Preface

The symbiotic relationship between mathematics and physics has seldom been more apparent than in the development of quantum mechanics and the spectral theory of self-adjoint operators in Hilbert spaces since the early years of the last century. The lofty position that quantum mechanics has attained in science has corresponded with the creation of powerful and elegant theories in functional analysis and operator theory, each feeding voraciously on the problems and discoveries of the other. At the core of this activity, the spectral analysis of the Schrödinger operator has been intensively studied by many and the achievements have been impressive. These range from the detailed description of the spectral properties of atoms subject to electrostatic and magnetic forces, to the study initiated by Dyson and Lenard, and then by Lieb and Thirring, concerning the stability of matter governed by systems of particles under the influence of internal Coulomb forces and external fields.

Attempts to incorporate relativistic effects when appropriate in the theory have encountered many difficulties. Dirac's equation describes the electron and positron as a pair and this yields an operator which is unbounded above and below. Such operators are harder to deal with than those which are semi-bounded, as is typically the case with the Schrödinger operator. Also it results in the Dirac operator not being a suitable model to describe relativistic systems of many particles because their spectrum occupies the whole of the real line and bound states are not defined. In an effort to bypass these problems with the Dirac operator, various alternatives and approximations have been suggested and studied for the kinetic energy term in the total energy Hamiltonian, which preserve some essential features. The so-called quasi-relativistic operator $\sqrt{-\Delta+1}$ (in appropriate units) shares with the free Dirac operator $\mathbb{D}_0$ the property that its square is the Schrödinger operator, and has the advantage that $\sqrt{-\Delta+1}-\gamma/|\mathbf{x}|$, where $\gamma/|\mathbf{x}|$ represents the Coulomb potential due to the electron-nuclear interaction, is bounded below for a range of constants γ. This operator was studied by Herbst in [Herbst (1977)] and Weder in [Weder (1974, 1975)], and their work is included within the discussion in this book. The other main operator studied in depth in this book is that introduced by Brown and Ravenhall in [Brown and Ravenhall (1951)], studied by Hardekopf and Sucher in [Hardekopf and Sucher (1985)], which attempts to split the Dirac operator into positive and negative spectral parts. The basic Brown–Ravenhall operator is of the form $\Lambda_+ (\mathbb{D}_0 - \gamma/|\mathbf{x}|) \Lambda_+$, where Λ_+ is the projection onto the positive spectral subspace of the free Dirac operator, and sensationally, it is bounded below, and indeed positive, for all known elements. It is the restriction of the quasi-relativistic operator to a subspace of the underlying L^2 space and there is justification in regarding it as a better physical model than the quasi-relativistic operator.

The book is primarily designed for the mathematician with an interest in the spectral analysis of the operators of mathematical physics, but we hope that other scientists will find topics of interest here, and we have written the book with that in mind. The topics covered naturally reflect our own interests and areas of expertise, and are mainly those with which we have been closely associated during the last fifteen years. A knowledge of basic functional analysis and operator theory is assumed, but the first chapter gives a brief survey of the necessary background material to help the reader who is not familiar with, or needs reminding of, the material and techniques in the following chapters. Much of Chapter 2 is taken up by precise descriptions and the establishment of basic properties of the Dirac, quasi-relativistic and Brown–Ravenhall operators with Coulomb potentials. This involves the definition of self-adjoint realisations in an appropriate Hilbert space, these being either defined uniquely in the case of essential self-adjointness, or otherwise as a Friedrichs extension, or some other physically relevant self-adjoint operator, associated with a lower semi-bounded quadratic form. Of particular concern is the determination of optimal conditions on the Coulomb potential for which the different types of self-adjoint realisations are valid. The nature of the spectrum of these operators in turn is addressed in Chapter 3, in particular the location of the essential spectrum, and the existence of eigenvalues, which are either isolated from the essential spectrum or embedded in it. The analysis of embedded eigenvalues is based on a simple abstract virial theorem, modelled on a celebrated result of J. Weidmann for Schrödinger operators, and this is then applied to each of the three types of operator in turn. The stability of matter is a problem that has attracted a great deal of attention, and in this context the Pauli operator in particular presents some interesting challenges. Chapter 4 deals with some of these. There is a brief outline of some of the highlights of what has been achieved over the last three decades, but the focus is mainly on important auxiliary issues and techniques, which are of intrinsic interest. In particular the existence or otherwise of magnetic fields that give rise to zero modes of the Pauli operator is examined in detail. Zero modes have some profound physical and mathematical consequences, and because of their importance they merit substantial coverage. Topics covered include the following: a discussion of some known examples with a description of techniques developed for their construction based on quaternions; a detailed analysis of a class of magnetic potentials that give rise to zero modes; growth rates and asymptotic limits of the magnetic potentials; and the relevance of zero modes to some spectral, Dirac–Sobolev and Dirac–Hardy inequalities. Also techniques that have proved to be effective in establishing stability of matter results, including Lieb–Thirring inequalities, are discussed.

Chapters are divided into sections and most sections into subsections. Theorems, corollaries, lemmas, remarks and equations are numbered consecutively within a section. Thus equation (3.2.15) is the fifteenth equation in Section 3.2, which is the second section in Chapter 3. Section 3.2.4 refers to the fourth subsection within Section 3.2.

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Basic Notation

- $\mathbb{C}$ complex plane; $\mathbb{C}^n$: n-dimensional complex space;
- $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Im z} > 0\}$; $\mathbb{C}_- = \{\text{z} \in \mathbb{C} : \text{Im z} < 0\}$;
- $\mathbb{R}$ real line; $\mathbb{R}^n$: n-dimensional Euclidean space;
- $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}_- = (-\infty, 0)$;
- $\mathbb{S}^n$: n-dimensional sphere;
- $\omega_n = \frac{\pi^n/2}{\Gamma(1+\frac{1}{2}n)}$, the volume of the unit ball in $\mathbb{R}^n$;
- $\mathbb{N}$: positive integers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $\mathbb{Z}$: all integers;
- $f(\mathbf{x}) \asymp g(\mathbf{x}) : c_1 \le f(\mathbf{x})/g(\mathbf{x}) \le c_2$ for some positive constants c_1, c_2.