

Chapter 1

Introduction

Since the Schrödinger equation for a hydrogenic atom or ion in a homogeneous electric field is separable in parabolic coordinates, the system is more easily accessible to an accurate theoretical treatment than other atoms or ions in electric fields. Furthermore, a Rydberg state of any atom or ion, i.e., a state in which one electron is in a highly excited state and the nucleus is shielded by the core of the other electrons, resembles a hydrogenic state in the sense that a single electron moves far away from an ionic core and does not penetrate into the core unless the magnetic quantum number m is small enough. Such an atom or ion is thus approximately hydrogenic. In the interstellar space there occur very highly excited Rydberg atoms, with values of the principal quantum number of the valence electron of the order of magnitude of one hundred. They are therefore closely hydrogenic and may be exposed to strong electric fields. With the ever increasing accuracy and sophistication of experiments the need for accurate analytical methods of analysis will increase, and the treatment of the Stark effect in hydrogenic atoms or ions may serve as a model problem for the treatment of Rydberg states. Thus, apart from the intrinsic interest of its own, the Stark effect problem for a hydrogenic atom or ion plays the role of a model problem, from which one can obtain information about the properties of Rydberg states. For comprehensive reviews of the properties of Rydberg states, see Gallagher (1988, 1994).

The quasistationary nature of the Stark resonances, due to the fact that the energy eigenvalue spectrum of one of the coupled differential

equations, resulting from the separation of the Schrödinger equation in parabolic coordinates, is continuous, with wave functions extending to infinity, makes the hydrogenic Stark effect problem intricate in spite of its seeming simplicity.

Brief review of different aspects studied and various methods used

Precise experimental results on Stark levels in atomic hydrogen have been reported by many authors, and a great number of theoretical papers have appeared in which different methods are used for the study of the Stark resonances. Semi-classical methods are adequate for highly excited states and have been used by several authors. The Stark effect for levels well below the top of the barrier was treated with the aid of Carlini (JWKB) technique, although rather crudely, already in the early days of quantum mechanics. Later there appeared improved treatments of that kind, in which also levels near the top of the barrier were considered. Another important method for determination of the positions of the Stark levels is the Rayleigh–Schrödinger perturbation theory, but it is not applicable in low orders for highly excited states and strong fields. However, perturbation theory of the Stark effect in atomic hydrogen has been made tractable to arbitrarily high orders by a restatement of the perturbation theory formulas that allows the perturbation series to be obtained from recursive relations run on a computer. The perturbation series is not convergent but asymptotic, and Borel summation together with the use of Padé approximants greatly accelerates the approach towards accurate energy values and is an efficient tool for obtaining accurate results for the Stark effect. It will, however, not be discussed in this book, since we restrict ourselves to the use of the phase-integral approximation generated from an appropriately chosen base function. This approach, which is capable of yielding *explicit analytical* formulas, is for the first time applied to the Stark effect in a systematic way in this book.

There occur in the literature different ways of defining the positions and the half-widths of the resonance levels. One finds definitions based on considerations of the probability amplitude, or based

on the rapid variation of the phase shift with energy, as well as definitions relating the real part of a complex energy eigenvalue to the position of the Stark level and its imaginary part to the width of the level. Different aspects of the last mentioned approach, i.e., the use of complex energy eigenvalues, and comparisons with different formulas for the half-width at low fields can be found in Yamabe, Tachibana and Silverstone (1977). For narrow levels the differing definitions yield essentially the same numerical results. However, for broad levels slightly below the top of the potential barrier and for autoionizing levels above the top of the barrier the differing definitions yield appreciable differences in the results. The broad levels are not of Lorentzian shape but are highly asymmetric, and hence the concept of half-width loses to some extent its precise meaning. As a consequence of this fact it is obvious that the methods based on complex energies are inadequate for broad levels.

