Chapter 4

The Black-Scholes Equation

The most important application of the Itô calculus, derived from the *Itô lemma*, in financial mathematics is the *pricing of options*. The most famous result in this area is the *Black-Scholes formulae* for pricing European vanilla call and put options. As a consequence of the formulae, both in theoretical and practical applications, *Robert Merton* and *Myron Scholes* were awarded the *Nobel Prize* for Economics in 1997 to honour their contributions to option pricing. Unfortunately, Fischer Black, who has also given his name and contributions, had passed away two years before.

In their famous work, in 1973, Black and Scholes transformed the option pricing problem into the task of solving a (parabolic) partial differential equation (PDE) with a final condition. The main conceptual idea of Black and Scholes lies in the construction of a riskless portfolio taking positions in bonds (cash), option, and the underlying stock. Such an approach strengthens the use of the *no-arbitrage* principle as well.

Derivation of a closed-form solution to the Black-Scholes equation depends on the fundamental solution of the *heat equation*. Hence, it is important, at this point, to transform the Black-Scholes equation to the heat equation by *change of variables*. Having found the closed-form solution to the heat equation, it is possible to transform it back to find the corresponding solution of the Black-Scholes PDE.

The connection between an *initial and/or boundary value problem* for differential equations, the so-called a *Cauchy problem*, and the computation of the *expected value* of a functional of a solution of an SDE is covered by the *Feynman-Kac* representation theorem. However, we leave it to interested readers, but apply the celebrated closed-form solutions to various examples.

Indeed, an important consequence of these closed-form solutions is the use of the *Greeks*: the partial derivatives of the value of an option with respect to the variables. The Greeks are used for *hedging* purposes, which is related to the sensitivity of the option prices to the parameters, such as the underlying asset prices, interest rates, time, and the volatility of the asset prices. Having solved the Black-Scholes equation, we have the opportunity to maintain the closed-form representations of these *Greeks.*

4.1 Derivation of the Black-Scholes Equation

This section applies the *Itô lemma* to derive the Black-Scholes equation, whose basic and the first assumption is a geometric Brownian motion for the asset price.

A direct consequence of the Itô lemma, Lemma 3.2 on page 96, follows for the geometric Brownian motion of the asset prices, where we have $X_t = S_t$, $a = \mu S_t$, and $b = \sigma S_t$. Hereafter, we will drop the subscript t for both a better understanding and simplicity. Assume that the asset price S follows the geometric Brownian motion,

$$dS = \mu S\, dt + \sigma S\, dW,$$

where μ and σ are constant, and W is a Wiener process. Let $V = V(S,t)$ denote the value of an option (or a contingent claim) that is sufficiently smooth, namely, its second-order derivatives with respect to S and first-order derivative with respect to t are continuous in the domain

$$\mathcal{D}_V = \{(S,t) \,:\, S \geq 0, \quad 0 \leq t \leq T\}\,. \tag{4.1}$$

Then, it immediately follows from the Itô lemma that

$$\boxed{dV = \left(\frac{\partial V}{\partial S}\,\mu S + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\,\sigma^2 S^2\right) dt + \frac{\partial V}{\partial S}\,\sigma S\, dW.} \tag{4.2}$$

This is in fact nothing more than a rephrasing of the Itô lemma, however, it will be used to derive the celebrated Black-Scholes equation in the sequel by applying the no-arbitrage principle.

Since both stochastic processes S and V are driven by the same Wiener process W, the stochastic term, $\sigma S \frac{\partial V}{\partial S}\, dW$, can be eliminated by constructing a portfolio that consists of the option and the underlying asset: a common exercise in finance. Let Π be the wealth of the portfolio that consists of one short position with value V and Δ units of the underlying asset with the price S. Assume that initially the portfolio wealth is Π_0, and hence, the value of the portfolio at time t can be determined from

$$\Pi = -V + \Delta\, S.$$

Therefore, the infinitesimal change in the portfolio becomes

$$\begin{aligned} d\Pi &= -dV + \Delta\, dS \\ &= -\left(\mu S\left[\Delta - \frac{\partial V}{\partial S}\right] + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \left(-\frac{\partial V}{\partial S} + \Delta\right)\sigma S\, dW. \end{aligned}$$

Note that the fluctuations caused by the increments of the underlying Wiener process have a coefficient, $\left(-\frac{\partial V}{\partial S} + \Delta\right)$, that depends on Δ, the number of shares of the underlying asset. Hence, by

$$\boxed{\Delta = \frac{\partial V}{\partial S}}$$

shares of asset, the infinitesimal change $d\Pi$ of the portfolio within the time interval dt is

$$\boxed{d\Pi = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt, \qquad (4.3)}$$

and it is purely *deterministic*. Indeed, more than that: *the drift rate* μ *has been cancelled out*! This represents the gain when Π_0, the initial wealth, is invested in the *risky*, but *frictionless* market[1] that consists of the option with value V and the underlying asset with S.

Furthermore, choosing $\Delta = \frac{\partial V}{\partial S}$ provides a strategy (*hedging*) to eliminate the risk in the portfolio due to the stochastic fluctuations and the drift coefficient μ of the underlying asset that has disappeared. In this sense, the modelling of V is risk-neutral. The remaining parameter σ reflects the stochastic behaviour in the Black-Scholes equation. Although it is assumed to be constant, its estimation is an important concept, known as the *implied volatility* in finance.

The same amount of wealth Π of the portfolio should gain the riskless interest rate in infinitesimal time. Under the assumption of a frictionless market without arbitrage and a constant risk-free interest rate r, the amount Π would grow to $\Pi = \Pi_0\, e^{r(t-t_0)}$. Hence, the change in infinitesimal time would be

$$d\Pi = r\Pi\, dt,$$

which is equivalent to

[1]This means that there are no transaction costs, the interest rates for borrowing and lending money are equal, all parties have immediate access to any information, and all securities and credits are available at any time and in any size. Further, individual trading will not influence the price.

$$d\Pi = r(-V + \Delta S)\, dt = \left(-rV + rS\frac{\partial V}{\partial S}\right) dt. \quad (4.4)$$

This infinitesimal change $d\Pi$ in the portfolio is due to the investment in the risk-free interest rate r, unlike the one in (4.3).

By the no-arbitrage principle and the possibility of an early exercise of an option, it is required that the riskless gain in (4.4) cannot be more than the gain in the risky market given by (4.3). Hence,

$$-rV + rS\frac{\partial V}{\partial S} \leq -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right).$$

Consequently, the inequality, due to Black and Scholes,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \leq 0 \quad (4.5)$$

must hold in the domain $\mathcal{D}_V$. This inequality is valid no matter if the considered option is European or American. Hence, an option price generally satisfies this *partial differential inequality*.

If the option is assumed to be a European one, then there is no possibility of early exercise, and the no-arbitrage principle implies that these gains must be equal at the end of the infinitesimal investment time interval. Hence, for European options the partial differential inequality in (4.5) is reduced to the celebrated *Black-Scholes* equation,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \quad (4.6)$$

in the domain $\mathcal{D}_V$.

Therefore, an option price $V = V(S,t)$ must solve either of the inequality or the equality depending on whether the option is, respectively, American or European. However, in (4.5) and (4.6) there is no μ, the drift rate of the asset. The drift rate μ has been replaced by the risk-free interest rate r under the assumption of no-arbitrage. This is known as the *risk-neutral valuation principle*, which is summarised in the following remark.

