{K}$ such that $\mu_{m_i} \Rightarrow \mu$[1] as $i \to \infty$. Fix $j \in \Gamma$ arbitrary so that $h_j \in \Gamma_m$ for all m sufficiently large. As h_j is continuous, $\mathbb{K}$ is compact and $\mu_{m_i} \Rightarrow \mu$, we have

$$\int_{\mathbb{K}} h_j \, d\mu \;=\; \lim_{i \to \infty} \int_{\mathbb{K}} h_j \, d\mu_{m_i} \;\leqq \gamma_j,$$

and so, as j was arbitrary, μ is feasible for the moment problem. Moreover, as f is upper semicontinuous, it is bounded above on $\mathbb{K}$, and

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[1] The notation $\mu_{n_j} \Rightarrow \mu$ (standard in probability) stands for the weak convergence of measures; see Definition B.1.

$$\rho_{\mathrm{mom}} \leq \limsup_{i\to\infty} \rho_{\mathrm{mom}}^{m_i} = \limsup_{i\to\infty} \int_{\mathbb{K}} f\, d\mu_{m_i} \leq \int_{\mathbb{K}} f\, d\mu$$

where the last inequality on the right follows from Proposition 1.4.18 in Hernández-Lerma and Lasserre (2003). This proves that μ is a primal solution of the moment problem (with Γ countable), and so $\rho_{\mathrm{mom}}^{m_i} \downarrow \rho_{\mathrm{mom}}$, which also implies $\rho_{\mathrm{mom}}^{m} \downarrow \rho_{\mathrm{mom}}$ because the sequence $(\rho_{\mathrm{mom}}^{m})$ is monotone. Finally, as $\rho_{\mathrm{pop}}^{m} = \rho_{\mathrm{mom}}^{m}$, we also get $\rho_{\mathrm{pop}}^{m} \downarrow \rho_{\mathrm{mom}}$. □

1.3 Computational Complexity

In this section, we consider the complexity of a variant of problem (1.1) with only *inequality* moment constraints:

$$\begin{aligned} \rho_{\mathrm{mom}}^{\leq} = \sup_{\mu\in\mathscr{M}(\mathbb{K})_+} & \int_{\mathbb{K}} f d\mu \\ \text{s.t.} & \int_{\mathbb{K}} h_j d\mu \leq \gamma_j \quad j\in\Gamma, \end{aligned} \tag{1.9}$$

(i.e., $\Gamma = \Gamma_+$), and the corresponding dual problem becomes:

$$\begin{aligned} \rho_{\mathrm{pop}}^{\leq} = \inf & \sum_{j\in\Gamma} \gamma_j \lambda_j \\ \text{s.t.} & \sum_{j\in\Gamma} \lambda_j h_j(\mathbf{x}) \geq f(\mathbf{x}), \quad \forall\, \mathbf{x}\in\mathbb{K}, \\ & \lambda_j \geq 0, \quad j\in\Gamma. \end{aligned} \tag{1.10}$$

All the results we presented earlier regarding strong duality continue to hold for the pair of primal and dual problems (1.9) and (1.10). In order to study the complexity of problem (1.9), we consider the separation problem associated with problem (1.10):

Definition 1.2. The **separation problem** for problem (1.10): Given $\lambda_j \geq 0, \quad j\in\Gamma$, check whether

$$\sum_{j\in\Gamma} \lambda_j h_j(\mathbf{x}) \geq f(\mathbf{x}), \quad \forall\, \mathbf{x}\in\mathbb{K},$$

and if not, find a violated inequality.

Given the equivalence of separation and optimization, as well as strong duality, we need to consider conditions under which the separation problem is solvable in polynomial time.

