Preface

Introduction

This is the text for the course in geometric mechanics taught by the author for undergraduates in their third year of mathematics at Imperial College London.

A brief history of geometric mechanics

The ideas underlying geometric mechanics first emerged in the principles of optics formulated by Galileo, DesCartes, Fermat and Huygens. These underlying ideas were developed in optics and particle mechanics by Newton, Euler, Lagrange and Hamilton, with added contributions from Gauss, Poisson, Jacobi, Riemann, Maxwell and Lie, for example, then later by Poincaré, Noether, Cartan and others. In many of these contributions, optics and mechanics held equal sway.

Fermat's principle (that light rays follow extremal time paths) is complementary to Huygens principle (that a later wave front emerges as the envelope of wavelets emitted from the present wave front). Both principles are only models of reality, but they are models in the best sense. Both are transcendent fabrications that intuited the results of a more fundamental principle (Maxwell's equations) and gave accurate predictions at the level of physical perception in their time. Without being the full truth by being physically tenable themselves, they fulfilled the tasks for which they were developed and they laid the foundations for more fundamental theories. Light rays do not exist and points along a light wave do not emit light. However, both principles work quite well in the design of optical instruments! In addition, both principles are still interesting now as the mathematical *definitions* of rays and wave fronts, respectively, although neither one properly represents the physical principles of optics.

The duality between tangents to extremal paths (Fermat) and normals to wave fronts (Huygens) in classical optics corresponds in geometric mechanics to the duality between velocities and momenta. This duality between ray paths and wave fronts may remind us of the duality between complementary descriptions of particles and waves in quantum mechanics. The bridge from the wave description to the ray description is crossed in the geometric-optical high-wavenumber limit ($k \to \infty$). The bridge from quantum mechanics to classical mechanics is crossed in another type of geometric-optical limit ($\hbar \to 0$) as Planck's constant tends to zero. In this course we arrive at the threshold of the bridge to quantum mechanics when we write the Maxwell-Bloch equations for the two-level qubit of quantum computing. Although we do not cross over this bridge, its presence reminds us that the conceptual unity in the historical developments of geometrical optics and classical mechanics is still of interest today. Indeed, Hamilton's formulations of optics and mechanics were guiding lights in the development of the quantum mechanics of atoms and molecules, and the quantum version of the Hamiltonian approach is still used today in scientific research on the interactions of photons, electrons and beyond.

Building on the earlier work by Hamilton and Lie, in a series of famous studies during the 1890s, Poincaré laid the geometric foundations for the modern approach to classical mechanics. One study by Poincaré addressed the propagation of polarised optical beams. For us, Poincaré's representation of the oscillating polarisation states of light as points on a sphere turns out to inform the geometric mechanics of nonlinearly coupled resonant oscillators. Following Poincaré, we shall represent the dynamics of coupled resonant oscillators as flows along curves on manifolds whose points are resonantly oscillating motions. Such orbit manifolds are *fibrations* (local factorisations) of larger spaces.

Lie symmetry is the perfect method for applying Poincaré's geometric approach. In mechanics, a Lie symmetry is an invariance of the Lagrangian or Hamiltonian under a Lie group; that is, under a group of transformations that depend smoothly on a set of parameters. The effectiveness of Lie symmetry in mechanics is seen on the Lagrangian side in Noether's theorem. Noether's theorem states that each Lie symmetry of the Lagrangian in Hamilton's principle of least action implies a conservation law. On the Hamiltonian side, Noether's theorem states that each Lie symmetry of the Hamiltonian summons a momentum map, which maps the canonical phase space to the dual of the Lie symmetry algebra. When the Lie group is a symmetry of the Hamiltonian, the momentum map is conserved and the dynamics is confined to its level sets. Of course, there is much more geometry in this idea than the simple restriction of the dynamics to a level set. In particular, restriction to level sets of momentum maps culminates in the reduction of phase space to manifolds whose points are orbits of symmetries. In some cases, this geometric approach to the separation of motions may produce complete integrability of the original problem.