Remark 4.1. For pricing options the return rate μ of the underlying asset that pays no dividend is replaced by the risk-free interest rate r. In other words, $\mu = r$ is assumed.

The remark above still remains valid if, further, *dividends* are assumed to be paid with continuously compounding yield, say δ. The continuous flow of dividends, however, can be modelled easily by a decrease of the asset price, S, in each infinitesimal time interval dt. This decrease in S is equal to the amount paid out by the dividend: $\delta\, S\, dt$ with a constant $\delta \geq 0$. This is due to the no-arbitrage principle: otherwise, by purchasing the asset at time t and selling it immediately after receiving the dividend one would make a risk-free profit of amount $\delta\, S\, dt$. The continuously compounding dividend yield can easily be inserted into the Black-Scholes framework: the drift coefficient of the asset price model changes to $\mu - \delta$ rather than μ only. That is, the geometric Brownian motion of the asset price is generalised to

$$dS = (\mu - \delta) S\, dt + \sigma S\, dW. \tag{4.7}$$

Hence, carrying out a similar argument,[2] the corresponding Black-Scholes equation for a European option price $V(S,t)$ with the domain $\mathcal{D}_V$ becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta) S \frac{\partial V}{\partial S} - rV = 0 \tag{4.8}$$

instead of the Black-Scholes PDE in (4.6). For American options, the equality sign "=" in (4.8) must be changed to an inequality sign "$\leq$" to allow possible early exercise opportunities.

Outlook

Derivation of the Black-Scholes equation is originally proposed in [Black and Scholes (1973)] and [Merton (1973)] based on the no-arbitrage principle or the *delta-hedging* argument. The riskless portfolio

$$\Pi = -V + \Delta\, S \quad \text{with} \quad \Delta = \frac{\partial V}{\partial S},$$

is sometimes called the *delta-hedge* portfolio. See Section 4.3 for more on hedging.

It is important to emphasise again that in the Black-Scholes equation μ does not appear due to the riskless portfolio. The risk-neutral valuation principle in Remark 4.1 is indeed based on a more mathematical

[2]Readers are encouraged to derive the Black-Scholes equation with continuous dividend yield by considering a portfolio strategy.

setting: existence of a *risk-neutral* measure. Girsanov theorem (see for instance (Shreve, 2004b, p. 212)) states that there exists a unique measure $\mathbb{Q}$ under which

$$\tilde{W}_t = W_t + \frac{\mu - r}{\sigma} t$$

becomes a Brownian motion. Here, the term $(\mu - r)/\sigma$ is called the *market price of risk*. Rearranging the terms and using the geometric Brownian motion for the asset prices S_t driven by the standard Brownian motion W_t, we obtain

$$dS_t = rS_t \, dt + \sigma S_t \, d\tilde{W}_t.$$

In this setting the pricing is maintained by the risk-neutral probability $\mathbb{Q}$ rather than the market probability $\mathbb{P}$.

4.2 Solution of the Black-Scholes Equation

The Black-Scholes equation admits a closed-form solution and, hence, this solution made the founders well-known and respected. In fact, the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta) S \frac{\partial V}{\partial S} - rV = 0 \tag{4.9}$$

for a European option $V(S,t)$ is of the type of a *parabolic* partial differential equation in the domain $\mathcal{D}_V$, where

$$\mathcal{D}_V = \{(S,t) \,:\, S > 0, \quad 0 \le t \le T\}. \tag{4.10}$$

Hence, by a suitable transformation of the variables the Black-Scholes equation is equivalent to the *heat* equation,

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{4.11}$$

for $u = u(x, \tau)$ for x and t in the domain

$$\mathcal{D}_u = \left\{ (x,\tau) \,:\, -\infty < x < \infty, \quad 0 \le \tau \le \frac{\sigma^2}{2} T \right\}. \tag{4.12}$$

In general, the classical heat equation may be considered in a larger domain, $x \in \mathbb{R}$ and $\tau \ge 0$. However, since the option expires at maturity T, and the time when the option contract is signed is assumed to be $t_0 = 0$, then the transformed heat equation will naturally have a bounded τ. On the other hand, although in the domain of the Black-Scholes equation the variable S lies on the positive real axis, the variable x in the domain of the heat equation lies on the whole real axis. These are all due to the transformations used in the sequel.

4.2.1 Transforming to the Heat Equation

Consider the transformations of the *independent* variables

$$S = K\, e^x, \quad \text{and} \quad t = T - \frac{\tau}{\sigma^2/2},$$

and the *dependent* variable

$$v(x,\tau) = \frac{1}{K} V(S,t) = \frac{1}{K} V\left(K e^x, T - \frac{\tau}{\sigma^2/2}\right).$$

In fact, the change of the independent variables ensures that the domain of the new dependent variable $v = v(x,\tau)$ is $\mathcal{D}_u$.

By the chain rule for functions of several variables, these changes of variables give

$$\begin{aligned}
\frac{\partial V}{\partial t} &= K \frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{\sigma^2}{2} K \frac{\partial v}{\partial \tau}, \\
\frac{\partial V}{\partial S} &= K \frac{\partial v}{\partial x}\frac{\partial x}{\partial S} = \frac{K}{S}\frac{\partial v}{\partial x}, \\
\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \frac{K}{S^2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right).
\end{aligned}$$

Inserting the derivatives in the Black-Scholes equation (4.9) transforms it to a *constant coefficient* one:

$$v_\tau = v_{xx} + \left(\frac{r-\delta}{\sigma^2/2} - 1\right) v_x - \frac{r}{\sigma^2/2} v,$$

where the subscripts represents the partial derivatives with respect to the corresponding variables. Define the following new constants,

$$\kappa = \frac{r-\delta}{\sigma^2/2}, \quad \text{and} \quad \ell = \frac{\delta}{\sigma^2/2},$$

so that the transformed PDE turns into a simpler form

$$v_\tau = v_{xx} + (\kappa - 1) v_x - (\kappa + \ell) v, \tag{4.13}$$

the coefficients of which involve the new two constants κ and ℓ. This constant coefficient PDE must be transformed further to the heat equation by some other change of the independent variables.

In order to simplify the final transformation of the dependent variable v, let us first define the following constants:

$$\gamma = \frac{1}{2}(\kappa - 1), \quad \text{and} \quad \beta = \frac{1}{2}(\kappa + 1) = \gamma + 1,$$

so that

$$\beta^2 = \gamma^2 + \kappa.$$

In terms of these new constants, now the transformation can be defined by

$$v(x, \tau) = e^{-\gamma x - (\beta^2 + \ell)\tau} u(x, \tau),$$

for all (x, τ) in $\mathcal{D}_u$. Hence, the partial derivatives with respect to τ and x can be calculated as

$$\begin{aligned} v_\tau &= e^{-\gamma x - (\beta^2 + \ell)\tau} \left\{ -(\beta^2 + \ell) u + u_\tau \right\}, \\ v_x &= e^{-\gamma x - (\beta^2 + \ell)\tau} \left\{ -\gamma u + u_x \right\}, \\ v_{xx} &= e^{-\gamma x - (\beta^2 + \ell)\tau} \left\{ \gamma^2 u - 2\gamma u_x + u_{xx} \right\}. \end{aligned}$$

Thus, substituting these derivatives into (4.13) yields

$$u_\tau = u_{xx} + (-2\gamma + \kappa - 1)\, u_x + \gamma\,(2\gamma - \kappa + 1)\, u,$$

after having used the fact that $\beta^2 = \gamma^2 + \kappa$. Notice that the coefficients of the terms u_x and u in the equation above vanishes by the choice of γ as $\frac{1}{2}(\kappa - 1)$. Consequently, the equation that is to be satisfied by the transformed dependent variable $u = u(x, \tau)$ is the dimensionless form of the heat equation,

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \tag{4.14}$$

that is to be solved on the domain $\mathcal{D}_u$. This shows the equivalence between the Black-Scholes equation (4.9) and the heat equation (4.11).