The advent of tunable lasers created a radically new situation as to the possibility for selective excitation of high Rydberg states and for making precise measurements on their properties. Highly excited atoms are very sensitive to external fields, and currently used field ionization methods are very powerful for detecting Rydberg states. As a consequence of these circumstances, such an old problem as the Stark effect in atomic hydrogen attracted a renewed interest.

Brief account of the background of this book

Papers concerning the Stark effect of a hydrogenic atom in a homogeneous electric field appeared already in the early days of quantum mechanics. On the basis of the matrix mechanics invented by Heisenberg (1925), Born and Jordan (1925), Dirac (1925), and Born, Heisenberg and Jordan (1926), Pauli (1926) obtained for the spectrum of the hydrogen atom and for the Stark effect of that atom results that agreed with experimental data. In connection with his development of wave mechanics Schrödinger (1926) made an application to the Stark effect in atomic hydrogen. He separated the time-independent Schrödinger equation for the problem in question in parabolic coordinates and used first-order perturbation theory to treat the two resulting ordinary differential equations.

In the paper where Wentzel (1926) presented his rediscovery of the Carlini (JWKB) approximation, he applied this approximation to the treatment of the two ordinary differential equations just mentioned. Waller (1926) treated instead these two ordinary differential equations by expressing their solutions with the use of series expansions in powers of the field strength. By successive approximations he obtained a second-order formula for the energy levels of a hydrogen-like ion in a homogeneous electric field. Only slightly later Epstein (1926) also presented a theory for the Stark effect in a hydrogen-like ion, based on the time-independent Schrödinger equation, which he, after separation in parabolic coordinates, treated by successive approximations and obtained results up to the second order in the electric field strength. Van Vleck (1926) used the formula for the energy levels of a hydrogen atom in an electric field, obtained independently by Waller and Epstein, to calculate the dielectric constant of atomic hydrogen. The Stark effect in hydrogenic atoms or ions was thus treated by means of quantum mechanics very soon after its discovery.

Oppenheimer (1928) developed a method for computing the probabilities for transitions between states of the same energy, represented by almost orthogonal eigenfunctions, and applied the resulting formula to treat the ionization of hydrogen atoms in a homogeneous electric field. Somewhat later Lanczos (1930a, 1930b) treated the Stark effect for a hydrogen atom in a strong electric field by deriving an approximate asymptotic solution for the one of the previously mentioned ordinary differential equations that has a continuous energy spectrum. He pointed out that the Stark levels are not sharp but have a finite width which he discussed. Lanczos (1930c) also improved the method of asymptotic treatment of the Stark effect for a hydrogen atom, with the magnetic quantum number m equal to zero, in a strong electric field. The asymptotic method he used is closely related to the first order of the Carlini (JWKB) approximation along with Jeffreys' (1925) connection formulas for that approximation, the one-directional validity of which is, however, not discussed. For the positions of the Stark levels Lanczos arrived at a quantization condition of the Bohr–Sommerfeld type, which he expressed in terms of complete elliptic integrals of the first and second kind. He also

discussed the breakdown of perturbation treatments for strong electric fields. On the basis partly of the time-independent and partly of the time-dependent Schrödinger equation, Lanczos (1931) discussed, although in a not quite clear way, the weakening of the intensities of the spectral lines and the ionization of atomic hydrogen in strong electric fields. For the disintegration constant he obtained an expression in terms of complete elliptic integrals of the first and second kind.

Publications with relevance to this book

We shall consider mainly publications in which asymptotic methods are used, but we also mention numerical methods, since we use numerical results for comparison with our phase-integral results. For a general review of the field we refer to Bethe and Salpeter (1957), Ryde (1976), Bayfield (1979), Koch (1981), Gallas, Leuchs, Walther and Figger (1985), Lisitsa (1987) and Gallagher (1988, 1994).