Theorem 1.5.
(a) *If f is concave, h_j is convex for every $j \in \Gamma$, and $\mathbb{K}$ is a convex set, then problem (1.9) is solvable in polynomial time.*
(b) *If f and h_j, $j \in \Gamma$ are quadratic or piecewise linear functions over p polyhedra $\mathbb{K}_i$, $i = 1, \ldots, p$, that form a partition of $\mathbb{K} = \mathbb{R}^n$ (with p being a polynomial in n and $|\Gamma|$), then problem (1.9) is solvable in polynomial time.*

Proof. **(a)** The separation problem becomes

$$\inf_{\mathbf{x}\in\mathbb{K}} \sum_{j\in\Gamma} \lambda_j h_j(\mathbf{x}) - f(\mathbf{x}),$$

which in this case is a convex optimization problem, solvable efficiently using the ellipsoid method.
(b) We present the case in which $f(\mathbf{x}) = \mathbf{x}'\mathbf{Q}\mathbf{x} + \mathbf{b}'\mathbf{x} + c$ and $h_j(\mathbf{x}) = \mathbf{x}'\mathbf{Q}_j\mathbf{x} + \mathbf{b}_j'\mathbf{x} + c_j$, $j \in \Gamma$. We have

$$\sum_{j\in\Gamma} \lambda_j h_j(\mathbf{x}) - f(\mathbf{x}) = \mathbf{x}'\overline{\mathbf{Q}}\mathbf{x} + \overline{\mathbf{b}}'\mathbf{x} + \overline{c},$$

where

$$\overline{\mathbf{Q}} = \sum_{j\in\Gamma} \lambda_j \mathbf{Q}_j - \mathbf{Q},$$
$$\overline{\mathbf{b}} = \sum_{j\in\Gamma} \lambda_j \mathbf{b}_j - \mathbf{b},$$
$$\overline{c} = \sum_{j\in\Gamma} \lambda_j c_j - c.$$

To solve the separation problem, we check whether $\overline{\mathbf{Q}}$ is positive semidefinite. If it is not, we decompose $\overline{\mathbf{Q}} = \mathbf{U}'\,\boldsymbol{\Theta}\,\mathbf{U}$, where $\boldsymbol{\Theta}$ is the diagonal matrix of the eigenvalues of $\overline{\mathbf{Q}}$. Let $\theta_i < 0$ be a negative eigenvalue of $\overline{\mathbf{Q}}$. Let $\mathbf{u}$ be a vector with $u_j = 0$, $j \neq i$, and u_i large enough so that

$\theta_i u_i^2 + (\mathbf{U}\overline{\mathbf{b}})_i u_i + \overline{c} < 0$. Let $\mathbf{x}_0 = \mathbf{U}'\mathbf{u}$. Then,

$$\begin{aligned}
\sum_{j\in\Gamma} \lambda_j h_j(\mathbf{x}_0) - f(\mathbf{x}_0) &= \mathbf{x}_0'\overline{\mathbf{Q}}\mathbf{x}_0 + \overline{\mathbf{b}}'\mathbf{x}_0 + \overline{c} \\
&= \mathbf{u}'\mathbf{U}\mathbf{U}'\,\boldsymbol{\Theta}\,\mathbf{U}\mathbf{U}'\mathbf{u} + \overline{\mathbf{b}}'\mathbf{U}'\mathbf{u} + \overline{c} \\
&= \mathbf{u}'\boldsymbol{\Theta}\mathbf{u} + \overline{\mathbf{b}}'\mathbf{U}'\mathbf{u} + \overline{c} \\
&= \sum_{j=1}^{n} \theta_j u_j^2 + \sum_{j=1}^{n} (\mathbf{U}\overline{\mathbf{b}})_j u_j + \overline{c} \\
&= \theta_i u_i^2 + (\mathbf{U}\overline{\mathbf{b}})_i u_i + \overline{c} < 0,
\end{aligned}$$

which produces a violated inequality.