To sum up, in order to transform the Black-Scholes equation to the classical dimensionless heat equation, the constants used above are defined to be

$$\begin{aligned} \kappa &= \frac{r - \delta}{\sigma^2/2}, & \ell &= \frac{\delta}{\sigma^2/2}, \\ \gamma &= \frac{1}{2}(\kappa - 1), & \beta &= \frac{1}{2}(\kappa + 1) = \gamma + 1. \end{aligned} \tag{4.15}$$

On the other hand, the transformations of the dependent and the independent variables that use those constants are given by

$$S = K\,e^x, \qquad t = T - \frac{\tau}{\sigma^2/2}, \qquad V(S,t) = K\,v(x,\tau), \qquad v(x,\tau) = e^{-\gamma x - (\beta^2+\ell)\tau}\,u(x,\tau). \tag{4.16}$$

Under these changes of variables, the domain $\mathcal{D}_V$ is mapped to $\mathcal{D}_u$.

The *fundamental solution* of the dimensionless heat equation $u_\tau = u_{xx}$ is given by

$$G(x,\tau) = \frac{1}{\sqrt{4\pi\tau}}\exp\left\{-\frac{x^2}{4\tau}\right\} \tag{4.17}$$

which satisfies the equation for all $\tau > 0$ and $x \in \mathbb{R}$. This can be easily shown by direct substitution into the equation. Note also that $G(x,\tau) = \phi_{0,\sqrt{2\tau}}(x)$, that is, it is the probability density function of the normal distribution with mean zero and variance 2τ.

Moreover, for a given *initial* condition,

$$u(x,0) = u_0(x), \qquad -\infty < x < \infty, \tag{4.18}$$

at $\tau = 0$, the solution of the heat equation can be written as a convolution integral of G and u_0 as

$$u(x,\tau) = \int_{-\infty}^{\infty} G(x-\xi,\tau)\,u_0(\xi)\,d\xi \tag{4.19}$$

for $\tau > 0$. With this representation, the function $G(x-\xi,\tau)$ is also called the *Green's function* for the diffusion equation. It is not too difficult to show that $u = u(x,\tau)$ represented by the convolution integral above is indeed a solution of the heat equation and satisfies

$$\lim_{\tau\to 0^+} u(x,\tau) = u_0(x).$$

We leave these details to the readers.

Consequently, the solution of the heat equation which satisfies the initial condition (4.18) can be represented by (4.19) or, using (4.17), by

$$u(x,\tau) = \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4\tau}}\,u_0(\xi)\,d\xi. \tag{4.20}$$

Therefore, in order to solve the Black-Scholes equation we need to determine what the initial function $u_0(x) = u(x,0)$ corresponds to in the original setting. This initial function is given for $\tau = 0$, and hence, there is the corresponding given function at maturity $t = T$, the payoff function. Due to the transformations in (4.16), the payoff function of the contingent claim stands for the *terminal condition* of the Black-Scholes equation.

If the terminal condition of the Black-Scholes equation is given by $V(S,T) = P(S)$ at maturity $t = T$, then it must be transformed to find the corresponding initial condition, $u(x,0) = u_0(x)$, of the heat equation. By plugging it in (4.20) and, if possible, performing the integration, the solution to the heat equation can be found. Consequently, using the transformations (4.16), the computed solution must be interpreted using the original variables S, t and V involved in the Black-Scholes PDE.

4.2.2 *Closed-Form Solutions of European Call and Put Options*

The well-known Black-Scholes formulae for European call and put options can be derived from the solution represented in (4.20) for the heat equation. In fact, there are many cases where closed-form solutions can be derived by using these integral representations. However, in most cases, the closed-form solutions of the European call and put options are at the centre, and they can be used to derive others. Moreover, due to the *put-call parity* of European options, it is preferable to look for a closed-form solution of either a call or a put option. Using the solution of the heat equation, however, the corresponding closed-form solutions of both, call and put options, can easily be derived.

In order to derive these formulae the payoff functions must be transformed by the change of variables in (4.16) into the corresponding initial conditions for the heat equation. Let us denote by $V_C(S,t)$ and $V_P(S,t)$ the values of the European call and put options, respectively. Then, the payoff functions are

$$V_C(S,T) = \max\{S - K,\ 0\}, \tag{4.21}$$

$$V_P(S,T) = \max\{K - S,\ 0\}, \tag{4.22}$$

where K is the strike price. Using the transformations in (4.16), the payoff of a call option, for instance, is easily converted to

$$\begin{aligned} u_C(x,0) &= \frac{1}{K} e^{\gamma x} V_C(Ke^x, T) \\ &= \frac{1}{K} e^{\gamma x} \max\{Ke^x - K,\ 0\} \\ &= \max\left\{ e^{(\gamma+1)x} - e^{\gamma x},\ 0 \right\}. \end{aligned}$$

Similar calculations can be carried out for the payoff function of a put option. Using the constant $\beta = \gamma + 1$ in (4.15), the corresponding initial conditions at $\tau = 0$ for the heat equation become

$$u_C(x,0) = \max\left\{ e^{\beta x} - e^{\gamma x},\ 0 \right\}, \tag{4.23}$$

$$u_P(x,0) = \max\left\{ e^{\gamma x} - e^{\beta x},\ 0 \right\}. \tag{4.24}$$

Substitution of these functions into the integral solution in (4.20) will then yield the solution $u = u(x, \tau)$ for the transformed dependent variable. For example, substituting the initial condition (4.23) for a European call option into the solution formula gives

$$\begin{aligned} u_C(x,\tau) &= \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4\tau}} \max\left\{ e^{\beta x} - e^{\gamma x},\ 0 \right\} d\xi \\ &= \frac{1}{\sqrt{4\pi\tau}} \int_{0}^{\infty} e^{-\frac{(x-\xi)^2}{4\tau}} \left(e^{\beta x} - e^{\gamma x} \right) d\xi \\ &= I_\beta - I_\gamma, \end{aligned} \tag{4.25}$$

where the last integrals are defined by

$$I_\alpha = \frac{1}{\sqrt{4\pi\tau}} \int_0^\infty e^{-\frac{(x-\xi)^2}{4\tau} + \alpha\xi} \, d\xi \tag{4.26}$$

for each $\alpha = \beta, \gamma$. Calculation, or simplification of the integral I_α can further be carried out by a change of variables as follows:

$$\begin{aligned} I_\alpha &= \frac{1}{\sqrt{4\pi\tau}} \int_0^\infty e^{-\frac{[(x+2\tau\alpha)-\xi]^2}{4\tau} + \alpha\xi} \, e^{\alpha x + \alpha^2\tau} \, d\xi \\ &= e^{\alpha x + \alpha^2 \tau} \int_{-\infty}^{\frac{x+2\tau\alpha}{\sqrt{2\pi}}} \frac{1}{\sqrt{2\pi}} e^{-\eta^2/2} \, d\eta, \end{aligned}$$

where the change of variable $\eta = \frac{x+2\tau\alpha-\xi}{\sqrt{2\pi}}$ is used. Note that the last integral contains the probability density function of the standard normal distribution. Hence, using the distribution function Φ,

$$\Phi(\zeta) = \int_{-\infty}^{\zeta} \phi(\eta)\, d\eta = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\zeta} e^{-\eta^2/2} \, d\eta, \tag{4.27}$$

of the normal distribution with mean zero and variance one, the integral I_α can be written in closed-form as

$$I_\alpha = e^{\alpha x + \alpha^2 \tau}\, \Phi\left(\frac{x + 2\tau\alpha}{\sqrt{2\pi}}\right). \tag{4.28}$$