The language of Lie groups, especially Lie derivatives, is needed to take advantage of Poincaré's geometrical framework for mechanics. The text also provides an introduction to exterior differential calculus; so that the student will have the language to go further in geometric mechanics. The lessons here are only the first steps – the road to geometric mechanics is long and scenic, even beautiful, for those who may take it. It leads from finite to infinite dimensions. Anyone taking this road will need these basic tools and the language of Lie symmetries, in order to interpret the concepts that will be met along the way.

The approach of the text

The text surveys a small section of the road to geometric mechanics, by treating several examples in classical mechanics, all in the same geometric framework. These example problems include:

- Fermat's principle for ray optics to travelling waves propagating by self-induced transparency (the Maxwell-Bloch equations);

- Bifurcations in the behaviour of resonant oscillators and polarised travelling wave pulses in optical fibres;

- The bead sliding on a rotating hoop, the spherical pendulum and the elastic spherical pendulum. The approximate solution of the elastic spherical pendulum via a phase-averaged Lagrangian shares concepts with molecular oscillations of CO_2 and with 2nd harmonic generation in nonlinear laser optics;

- Divergenceless vector fields and stationary patterns of fluid flow on invariant surfaces.

In each case, the results of the geometric analysis eventually reduce to divergence free flow in $\mathbb{R}^3$ along intersections of level surfaces of constants of the motion. On these level surfaces, the motion is symplectic, as guaranteed by the Marsden-Weinstein theorem [MaWe74].

How to read this book

The book is organised into six chapters and one appendix. Chapter 1 treats Fermat's principle for ray optics in refractive media as a detailed example that lays out the strategy of Lie symmetry reduction in geometric mechanics to be applied in the remainder of the text. Chapter 2 summarises the contributions of Newton, Lagrange and Hamilton to geometric mechanics. Chapter 3 discusses Lie symmetry reduction in the language of the exterior calculus of differential forms. The strategy of Lie symmetry reduction laid out in Chapter 1 in the example of ray optics is applied to resonant oscillator dynamics in Chapter 4, to the elastic spherical pendulum in Chapter 5 and to a special case of the Maxwell-Bloch equations for laser excitation of matter in Chapter 6. The Appendix contains a compendium of example problems which may be used as topics for homework and enhanced coursework.

The first chapter treats Fermat's principle for ray optics as an example that lays out the strategy of Lie symmetry reduction for all of the other applications of geometric mechanics discussed in the course. This strategy begins by deriving the dynamical equations from a variational principle and Legendre transforming from the Lagrangian to the Hamiltonian formulation. Then the implications of Lie symmetries are considered. For example, when the medium is symmetric under rotations about the axis of optical propagation, the corresponding symmetry of the Hamiltonian for ray optics yields a conserved quantity called *skewness*, which was first discovered by Lagrange.

The second step in the strategy of Lie symmetry reduction is to transform to invariant variables. Writing the ray optics Hamiltonian as a function of the axisymmetric invariants that are bilinear in the optical phase space variables has the effect of quotienting out the angular dependence by introducing a variant of the polar coordinate representation. The transformation to bilinear axisymmetric invariants is called the *quotient map*. The quotient map takes the four-dimensional optical phase space into the three-dimensional real space $\mathbb{R}^3$. The image of the quotient map in $\mathbb{R}^3$ is conveniently represented as the zero level set of a function. This zero level set is the two-dimensional *orbit manifold*, on which each point represents a circle in phase space corresponding to the orbit of the axial rotations.

In the third step, the canonical Poisson brackets of the axisymmetric invariants with the phase space coordinates produce *Hamiltonian vector fields*, whose flows on phase space yield the diagonal action of the symplectic Lie group $Sp(2,\mathbb{R})$ on the optical position and momentum. The Poisson brackets of the axisymmetric invariants *close among themselves* as linear functions of these invariants, thereby yielding a *Lie-Poisson bracket* dual to the symplectic Lie algebra $sp(2,\mathbb{R})$, represented as divergenceless vector fields on $\mathbb{R}^3$. The Lie-Poisson bracket reveals the geometry of the solution behaviour in axisymmetric ray optics as flows along the intersections of the level sets of the Hamiltonian and the orbit manifold in the $\mathbb{R}^3$ space of axisymmetric invariants. This is coadjoint motion.