Rice and Good (1962) calculated the positions of the Stark levels of atomic hydrogen in a homogeneous electric field by using the Carlini (JWKB) approximation combined with comparison equation technique for the treatment of the time-independent Schrödinger equation separated in parabolic coordinates. They considered in particular the case when the energy lies close to the top of the barrier. For the positions of the energy levels the authors obtained quantization conditions expressed in terms of complete elliptic integrals of the first and second kind. Furthermore, they improved Lanczos' (1930b, 1930c, 1931) estimate of the dependence of the lifetime on field ionization and calculated also the half-width of the Stark levels. Thus they obtained formulas for the lifetime and the half-width of the Stark levels in terms of complete elliptic integrals of the first and second kind. Due to a need for explicit values of the ionization probabilities up to very high energy levels, several electric field ionization probabilities for a hydrogen atom in an electric field were calculated by Bailey, Hiskes and Riviere (1965) by the methods of Lanczos (1931) and Rice and Good (1962). The results were presented graphically and in a table. Guschina and Nikulin (1975) calculated the resonance energy and the

decay probability for a particular quasistationary state of a hydrogen atom in a homogeneous electric field by numerical integration of the two coupled differential equations obtained by separation of the time-independent Schrödinger equation in parabolic coordinates. The values obtained for the resonance energy agree perfectly with values obtained by Rayleigh–Schrödinger perturbation theory up to the fourth power in the electric field strength and rather well with values obtained by Bailey, Hiskes and Riviere (1965). The values of the decay probability agree rather well with those obtained by Bailey, Hiskes and Riviere (1965). To solve the two differential equations, obtained by separation in parabolic coordinates of the Schrödinger equation for a hydrogenic atom in a homogeneous electric field, Bekenstein and Krieger (1969) used the Carlini (JWKB) approximation and derived quantization conditions in the fifth order of that approximation. From these quantization conditions the authors obtained for the positions of the Stark energy levels a series up to the fourth power of the electric field strength. This series agrees, for those states for which comparison could be made, with the corresponding series obtained by perturbation theory. The general conclusion of Bekenstein and Krieger seems to be that the use of the Carlini (JWKB) approximation is superior to the use of perturbation theory for all Stark levels of a hydrogenic atom. Alliluev and Malkin (1974) derived the perturbation series for the Stark effect of atomic hydrogen up to the fourth power of the electric field strength. They find that their result is in complete agreement with the results of previous authors up to the third-order correction. Although they find a disagreement in their fourth-order correction with the result obtained by Bekenstein and Krieger (1969), they express the opinion that the correct Carlini (JWKB) approximation and perturbation theory lead to identical results in the case of weak electric fields. Furthermore, Alliluev and Malkin (1974) quote Basu's (1934) result for the fourth-order correction, which is published in a journal that is almost inaccessible, and point out that his fourth-order formula contains errors. Herrick (1976) confirms on page 3534 that Alliluev and Malkin (1974) corrected errors in both the Basu (1934) formula and in the WKB expansion of Bekenstein and Krieger (1969). Yamabe, Tachibana and Silverstone (1977) developed the theory of

the ionization of a hydrogen atom in an electric field analytically and corrected Oppenheimer's (1928) formula for the ionization in a weak electric field. As a general conclusion of this and other results, the authors state that the field ionization of hydrogen is unsuspectedly insidious, having left a legacy of errors. Drukarev (1978) calculated in the quasiclassical approximation the energies and widths of energy levels of a hydrogen atom in a homogeneous electric field. Later Drukarev (1982) considered the Stark effect when the energy level lies at the top of the barrier. Gallas, Walther and Werner (1982a) used the first-order Carlini (JWKB) approximation to treat the Stark effect in a hydrogen atom for arbitrary values of the magnetic quantum number m . In the two coupled, ordinary differential equations, obtained after separation in parabolic coordinates, these authors erroneously replaced $m^2 - 1$ by m^2 and obtained differential equations that are not correct. Many other authors have also made this serious mistake, and therefore it is important to emphasize that the replacement of $m^2 - 1$ by m^2 , or $l(l + 1)$ by $(l + 1/2)^2$ in a radial problem, is not to be made in the *differential equations* but only in the *first-order Carlini (JWKB) approximation*, and that this replacement in the higher-order corrections does not give a correct result. For the positions of the Stark levels well below the top of the barrier the authors obtained quantization conditions expressed in terms of complete elliptic integrals of the first, second and third kind, which they extended in an unsatisfactory way to energy levels above the top of the barrier. Somewhat later Gallas, Walther and Werner (1982b) used the first-order Carlini (JWKB) approximation and handled the three-turning-point problem also when the energy may lie close to the top of the barrier, but they made the same mistake as in their previous paper (1982a). For the ionization rate of a hydrogenic atom or ion in an electric field they obtained a simple formula, expressed in terms of complete elliptic integrals of the first, second and third kind, which they found to be in excellent agreement with results obtained from numerically exact calculations, and which for energies well below the top of the barrier agrees with the formula obtained by Rice and Good (1962). On the basis of, on the one hand the first-order Carlini (JWKB) approximation combined with comparison equation results, and on the other hand a