Therefore, the solution $u_C(x, \tau)$ represented by the difference of two integrals, as in (4.25), is simplified to

$$u_C(x,\tau) = e^{\beta x + \beta^2 \tau}\, \Phi\left(\frac{x + 2\tau\beta}{\sqrt{2\pi}}\right) - e^{\gamma x + \gamma^2 \tau}\, \Phi\left(\frac{x + 2\tau\gamma}{\sqrt{2\pi}}\right). \tag{4.29}$$

Similar calculations carried out for the transformed initial condition $u_P(x, 0)$ in (4.24) for the put option shows that

$$u_P(x,\tau) = e^{\gamma x + \gamma^2 \tau}\, \Phi\left(-\frac{x + 2\tau\gamma}{\sqrt{2\pi}}\right) - e^{\beta x + \beta^2 \tau}\, \Phi\left(-\frac{x + 2\tau\beta}{\sqrt{2\pi}}\right). \tag{4.30}$$

What remains only is that the solutions represented by equations (4.29) and (4.30) must be transformed back in order to write the solutions of the Black-Scholes equation for the European call and put options, respectively. This can be done by using the transformations defined by (4.16) that are accompanied with the notations in (4.15). Let us define

$$d_1 = \frac{x + 2\tau\beta}{\sqrt{2\pi}}, \quad \text{and} \quad d_2 = \frac{x + 2\tau\gamma}{\sqrt{2\pi}}. \tag{4.31}$$

Then, in terms of the original variables $S = Ke^x$ and $t = T - \frac{\tau}{\sigma^2/2}$ of the Black-Scholes equation, d_1 and d_2 can easily be obtained as

$$d_1 = \frac{\log(S/K) + \left(r - \delta + \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}}, \tag{4.32}$$

$$d_2 = \frac{\log(S/K) + \left(r - \delta - \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}}. \tag{4.33}$$

Recall that the constants β and γ were defined by (4.15). For ease of reference, they were

$$\gamma = \frac{1}{2}(\kappa - 1), \quad \text{and} \quad \beta = \frac{1}{2}(\kappa + 1) = \gamma + 1,$$

where $\kappa = \frac{r-\delta}{\sigma^2/2}$. Note also that d_2 can be defined via d_1 as

$$d_2 = d_1 - \sigma\sqrt{T-t}. \qquad (4.34)$$

On the other hand, the transformation used for the dependent variable $V(S,t)$, the value of an option, was

$$V(S,t) = K\,v(x,t), \qquad v(x,\tau) = e^{-\gamma x-(\beta^2+\ell)\tau}\,u(x,\tau),$$

where $\ell = \frac{\delta}{\sigma^2/2}$. Hence, the value of a European call option can be converted back from (4.29) as

$$\begin{aligned} V_C(x,t) &= Ke^{-\gamma x-(\beta^2+\ell)\tau}\left\{e^{\beta x+\beta^2\tau}\,\Phi(d_1) - e^{\gamma x+\gamma^2\tau}\,\Phi(d_2)\right\} \\ &= Ke^{(\beta-\gamma)x-\ell\tau}\Phi(d_1) - Ke^{(\gamma^2-\beta^2-\ell)\tau}\Phi(d_2). \end{aligned}$$

Here, notice that

$$\beta - \gamma = 1 \quad \text{and} \quad \ell\tau = \delta(T-t)$$

so that $Ke^{(\beta-\gamma)x-\ell\tau} = Se^{-\delta(T-t)}$. Moreover,

$$(\gamma^2 - \beta^2 - \ell)\tau = -(\ell+\kappa)\tau = -r(T-t),$$

hence, $Ke^{(\gamma^2-\beta^2-\ell)\tau} = Ke^{-r(T-t)}$. Therefore, replacing the values of the parameters and the independent variables x and τ with the original ones, S and t, gives

$$V_C(S,t) = Se^{-\delta(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \qquad (4.35)$$

which is the celebrated *Black-Scholes formula* for a European call option.

Similar calculations show that the value of a European put option $V_P(S,t)$ can be written as

$$V_P(S,t) = Ke^{-r(T-t)}\Phi(-d_2) - Se^{-\delta(T-t)}\Phi(-d_1). \qquad (4.36)$$

On the other hand, this closed-form formula for the value of a European put option can also be obtained from the *put-call* parity

$$V_P(S,t) = V_C(S,t) - Se^{-\delta(T-t)} + Ke^{-r(T-t)} \qquad (4.37)$$

by using the relation $\Phi(-\zeta) = 1 - \Phi(\zeta)$, which can be proved easily, and is left as an exercise.

Exercise 4.1. Using the definition of Φ show that

$$\Phi(-\zeta) = 1 - \Phi(\zeta) \qquad (4.38)$$

holds for all $\zeta \in \mathbb{R}$.

Exercise 4.2. Show that the closed-form solution $V_{con}(S,t)$ of a *cash-or-nothing* option is given by

$$V_{con}(S,t) = B\, e^{-r(T-t)}\, \Phi(d_2).$$

A cash-or-nothing option has the payoff function

$$V_{con}(S,T) = \begin{cases} B & \text{if } S > K, \\ 0 & \text{if } S \le K. \end{cases}$$

That is, the reward B is paid if the asset price is more than the bet K at maturity T.

Exercise 4.3. Show that the value $V(S,t)$ of a European option can be expressed as the discounted, expectation of the payoff $V(S,T)$ under the risk-neutrality condition: $\mu = r$. In other words, show that

$$\begin{aligned} V(S,t) &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}\left[V(S,T)\right] \\ &= e^{-r(T-t)} \int_0^\infty V(s,T)\, p(s;T,S,t)\, ds, \end{aligned}$$

where $p = p(s;T,S,t)$ is the density function of a lognormal distribution, and it is defined by

$$p(s;T,S,t) = \frac{1}{s\sigma\sqrt{2\pi(T-t)}}\, e^{-\frac{\left[\log(s/S) - \left(r - \delta - \frac{1}{2}\sigma^2\right)(T-t)\right]^2}{2\sigma^2(T-t)}}.$$

This is sometimes called the *transition probability* density.

Although the formulae (4.35) and (4.36) are the closed-form solutions of the Black-Scholes equation for European call and put options, respectively, they still require evaluation of improper integrals. This can be done, however, numerically, in most cases. Hence, truncation of the domains of the integrals is unavoidable for numerical calculations.