If $\overline{\mathbf{Q}}$ is positive semidefinite, then we solve the convex quadratic optimization problem

$$\rho_0 = \min_{\mathbf{x}\in\mathbb{R}^n} \mathbf{x}'\overline{\mathbf{Q}}\mathbf{x} + \overline{\mathbf{b}}'\mathbf{x} + \overline{c},$$

which can be solved efficiently (e.g. as a semidefinite program after some transformation). If $\rho_0 < 0$ and it is attained as some $\mathbf{x}_0$, then $\mathbf{x}_0$ produces a violated inequality. Otherwise, the given dual solution is feasible. The case of piecewise linear functions is addressed in Exercise 1.3. □

1.4 Summary

A key objective in this book is to provide algorithms for solving problem (1.1). In order to make progress we will restrict ourselves to cases where the functions f and h_j, $j \in \Gamma$, are polynomials (or is some cases, rational functions or piecewise polynomials) and the set $\mathbb{K}$ is a basic semi-algebraic set, i.e.,

$$\mathbb{K} = \{\mathbf{x} \in \mathbb{R}^n \,:\, g_i(\mathbf{x}) \geq 0,\ i = 1,\ldots,m\},$$

where $g_i \in \mathbb{R}[\mathbf{x}]$, for all $i = 1,\ldots,m$.

In particular, in the case where f, $\{h_j\}_{j\in\Gamma} \in \mathbb{R}[\mathbf{x}]$, and $\mathbb{K}$ is a basic semi-algebraic set, then problem (1.6) asks for a polynomial to be nonnegative for all $\mathbf{x} \in \mathbb{K}$. This naturally leads us in Chapter 2 to study nonnegative polynomials (and polynomials nonnegative on a basic semi-algebraic set) a topic of central importance in the development of 20th century mathematics. Similarly, when f, $\{h_j\}_{j\in\Gamma} \in \mathbb{R}[\mathbf{x}]$, then in problem (1.1), only the moments of the unknown measure μ are involved and not μ

itself, which naturally leads us in Chapter 3 to the study of the $\mathbb{K}$-moment problem. In fact, we will see that there is a nice and beautiful duality between the theory of moments and the theory of positive polynomials.

1.5 Exercises

Exercise 1.1. Given vectors $\mathbf{x}_i \in \mathbb{R}^n$ and scalars α_i, $i = 1, \ldots, m$, β and γ, formulate the problem of finding a polynomial $f \in \mathbb{R}[\mathbf{x}]$ such that $f(\mathbf{x}_i) = \alpha_i$, $i = 1, \ldots, m$, $\beta \leq f(\mathbf{x}) \leq \gamma$ for all $\mathbf{x} \in \mathbb{R}^n$ as a generalized moment problem (1.1).

Exercise 1.2. When $\mathbb{K}$ is not compact, prove that Theorem 1.3 is also valid if the data satisfy the following: $-f$ is lower-semicontinuous, bounded from below and inf-compact (or a moment function); see Definition B.11. In addition, for each $\boldsymbol{\alpha} \in \Gamma$, when the moment constraint is an inequality $\int h_{\boldsymbol{\alpha}} d\mu \leq \gamma_{\boldsymbol{\alpha}}$, then $h_{\boldsymbol{\alpha}}$ is a nonnegative lower-semicontinuous function, and h_α is continuous otherwise. (Hint: Use Theorem B.9 and Proposition B.10.)

Exercise 1.3. Prove Theorem 1.5(b) for the case that f and h_j, $j \in \Gamma$, are piecewise linear functions over p convex sets $\mathbb{K}_i$ that form a partition of $\mathbb{K} = \mathbb{R}^n$.

1.6 Notes and Sources

1.1. The fact that no-arbitrage is equivalent to the existence of a martingale measure was originally proved by Harrison and Kreps (1979). For a derivation from linear optimization duality see Bertsimas and Tsitsiklis (1997).

1.2. For a survey on duality results for conic linear problems see Shapiro (2001).

1.3. The ellipsoid method was developed by Shor (1970), and Yudin and Nemirovski (1977). The polynomial time complexity of the method was shown by Khachian (1979). For the equivalence of separation and optimization see Grötschel *et al.* (1988). Theorem 1.5 is from Bertsimas and Sethuraman (2000, Theor. 16.4.4).

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