purely numerical method, Farrelly and Reinhardt (1983) performed calculations of complex energy eigenvalues for a hydrogen atom in a homogeneous electric field. They demonstrated the efficiency and remarkable accuracy of the Carlini (JWKB) approximation already in the first-order approximation and pointed out that previous discrepancies between results obtained by the use of that approximation and accurate numerical results, which had usually been attributed to the break-down of the approximation, are rather due to a failure to use the approximation in a correct and uniform way. The authors concluded that an appropriate approach based on the approximation in question is an efficient and highly accurate method for the calculation of complex energy eigenvalues for the Stark problem. Korsch and Möhlenkamp (1983) performed, independently of Farrelly and Reinhardt (1983), a similar investigation. By means of comparison equation technique Kolosov (1983) determined the energy and the ionization probability of a hydrogen atom in a homogeneous electric field, when the energies of the differential equation describing tunneling through the potential barrier lie near the top of the barrier. Formulas for the energy and the ionization probability in some previous papers are characterized as either erroneous or too complicated.

Though the exact solution of the Stark effect problem for hydrogenic atoms or ions can in principle be obtained by numerical integration of the two ordinary differential equations, resulting from the separation of the three-dimensional Schrödinger equation in parabolic coordinates, exact calculations encounter computational difficulties and have hence been rather few [see, however, Alexander (1969) and Hirschfelder and Curtiss (1971)] until the extensive calculations of positions and widths of Stark levels by Damburg and Kolosov (1976a, 1976b, 1977, 1978a, 1978b, 1979, 1980, 1981, 1982) and by Kolosov (1983, 1987) began to appear. A numerical method for calculating normalized wave functions and absolute values for the density of oscillator strengths in the photoabsorption spectrum of hydrogenic atoms or ions in the presence of a homogeneous electric field has been presented by Luc-Koenig and Bachelier (1980a,b).

Treatment in this book

In the present book we treat the Stark effect for a hydrogenic atom or ion in a homogeneous electric field with the use of the phase-integral approximation generated from a conveniently chosen base function; see for a detailed presentation of that approximation Chapter 1 in Fröman and Fröman (1996) and for a summary Section 4.1 in the present book. We shall give a non-relativistic treatment of the problem in which electron spin, fine structure and hyperfine structure are not taken into account. Furthermore, we assume that the time for electric field ionization due to the Stark effect is much smaller than the time for emission of a photon from the state in question. Some previous authors have combined the Carlini (JWKB) approximation and comparison equation technique; see for instance Rice and Good (1962), Bailey, Hiskes and Riviere (1965) and Harmin (1981). We do not proceed in a corresponding way, since comparison equation technique has already been used to obtain the general, analytic, arbitrary-order phase-integral formulas on which we base our treatment of the Stark effect; see Fröman and Fröman (1996). In particular we use an arbitrary-order phase-integral formula for barrier transmission (Fröman and Fröman 2002), which allows the energy to lie close to and even above the top of the barrier. Our treatment is thus in several respects more satisfactory and more straightforward than previous asymptotic treatments. Finally we arrive at phase-integral formulas, expressed in terms of complete elliptic integrals of the first, second and third kind, for Stark level profiles, positions and half-widths.