Fortunately, a numerous numerical software includes libraries to calculate the *error function*, which is denoted by erf, and is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt. \tag{4.39}$$

In fact, this error function is rather similar to the distribution function of the standard normal distribution. It is easy to write the latter in terms of the former. For,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\xi^2}\, d\xi = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{2}} e^{-t^2}\, dt$$
$$= \frac{1}{\sqrt{\pi}} \left(\int_{-\infty}^{0} e^{-t^2}\, dt + \int_{0}^{x/\sqrt{2}} e^{-t^2}\, dt \right).$$

By using the well-known integral

$$\int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi},$$

as well as the definition (4.39) of the error function, it follows that

$$\Phi(x) = \frac{1}{2} \left\{ 1 + \mathrm{erf}\left(x/\sqrt{2} \right) \right\}. \tag{4.40}$$

In most cases, since the error function is available in MATLAB, calculation of the value $\Phi(x)$ at x will be done by using (4.40). However, there is no explicit form for the calculation of neither $\Phi(x)$ nor $\mathrm{erf}(x)$, but there are some well-known approximations collected in [Abramowitz and Stegun (1972)]. The algorithm given in the exercise below is frequently used and relatively fast besides its accuracy. The implementation of the algorithm is left to the readers.

Exercise 4.4. Write a program that computes the value of the standard normal distribution function $\Phi(x)$ at a given point x. First, by using (4.40) if possible. Second, by using the following procedure.

(1) Let $\gamma = 0.2316419$.
(2) Calculate $z = \dfrac{1}{1+\gamma x}$, for $x \geq 0$.
(3) Let the coefficients be

$$a_1 = 0.319381530, \quad a_2 = -0.356563782, a_3 = 1.781477937,$$
$$a_4 = -1.821255978, a_5 = 1.330274429.$$

(4) Then, the approximate value of $\Phi(x)$ for $x \geq 0$ is

$$\Phi(x) \approx 1 - \phi(x)\, z\, ((((a_5 z + a_4)z + a_3)z + a_2)z + a_1),$$

where $\phi(x)$ is the value of the density function at x. If $x < 0$, then apply $\Phi(x) = 1 - \Phi(-x)$.

Outlook

Within a more general mathematical setting, the risk-neutral expected discounted payoff is linked to the solution of the Black-Scholes equation by the Feynman-Kac theorem. To see this close relation we refer to (Shreve, 2004b, pp. 268–272). For an intuitive and well-illustrated introduction to the relation between partial differential equations and stochastic processes, [Neftci (2000)] seems to be a good reference.

A clear and concise reference for the heat equation and its qualitative properties we refer to [John (1991)], which also includes the Green's functions, *fundamental* solutions, and Fourier transforms.

For similar transformations applied to the Black-Scholes equation in order to get the classical heat equation, readers can refer to [Barraquand and Pudet (1996); Seydel (2002); Wilmott *et al.* (1995)]. In this section, we skipped the transformations of the boundary conditions for options in order to avoid some technical definitions for function spaces in which the solutions are sought. However, readers may have a glance on the literature referenced above, or Chapter 6 in advance, for detailed discussions on some specific options; preferably, European call and put options.

4.3 Hedging Portfolios: The Greeks

This section briefly considers the sensitivity of option price to the underlying parameters, such as asset prices, volatility, interest rates, and so on. Changes in the values of these parameters will certainly change values of the options considerably. A portfolio consisting of options is liable to changes of these parameters and, thus, should be hedged, and the risk it is exposed to should be reduced.

Recall that the portfolio

$$\Pi = -V + \Delta\, S \tag{4.41}$$

was considered in Section 4.1 when deriving the Black-Scholes PDE. This portfolio was made riskless, in other words, it did not change its value by the stochastic fluctuations caused by the asset prices. This was achieved by choosing a Δ number of shares from the underlying asset as

$$\Delta = \frac{\partial V}{\partial S}. \tag{4.42}$$

However, mathematically, this corresponds to the *rate of change* of the option value due to the changes of the underlying asset prices. It is a measure of the *sensitivity* of an option price to the asset prices, which is called by the Greek name: the *delta* of the option.

The *delta* of an option is particularly important in *hedging* portfolios. For instance, an investor likes to have a portfolio that is not affected by the changes in the asset prices. That is, he wishes to manage a portfolio Π whose rate of change

$$\Delta_\Pi = \frac{\partial \Pi}{\partial S} \tag{4.43}$$

with respect to asset prices S is *zero*: $\Delta_\Pi = 0$. This is called the *delta*-hedging of the portfolio.

Suppose that you are in a *short* position in an option with the value V, and you want to protect yourself by taking positions in the asset because of the changes of the underlying asset prices S. Then, you would construct the portfolio in (4.41), where the Δ represents the number of shares of the asset that you need to purchase. Thus, in order to hedge the portfolio with respect to the changes of the prices, you would require the delta of the portfolio to vanish. That is,

$$0 = \frac{\partial \Pi}{\partial S} = -\frac{\partial V}{\partial S} + \Delta.$$

However, this leads to the same Δ defined in (4.42), the delta of the *option* in the portfolio.

A portfolio that has to be hedged may contain several parameters, even if it has only a single option. Of course, a portfolio may have many other financial derivatives and, hence, completely different parameters than that of an option. However, the sensitivities of a portfolio to the parameters of an option are particularly important in hedging. These sensitivities are named after Greek names, and simply called the *Greeks* of a portfolio. The Greeks for a portfolio Π are defined as

Delta: $\Delta_\Pi = \dfrac{\partial \Pi}{\partial S}$, **Gamma:** $\Gamma_\Pi = \dfrac{\partial^2 \Pi}{\partial S^2}$, **Theta:** $\Theta_\Pi = \dfrac{\partial \Pi}{\partial t}$,

Vega: $\mathcal{V}_\Pi = \dfrac{\partial \Pi}{\partial \sigma}$, **Rho:** $\rho_\Pi = \dfrac{\partial \Pi}{\partial r}$.

Remark 4.2. Sometimes, the Greek *theta*, Θ_Π, of a portfolio Π is defined to be

$$\Theta_\Pi = \frac{\partial \Pi}{\partial \tau_m},$$

where $\tau_m = T - t$ is the time to maturity. It is easy to use the chain rule and obtain the relation,

$$\frac{\partial \Pi}{\partial t} = \frac{\partial \Pi}{\partial \tau_m}\frac{\partial \tau_m}{\partial t} = -\frac{\partial \Pi}{\partial \tau_m}.$$

Depending on his preferences, an investor may wish to hedge a portfolio that is liable to the changes in any, or several of the parameters. Thus, knowing the Greeks for the options is particularly important. Thanks to the closed-form solutions of the Black-Scholes equation. By using the closed-form solutions, it is possible to derive the corresponding closed-form representations for the Greeks of the European call and put options.