In the final step, the angle variable is reconstructed. This angle turns out to be the sum of two parts: one part is called *dynamic*, because it depends on the Hamiltonian. The other part is called *geometric* and is equal to the area enclosed by the solution on the orbit manifold.

The geometric-mechanics treatment of Fermat's principle identifies two *momentum maps* admitted by axisymmetric ray optics. The first is the map from optical phase space (position and momentum of a ray on an image screen) to their associated area normal to the screen. This area is Lagrange's invariant in axisymmetric ray optics; it takes the same value on each image screen along the optical axis. The second momentum map transforms from optical phase space to the bilinear axisymmetric invariants by means of the quotient map. Because this transformation is a momentum map, the quotient map yields a valid Lie-Poisson bracket among the bilinear axisymmetric invariants.

Chapter 2 treats the geometry of rigid-body motion from the viewpoints of Newton, Lagrange and Hamilton, respectively. This is the classical problem of geometric mechanics, which makes a natural counterpoint to the treatment in Chapter 1 of ray optics by Fermat's principle. The treatments of the rigid body by these more familiar approaches also sets the stage for the introduction of the flows of Hamiltonian vector fields and their Lie-derivative actions on differential forms in Chapter 3.

The problem of a single, polarised, optical laser pulse propagating as a travelling wave in an anisotropic, cubically nonlinear, lossless medium is investigated in Chapter 4. This is a Hamiltonian system in $\mathbb{C}^2$ for the dynamics of two complex oscillator modes (the two polarisations). Since the two polarisations of a single optical pulse must have the same natural frequency, they are in $1:1$ resonance. An S^1 phase invariance of the Hamiltonian for the interaction of the optical pulse with the optical medium in which it propagates will reduce the phase space to the Poincaré sphere, S^2, on which the problem is completely integrable. In Chapter 4, the fixed points and bifurcations sequences of the phase portrait of this system on S^2 are studied as the beam intensity and medium parameters are varied. The corresponding Lie-symmetry reductions for the $n:m$ resonances is also discussed in detail.

Chapter 5 treats the swinging spring, or elastic spherical pendulum, from the viewpoint of Lie-symmetry reduction. In this case, averaging the Lagrangian for the system over its rapid elastic oscillations introduces the additional symmetry needed to reduce the problem to an integrable Hamiltonian system. This reduction results in the three-wave surfaces in $\mathbb{R}^3$ and thereby sets up the framework for predicting the characteristic feature of the elastic spherical pendulum, which is the step-wise precession of its swing plane.

Chapter 6 treats the Maxwell-Bloch laser-matter equations for self-induced transparency. The Maxwell-Bloch equations arise from a variational principle obtained by averaging the Lagrangian for the Maxwell-Schrödinger equations. As for the swinging spring, averaging the Lagrangian introduces the Lie symmetry needed for reducing the dimensions of the dynamics and thereby making it more tractable. The various Lie-symmetry reductions of the real Maxwell-Bloch equations to two-dimensional orbit manifolds are discussed and their corresponding geometric phases are determined in Chapter 6.

Exercises are sprinkled liberally throughout the text, often with hints or even brief explicit solutions. These are indented and marked with ★ and ▲, respectively. Moreover, the careful reader will find that many of the exercises are answered in passing somewhere later in the text in a more developed context.

> Key theorems, results and remarks are placed into frames (like this one).

The Appendix contains additional worked problems in geometric mechanics. These problems include the linear oscillator, planar pendulum, bead sliding on the rotating hoop, spherical pendulum, rigid body and the Duffing oscillator, as well as the study of pairs of nonlinearly-coupled resonant oscillators, second-harmonic generation and the dynamics of the rigid body with a flywheel, all treated from the viewpoint of geometric mechanics.

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