We share the opinion expressed by Farrelly and Reinhardt (1983) that discrepancies between Stark effect results obtained by the use of the Carlini (JWKB) approximation and by accurate numerical calculations cannot be attributed to the break-down of the approximation, but are due to a failure to use the approximation in a correct way. An appropriate approach based on the phase-integral approximation of arbitrary order generated from an appropriately chosen base function is a still more efficient and often highly accurate method for the treatment of several problems, not only in quantum mechanics, but in various fields of theoretical physics. With

great success it has for instance been used in the study of black-hole normal modes (Fröman, Fröman, Andersson and Hökback 1992; Andersson, Araújo and Schutz 1993) and in the study of cosmological perturbations during inflation (Rojas and Villalba 2007). See also Athavan *et al.* (2001a–c), which concerns the two-center Coulomb problem, and where the reason for the possibility of obtaining their accurate results is the presence of the unspecified base function from which the phase-integral approximation is generated. Such accurate results cannot be obtained by means of the Carlini (JWKB) approximation, since there is no unspecified base function in that approximation.

We shall now illustrate the accuracy of the energy values obtained by means of our phase-integral formulas. For 198 different Stark states of a hydrogen atom, with either different quantum numbers or the same quantum numbers but different electric field strengths, we present in the tables in Chapter 8 values of the energy and the half-width that have been calculated by means of the phase-integral approximation generated from an appropriate base function as well as by other methods. We emphasize that all results there have been obtained by neglecting fine structure corrections. Compared to the best energy values obtained by other methods, the optimum phase-integral energy values are for these states judged to be at least as accurate in more than half of the cases; see Fig. 1.1. The phase-integral formulas can sometimes give results of surprisingly great accuracy. Compared to the numerically obtained results, the phase-integral results in Chapter 8 can contain up to seven more digits for the energy eigenvalues. A more detailed presentation of the accuracy of the phase-integral energies versus the accuracy of the energies obtained by other methods is given in Fig. 1.1.

For large field strengths (thin barriers) the phase-integral method gives usually better results than for small field strengths (thick barriers). The phase-integral method is therefore an important complement to the numerical methods, which are in general less accurate for large field strengths than for small field strengths. For very thick barriers the numerical methods do not give good values of the half-widths, and for extremely thick barriers they may sometimes only give upper limits for the half-widths, while the phase-integral method