In fact, due to the *put-call* parity (4.37) for European options it is sufficient to know the Greeks only for call options in closed-form. The corresponding Greeks for put options can then be derived by using the put-call parity. The Black-Scholes closed-form solution for a European call option has been given in (4.35). For ease of reference, it is

$$V_C(S,t) = Se^{-\delta(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \qquad (4.44)$$

where Φ is the distribution function of the standard normal distribution whose density is $\Phi' = \phi$. Differentiating V_C with respect to S gives the delta Greek for the call option, which we will denote it by Δ_C, and

$$\Delta_C = \frac{\partial V_C}{\partial S} = e^{-\delta(T-t)}\Phi(d_1) + Se^{-\delta(T-t)}\phi(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}\phi(d_2)\frac{\partial d_2}{\partial S}.$$

The partial derivatives of d_1 and d_2 can be easily calculated by using their definitions in (4.32) and (4.33), respectively, and noticing the relation

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

in (4.34). Thus,

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}.$$

The delta Δ_C of a call option can further be simplified by the use of the following fact:

$$Se^{-\delta(T-t)}\phi(d_1) - Ke^{-r(T-t)}\phi(d_2) = 0. \qquad (4.45)$$

This can be proved by considering the relation,

$$\log\left(\frac{Se^{-\delta(T-t)}\phi(d_1)}{Ke^{-r(T-t)}\phi(d_2)}\right) = \log(S/K) + (r-\delta)(T-t) + \log\left(\frac{\phi(d_1)}{\phi(d_2)}\right) \quad (4.46)$$

and the definition of the probability density function which is

$$\phi(\xi) = \frac{1}{\sqrt{2\pi}}\, e^{-\xi^2/2}.$$

First, note that the last logarithm in (4.46) is simplified to

$$\log\left(\frac{\phi(d_1)}{\phi(d_2)}\right) = -\frac{1}{2}\left(d_1^2 - d_2^2\right)$$

by use of the definition of ϕ. Second, the difference $d_1^2 - d_2^2$ may be written as

$$\begin{aligned} d_1^2 - d_2^2 &= 2d_1\sigma\sqrt{T-t} - \sigma^2(T-t) \\ &= 2\log(S/K) + 2(r-\delta)(T-t) \end{aligned}$$

by using (4.32) and (4.34). Therefore, (4.46) is simplified as

$$\log\left(\frac{\phi(d_1)}{\phi(d_2)}\right) = -\log(S/K) - (r-\delta)(T-t).$$

Plugging the last expression into (4.46) proves the relation,

$$\log\left(\frac{Se^{-\delta(T-t)}\phi(d_1)}{Ke^{-r(T-t)}\phi(d_2)}\right) = 0,$$

which is equivalent to (4.45).

Hence, summarising the calculations above shows that the delta Δ_C of a European call option is simply

$$\Delta_C = \frac{\partial V_C}{\partial S} = e^{-\delta(T-t)}\Phi(d_1). \quad (4.47)$$

Note also that as $\tau_m = T-t$ approaches zero, d_1 and d_2 defined, respectively, by (4.32) and (4.33) are unbounded from above (tend to ∞) for $S > K$. Similarly, when $S < K$, they are unbounded from below (tend to $-\infty$) as $\tau_m = T-t$ approaches zero. Therefore, from (4.47) it follows that the delta of a call option has the limits:

$$\Delta_C \longrightarrow \begin{cases} 1, & \text{if } S > K \\ 0, & \text{if } S < K \end{cases} \quad \text{as} \quad \tau_m = T - t \to 0. \quad (4.48)$$

On the other hand, from the put-call parity (4.37) of European options, it is easy to calculate the corresponding delta Greek Δ_P for the put option. Differentiating both sides of the parity,

$$V_P(S,t) = V_C(S,t) - Se^{-\delta(T-t)} + Ke^{-r(T-t)},$$

with respect to S yields

$$\Delta_P = \frac{\partial V_P}{\partial S} = \Delta_C - e^{-\delta(T-t)}$$
$$= -e^{-\delta(T-t)}\left\{1 - \Phi(d_1)\right\}.$$

Hence, using the relation $\Phi(-\zeta) = 1 - \Phi(\zeta)$ the delta Δ_P of a European put option is

$$\Delta_P = \frac{\partial V_P}{\partial S} = -e^{-\delta(T-t)}\Phi(-d_1). \tag{4.49}$$

Furthermore, a similar argument as above shows that

$$\Delta_P \longrightarrow \begin{cases} 0, & \text{if } S > K \\ -1, & \text{if } S < K \end{cases} \quad \text{as} \quad \tau_m = T - t \to 0. \tag{4.50}$$

The closed-form representations for the other Greeks for European options can be calculated similarly. The following formulae are left as exercise to the readers. Let us define

$$\tau_m = T - t, \quad \text{and} \quad \eta = \begin{cases} 1, & \text{if } V \text{ is a European call} \\ -1, & \text{if } V \text{ is a European put} \end{cases} \tag{4.51}$$

for simplicity. Then, all the Greeks for a European option, no matter if it is a call or a put, are given by the following closed-form formulae.

Delta: $\Delta := \dfrac{\partial V}{\partial S}$,

$$\Delta = \eta e^{-\delta\tau_m}\,\Phi(\eta d_1), \tag{4.52}$$

Gamma: $\Gamma := \dfrac{\partial^2 V}{\partial S^2}$,

$$\Gamma = e^{-\delta\tau_m}\frac{1}{S\sigma\sqrt{\tau_m}}\,\phi(d_1), \tag{4.53}$$

Theta: $\Theta := \dfrac{\partial V}{\partial \tau_m}$,

$$\Theta = -\eta \left\{ \delta S e^{-\delta\tau_m} \Phi(\eta d_1) - rKe^{-r\tau_m} \Phi(\eta d_2) \right\} - e^{-\delta\tau_m} \frac{\sigma S}{2\sqrt{\tau_m}} \phi(d_1), \tag{4.54}$$

Vega: $\mathcal{V} := \dfrac{\partial V}{\partial \sigma}$,

$$\mathcal{V} = \sqrt{\tau_m} S e^{-\delta\tau_m} \phi(d_1), \tag{4.55}$$

Rho: $\rho := \dfrac{\partial V}{\partial r}$,

$$\rho = \eta \tau_m K e^{-\delta\tau_m} \Phi(\eta d_2). \tag{4.56}$$

There are, of course, other parameters in the Black-Scholes formulae, such as the strike price K and the dividend yield δ. However, they do not have Greek names, although they can effectively be used in hedging portfolios. For instance, the sensitivity of a portfolio may depend on the changes of the underlying strike prices. If this is to be hedged, then the sensitivity of the option value V to strike price K may be represented by the partial derivative $\frac{\partial V}{\partial K}$. The following exercise that considers this derivative, and the derivative with respect to the dividend yield, is helpful in this respect.

Exercise 4.5. For European call and put options, show that

$$\frac{\partial V}{\partial K} = -\eta e^{-r\tau_m} \Phi(\eta d_2)$$

and

$$\frac{\partial V}{\partial \delta} = -\eta \tau_m S e^{-\delta\tau_m} \Phi(\eta d_1)$$

hold, where η is 1 for call, and -1 for put options. Explain also how to use these sensitivity parameters in hedging.

Fig. 4.1 shows the calculation of the exact formulae of the prices of a European call as well as a put option. It also provides the values of the deltas corresponding to those options. In Fig. 4.2 we show the corresponding values versus the asset price S.

