

Preface

During the last quarter of the Twentieth Century, the study of nonlinear and complex systems has experienced unprecedented development. In the following pages, we present a first approach to standard results and recent advances in this field, applied to phenomena at our scale in our surrounding world. The pretext chosen is mainly instabilities in out-of-equilibrium systems, and more specifically the transition to turbulence in flowing fluids. It should however be clear that the methods used are of fully general use and can be applied as soon as a mathematical model of the problem at hand has been constructed, entering the dynamical framework that we shall develop.

The book is based on lecture notes for a short optional course given to second-year students in an engineering school, *École Nationale Supérieure de Techniques Avancées*, in Paris. This school is devoted to the training of high level engineers in fields including applied mathematics, mechanics and hydrodynamics, electronics, . . . , oceanography, and management. At the time of the course, students have not yet chosen their specialisation, so that the course has to be sufficiently general and without too specific requisites. Accordingly, the book should be of interest to nearly any science-oriented undergraduate student and, potentially, to everybody wanting to learn about recent advances in the field of applied nonlinear dynamics. Technicalities are not completely avoided but they are explained as simply as possible using heuristic arguments and specific worked examples, while openings on different topics can be gained by solving exercises at the end of each chapter using the same methods as those explained in the text.

At first, the problem of *chaos* that one has to face very early in this field may seem abstract and difficult. Even if the treatment of examples is not complete, the reader should get a concrete and operational mastery of concepts and techniques to be used from them. As far as the difficulty is

concerned, our aim is to transmit knowledge rather informally and without full mathematical rigour. With respect to mathematics and physics, only basic understanding is required, at the level of what is currently known after one or two years of undergraduate training. In mathematics, this does not go further than elementary algebraic calculus, basic notions of linear algebra and ordinary differential calculus. For physics, it should suffice to follow one's intuition and to admit a few fundamental equations without discussion. Adaptation of the approach to any other field of interest should thus be envisaged without excessive anxiety.

A first brief chapter situates the context of the study: evolutionary problems involving a specific independent variable called *time*, distinguishing *discrete* systems governed by finite sets of ordinary differential equations and *continuous* media described by partial differential equations. It serves as an introduction to the rest of the book, explaining in particular that continuous media driven out of equilibrium may experience instabilities inducing structures that further break down, leading to turbulence. It is concluded by a few remarks on the modelling of processes in more general systems, from chemistry to mathematical ecology, and beyond.

The second chapter is devoted to a preliminary study of dynamical systems *with a small number of degrees of freedom*. The archetype of such systems is the oscillator which serves to introduce the first manifestations of nonlinear effects, e.g. the occurrence of self-oscillations or the relation between amplitude and frequency.

The way to complex behavior is then apparently left at a too early stage, before the occurrence of chaos. In Chapter 3 we turn to a specific but particularly simple and intuitive physical problem, the stability of a fluid layer heated from below and entering a convection regime. The first part of the chapter is devoted to the elementary study of this paradigm of self-organising systems: the instability mechanism leading to the formation of convection cells out of thermal fluctuations, and an approximate determination of the threshold. In the second part, a description is given of the "death" of the so-formed *dissipative structure* and of the steps toward turbulence.

After this detour, we come back to the mathematics of the transition within the dynamical systems framework. A preliminary step has to be performed, namely a *mode reduction* resting on the distinction between *driving* and *enslaved* modes, and on the elimination of the latter. The emergence of complexity is then analyzed as a result of the increase of the number of driving modes. This is done in two steps, related to the role of

confinement effects, most easily understood in the context of convection: depending on its size, the convection box can contain a few cells or, on the contrary, many cells. In the first case, lateral boundaries freeze the spatial structure of the thermo-hydrodynamic fields and one is left with a few interacting modes characterised by their amplitudes as a function of time. This leads to the concept of *temporal chaos*, the occurrence of which is examined in Chapter 4 where we introduce classical *scenarios*. Tools used to identify chaos and measure its intensity are then presented in Chapter 5 where we also discuss empirical aspects and direct applications to chaotic mixing in fluids and chaos control.

The results and methods described up to that point will work provided that the underlying dynamics actually involves a small number of modes, which will only be recognised afterwards. In convection, when lateral walls are far enough and confinement is accordingly weak, many cells are present with variable sizes and positions. *Spatio-temporal chaos* becomes the relevant concept. Chapter 6 introduces the tools appropriate to describe *extended* systems, especially universal envelope equations accounting for modulations and *patterns* in the general case.

In common language, the third element of the title, *turbulence*, refers to the irregular, highly fluctuating, behavior of most of the flows surrounding us (the opposite situation of so-called *laminar* flows is rather exceptional). This problem is tackled in two steps. Instability and transition in *open flows* is examined in Chapter 7. In contrast to systems considered in Chapter 3, where the fluid remained confined to an enclosure, it now circulates from upstream to downstream, the consequences of which will be sketchily discussed. In Chapter 8, we consider *developed turbulence*, again along two paths. First we analyze the different scales, from the largest where energy is injected into eddies by instability mechanisms to the smallest where it is consumed by viscous dissipation. In a second instance, we turn to the statistical problem of predicting the average properties of a given turbulence flow, of utmost practical interest for an engineer.

Chapter 9 recapitulates the results and opens the perspective toward a complex dynamical system of contemporary interest, the climate of the Earth, and the problem of understanding/modeling its past and present trends.

A first appendix is devoted to a summary of linear algebra results that are useful throughout the book. As far as the understanding of nonlinear phenomena is concerned, the recourse to computers has been of considerable help at different levels. This is the reason why a second appendix,

introducing hands-on computer sessions, is devoted to elementary methods that can be developed, with sufficient common sense and no superfluous specialisation, to extract useful information from numerical simulations of simple, even simplistic, but well-designed generic models of nonlinear dynamics and pattern formation. The course was completed by laboratory sessions on topics, the theory of which is considered in some of the exercises.

Palaiseau, April 2010.