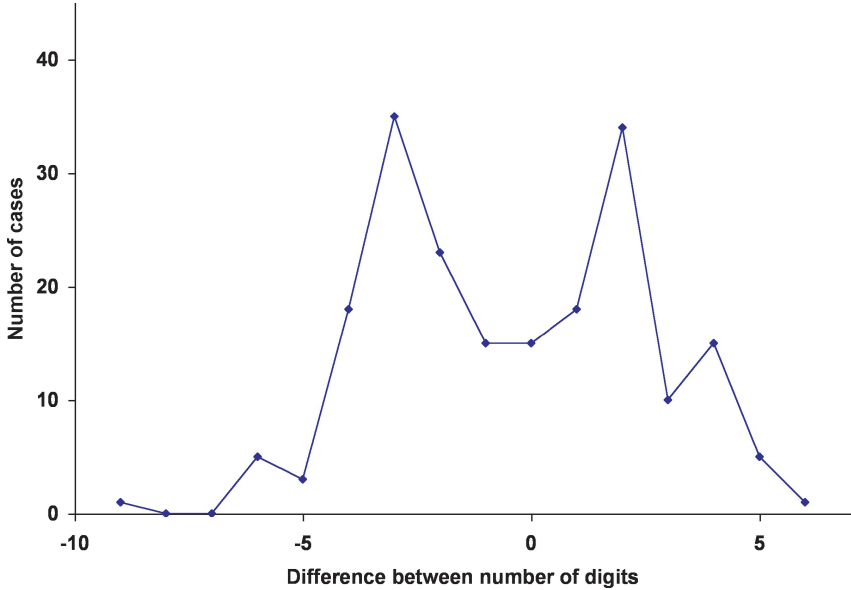


Fig. 1.1. For the 198 cases in the tables in Chapter 8 we have on the horizontal axis in this figure plotted the difference between the number of correct digits in the optimum phase-integral energy value, obtained with $\tilde{\phi}$ included, and the number of correct digits in the best energy value obtained by other methods, while on the vertical axis we have plotted the number of cases corresponding to different values of this difference. Recalling a property of asymptotic series, we have considered the error of the optimum phase-integral energy values E to be of the order of the smallest difference between the E -values for two consecutive optimum orders of the phase-integral approximation. When the phase-integral values of E are judged to be less accurate than the other values of E , we have in general considered all digits in the best of the other values to be correct. Since there is certainly an unknown number of exceptions from this assumption, the part of the figure that lies to the left of the origin underestimates the accuracy of the phase-integral method. According to this figure there are 98 cases in which the energy values obtained by the phase-integral method are at least as accurate as those obtained by other methods, but because of what has just been said, we believe that the energy values are obtained at least as accurately by the phase-integral method as by other methods in more than half of the cases.

gives rather accurate values; see Tables 8.10c, 8.10f and 8.10l in Chapter 8. For large values of the quantum number n_2 the phase-integral method gives often more accurate results than for small values of n_2 .

For some time there was a puzzling discrepancy between a phase-integral result obtained by us and a corresponding result obtained numerically by Damburg and Kolosov, and quoted by Silverstone and Koch (1979), for the position of the Stark level of a hydrogen atom with the quantum numbers $m = 0$, $n_1 = 0$, $n_2 = 29$ and $n = |m| + 1 + n_1 + n_2 = 30$ in a homogeneous electric field of the strength 800 V/cm. This is a state with the first-order barrier penetration phase-integral K (see Section 4.3) close to unity. It is thus situated not very far below the top of the barrier. Anders Hökback, who was research engineer at the Department of Theoretical Physics, University of Uppsala, found the explanation of this discrepancy, which turned out to be due to the seemingly innocent approximation of considering L , K and $\tilde{\phi}$, defined in (5.15a,b), (5.19a,b) and (4.46)–(4.48a,b), as constants over the width of the level, i.e., to use the quantization condition (5.33). When we avoided this approximation and used the more accurate quantization condition (5.44) and the formula (5.42) for Δ , the discrepancy disappeared completely. The state in question has been discussed in detail by Silverstone and Koch (1979), who reported results obtained by 24th-order Rayleigh–Schrödinger perturbation theory (RSPT) combined with a (12/12) Padé approximant (PA). For the same state Damburg and Kolosov improved their above-mentioned result by computing a more accurate numerical result (private communication from Damburg to Nanny Fröman in a letter dated 22 February 1985). The position of the Stark level in question is calculated by us by means of the accurate formula (5.44) with Δ given by (5.42), i.e., with the energy dependence of L , K and $\tilde{\phi}$ over the width of the level taken into account. Our results and the above-mentioned results obtained by Silverstone and Koch (1979) and by Damburg and Kolosov are presented in Table 1.1, which (except for the improved Damburg–Kolosov energy value obtained as private communication) is part of a table that was presented by Fröman and Fröman (1984) at a conference. It is seen that already the first-order phase-integral result is slightly more accurate than the result obtained by Silverstone and Koch (1979) by the use of 24th-order Rayleigh–Schrödinger perturbation theory combined with a (12/12) Padé approximant. Our third- and fifth-order energy values are at least as accurate as the value obtained