CallPut_Delta.m
```
function [C, Cdelta, P, Pdelta] = CallPut_Delta(S,K,r,sigma,tau,div)
% tau = time to expiry (T-t)

if nargin < 6
    div = 0.0;
end
if tau > 0
    d1 = (log(S/K) + (r + 0.5*sigma^2)*(tau)*ones(size(S))) / (sigma*sqrt(tau));
    d2 = d1 - sigma*sqrt(tau);
    N1 = 0.5*(1+erf(d1/sqrt(2))); N2 = 0.5*(1+erf(d2/sqrt(2)));
    C = exp(-div*tau) * S.*N1-K*exp(-r*(tau))*N2; Cdelta = exp(-div*tau) * N1;
    P = C + K*exp(-r*tau) - exp(-div*tau)*S; Pdelta = Cdelta - exp(-div*tau);
else
    C = max(S-K,0); Cdelta =  0.5*(sign(S-K) + 1);
    P = max(K-S,0); Pdelta = Cdelta - 1;
end
```

Fig. 4.1 The use of closed-form solution of the Black-Scholes equation, and the *delta* hedging parameter

As time to maturity approaches zero, the values of the options become closer to the corresponding payoff functions. On the other hand, the deltas of the options have a jump at the strike price ($K = 2$) when the maturity ($T = 5$) is reached.

The following example illustrates the delta Greeks for a portfolio of options. However, for simplicity, the options considered have the same underlying asset and the strike prices.

Example 4.1. Consider a portfolio Π consisting of a European call and a put option. Suppose the strike prices are the same: $K = 2$ for each. Let the interest rate r be $r = 0.03$ and the volatility σ of the underlying asset be $\sigma = 0.25$. Furthermore, assume that time to maturity is also the same: $T = 5$ for both options. A MATLAB script is shown in Fig. 4.3, which uses the function in Fig. 4.1.

The graphs of the values of the options and the portfolio are depicted in Fig. 4.4. First row in the figure shows the values, respectively, of the options and of the portfolio. The second row contains the graphs of the corresponding deltas of the options and the portfolio.

Notice that the delta of the portfolio, $\Delta_\Pi = \Delta_C + \Delta_P$, shown in Fig. 4.4 is *zero* for a nonzero value of the asset price. Indeed, adding some number of shares of the asset to the portfolio makes the portfolio riskless. Not a surprise! This number is $\Delta = -(\Delta_C + \Delta_P)$, which is obtained immediately

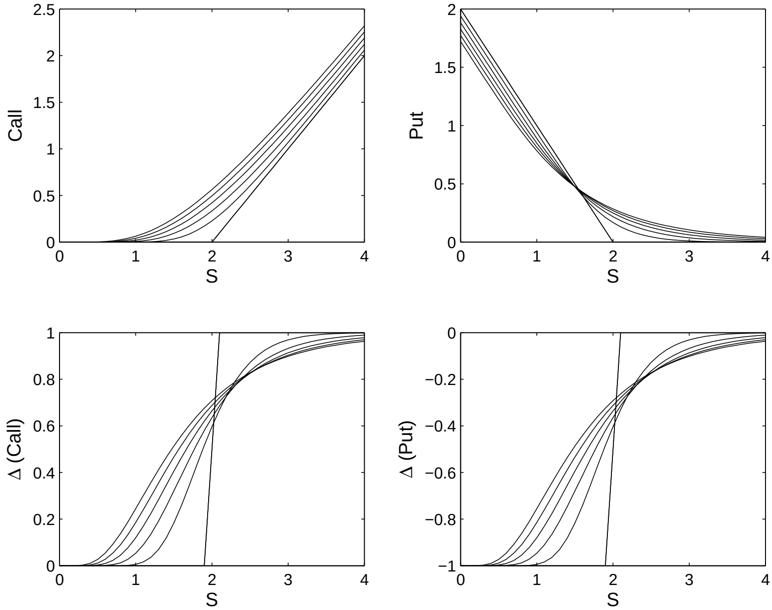

Fig. 4.2 Solutions obtained from the closed-form formulae of the Black-Scholes equation for the European option with varying time to maturity

by constructing the portfolio Π that includes a call and a put option, and Δ number of shares of the asset.

Outlook

In finance, a *hedge* is an investment that is taken out specifically to reduce or cancel out the risk in another investment. We refer to [Higham (2004); Joshi (2004)] for practical applications and brief discussions of the Greeks for hedging the risks associated with having a portfolio of derivatives. For more information on the Greeks, see also [Hull (2000); Kwok (1998)].

4.4 Implied Volatility

The Black-Scholes model has some restrictions. A constant risk-free interest rate r and a constant volatility σ do not seem to be realistic. After all, the derivation of the Black-Scholes equation, and hence, the closed-form solutions for some options, assume a continuously trading strategy which sumOfCallPut_Eg.m

```
% sumOfCallPut_Eg
clear all, close all
S = 0:0.1:4; K = 2; r = 0.03; sigma = 0.25; T = 5;

[c, cd, p, pd] = CallPut_Delta(S, K, r, sigma, T);
subplot(2,2,1), plot(S, c), hold on, plot(S, p, 'r--')
xlabel('S','Fontsize',12), ylabel('V','Fontsize',12); legend('V_C', 'V_P');
subplot(2,2,2), plot(S, c+p), xlabel('S','Fontsize',12)
ylabel('V_\Pi = V_C + V_P','Fontsize',12)
subplot(2,2,3), plot(S, cd), hold on, plot(S, pd, 'r--')
xlabel('S','Fontsize',12), ylabel('\Delta','Fontsize',12)
legend('\Delta_C', '\Delta_P');
subplot(2,2,4), plot(S, cd+pd), hold on, plot([0 4], [0 0], 'g-.')
xlabel('S','Fontsize',12), ylabel('\Delta_\Pi = \Delta_C + \Delta_P','Fontsize',12);
print -r900 -deps '../figures/sumOfCallPut_Eg'
```

Fig. 4.3 Value of a portfolio consisting of a call and a put option

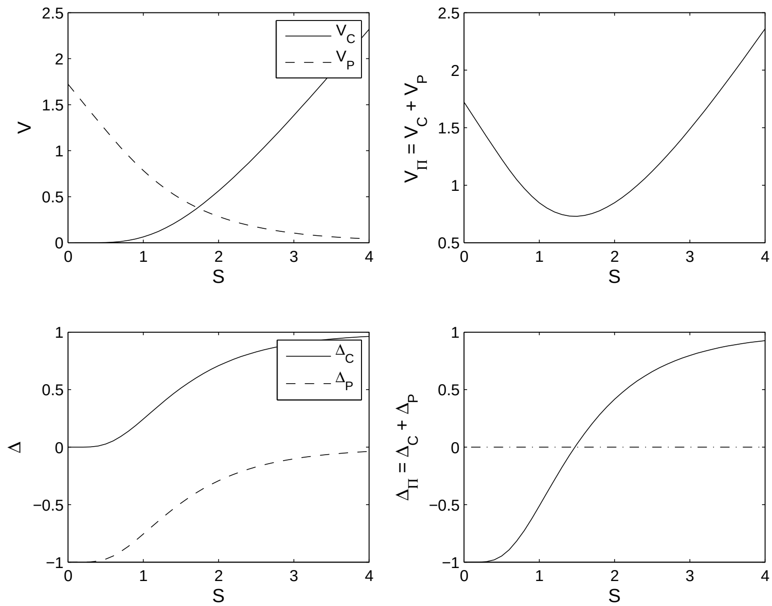

Fig. 4.4 Values and deltas of a portfolio consisting of a call and a put option

is not feasible in the market in order to hedge the portfolio that has been constructed. This is simply due to the changing number of shares $\Delta = \frac{\partial V}{\partial S}$ continuously in time. Furthermore, the model does not assume the presence of transaction costs.

In fact, you may possibly add more drawbacks to these deficiencies of the Black-Scholes setting. Despite these restrictions and deficiencies, however, the Black-Scholes model has become so popular and was awarded with a Nobel Prize! This is mainly due to the existence of a concrete, closed-form solutions to some options whose variants are traded at the market. Beyond professionals and experts in mathematical finance, a closed-form solution means a lot for academics and, especially, for practitioners, the actual players of the market. The Black-Scholes formulae have also the benefit of being very easy to use and understand: given the parameters that are involved in the Black-Scholes formulae, you may directly compute the price of the options.

The only trouble seems to be the estimation of the parameters, especially the estimation of the volatility σ from historical data. The estimation of μ may be easier than that of σ, even more, for pricing purposes μ disappears, and it is replaced by the risk-free interest rate r. It may be easier to estimate r for short term periods, and it may be a part of the option contract.

As it turns out, the empirical performance of the Black-Scholes formulae is reasonably good. For options with a strike price that is not too far from the current price of the underlying asset price, the Black-Scholes formulae anticipates the *observed prices* at the market rather well. However, for options that are deep *out of the money*, the observed prices are, in most cases, higher than the ones suggested by the formulae. This might be partly because of the difficulty of estimation of the parameters r and especially σ, which are assumed to be constant in the Black-Scholes setting. It does not appear to be the case that the volatility is constant over the life time of an option.

However, the option prices are quoted in the market so that the market implicitly knows or presumes the volatility. The volatility $\hat{\sigma}$ derived from these quoted prices for an option is called the *implied volatility.* Due to the closed-form solutions, the Black-Scholes setting is a good candidate model to estimate the volatility implied by the market.

If $\hat{V}$ denotes the quoted prices of an option, then the implied volatility $\hat{\sigma}$ is the value of the σ for which

$$\hat{V} = V(S, t, T, K, r, \sigma), \tag{4.57}$$

where $V = V(S, t, T, K, r, \sigma)$ denotes the model value of the option, which is mostly referred to as the *theoretical price*. Although the underlying model can be any challenging one, the use of the Black-Scholes formulae is easy and illustrative. Thus, it follows from (4.57) that the implied volatility $\hat{\sigma}$ is any of the zeros of the function

$$f(\sigma) = \hat{V} - V(S, t, T, K, r, \sigma), \tag{4.58}$$

which represents the difference between the observed and the theoretical prices. In other words, the roots of the equation $f(\sigma) = 0$ are sought. Indeed, a similar root finding problem was discussed in Example 2.8 on page 67. The premium of a pay-later contract was found to be the root of a certain function.

Example 4.2. This example presents the root finding problem for the implied volatility. The data shown in Table 4.1 are totally artificial and assumed to be observed for 9 call options in the market. Each row in the table shows the corresponding values of an option with the strike price K.

Table 4.1 Observed data

Option #	Strike price K	Call Option V_C
1	1.00	1.2098
2	1.25	1.0280
3	1.50	0.8677
4	1.75	0.7298
5	2.00	0.6132
6	2.25	0.5157
7	2.50	0.4349
8	2.75	0.3682
9	3.00	0.3134

Assume that the current price of the underlying asset of the options is $S = 2.00$ and the interest rate is $r = 3\%$. Also, suppose that the time to maturity of all options considered is the same: $T = 5$. These values and the observed data are also shown in Fig. 4.5.

The values of the implied volatility for the call options in Table 4.1 are calculated as

$$\hat{\sigma}_1 = 0.3507,\ \hat{\sigma}_2 = 0.3153,\ \hat{\sigma}_3 = 0.2973,\ \hat{\sigma}_4 = 0.2878,\ \hat{\sigma}_5 = 0.2826,$$
$$\hat{\sigma}_6 = 0.2798,\ \hat{\sigma}_7 = 0.2784,\ \hat{\sigma}_8 = 0.2780,\ \hat{\sigma}_9 = 0.2783,$$

respectively. The curve corresponding to the values of the implied volatility is depicted in Fig. 4.6.

impliedVola.m

```
% impliedVola
clear all, close all
K = [1.00 1.25 1.50 1.75 2.00 2.25 2.50 2.75 3.00];
Obs = [1.2098 1.0280 0.8677 0.7298 0.6132 0.5157 0.4349 0.3682 0.3134];
S = 2; r = 0.03; T = 5;
for i = 1:length(K)
    [implVola(i), value(i)] = fsolve(@(x) ...
        Obs(i) - CallPut_Delta(S, K(i), r, x, T), 0.3);
end
[implVola', value']
plot(K,implVola,'-o', S*ones(1,length(K)), implVola, 'r--'), hold on
text(S, 0.32, 'Current Asset Price');
xlabel('Strike Price','FontSize',12), ylabel('Implied Volatility','FontSize',12)
print -r900 -deps '../figures/impliedVola'
```

Fig. 4.5 Implied volatility calculation

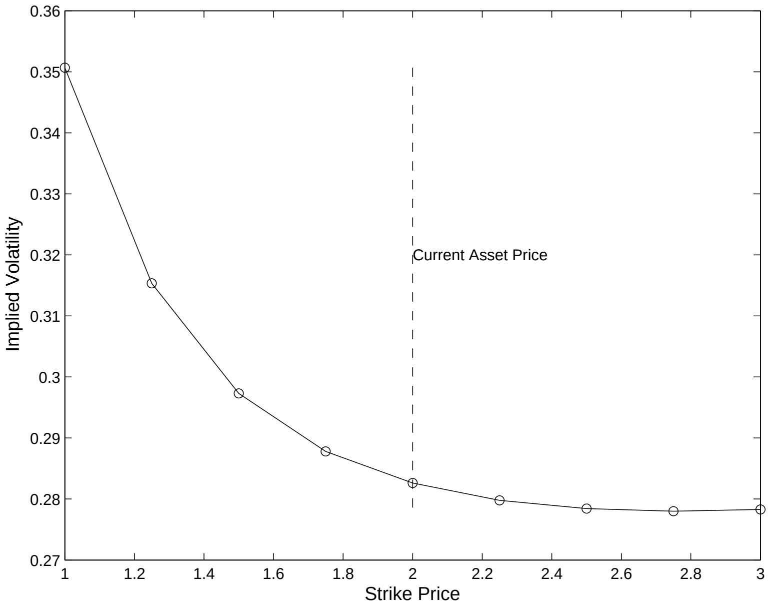

Fig. 4.6 Implied volatility due to data in Table 4.1

In fact, actual data from a market is expected to yield a similar graph for the implied volatility as in Fig. 4.6, which is called a *volatility smile* due to its shape. Of course, other shapes, such as *frowns* are possible for different options. However, it shows that the volatility is not constant at all, unlike the assumption in the Black-Scholes closed-form solutions.

Outlook

The changes of the volatility during the life time of the options cause hedging costs, hence, the volatility implied by the market has to be estimated by the traders. There are alternative models to the Black-Scholes model under which options are priced and used to estimate the implied volatility. See [Joshi (2004); Hull (2000); Kwok (1998)] for those alternative models, some of which assume a *stochastic* volatility.