Table 1.1. The data in this table, which have been taken from Table 8.7 in Chapter 8, refer to a hydrogen atom with the quantum numbers $m = 0$, $n_1 = 0$, $n_2 = 29$ and $n = |m| + 1 + n_1 + n_2 = 30$ in an electric field of the strength 800 V/cm. The value called RSPT + PA has been obtained by Silverstone and Koch (1979), who remarked that the two underlined digits are uncertain. In the value called Numerical, which has been obtained by Damburg and Kolosov (private communication from Damburg to Nanny Fröman in a letter dated 22 February 1985) there is some doubt about the underlined digit. Like Damburg and Kolosov we have used the conversion factor $1 \text{ au} = 5.142\,260\,3 \text{ V/cm}$.

| Method of calculation | $-E \times 10^4 \text{ au}$ | $\Gamma \times 10^7 \text{ au}$ |
|--|-----------------------------|---------------------------------|
| 1st-order phase-integral approximation | 7.844 656 | 2.849 |
| 3rd-order | 7.844 648 053 | 2.853 2 |
| 5th-order | 7.844 648 046 | 2.853 2 |
| Numerical (Damburg and Kolosov) | 7.844 648 <u>04</u> | |
| RSPT + PA (Silverstone and Koch) | 7.844 <u>68</u> | |

numerically by Damburg and Kolosov. The agreement between our results in the third and fifth orders of approximation indicates in fact that the last digit in the value obtained by Damburg and Kolosov may be wrong by one unit in the last digit. The last digit in the value obtained by Silverstone and Koch (1979) is wrong by three units. The number of reliable digits for E obtained by the phase-integral method is nine. We emphasize that the results in Table 1.1 have been obtained by disregarding the fine structure corrections, which may be of the order of 10^{-6} to 10^{-5} . Therefore the results in this table that are extremely accurate do not represent experimental reality; they are only intended to show the accuracy obtainable by different methods of calculation. There does not seem to exist an experimental value for the energy of a hydrogen atom with the quantum numbers $m = 0$, $n_1 = 0$, $n_2 = 29$ in an electric field with the strength 800 V/cm.

Brief account of the contents of this book

In Chapter 2 the time-dependent Schrödinger equation, describing the Stark effect of a hydrogenic atom or ion in a homogeneous electric field, is separated with respect to time dependence,

center of mass motion, and internal motion. The time-independent Schrödinger equation for the internal motion is then separated in parabolic coordinates. The result is a system of two coupled differential equations, one with a discrete energy spectrum, and the other with a continuous energy spectrum. Enclosing the independent variable η of the differential equation with a continuous energy spectrum in the large but finite interval, $0 \leq \eta \leq \rho$, i.e., imposing on the wave function $g(\eta)$ the boundary conditions $g(0) = 0$ and $g(\rho) = 0$, we perform an analysis of the properties of the eigenfunctions of the two coupled differential equations. In Chapter 3 we consider the development in time of the wave function for the internal motion. The result is an exact formula for the probability amplitude of a decaying state. The eigenfunctions of the above-mentioned two coupled differential equations appear in this formula, which provides the basis for the further treatment of the hydrogenic Stark effect by means of the phase-integral approximation generated from an appropriate base function. With the use of this approximation, which is briefly described in Chapter 4, we obtain in Chapter 5 a more explicit expression for the development in time of the probability amplitude of a decaying state. This expression, which is obtained in the limit $\rho \rightarrow \infty$, contains an energy-dependent quantity $(\Omega'/\Omega'')^2$, which can be interpreted as the level profile. It yields a natural definition of the position and (when the spectral line is not too broad) of the half-width of the Stark level. In Chapter 6 it is described how one transforms the phase-integral formulas derived in Chapter 5 into formulas expressed in terms of complete elliptic integrals of the first, second and third kind. The formulas thus obtained are collected in Chapter 7. These formulas along with well-known properties of complete elliptic integrals, such as for instance series expansions, can be exploited for analytic studies of the Stark effect. Complete elliptic integrals can be evaluated very rapidly by means of standard computer programs, and with the use of the formulas in Chapter 7 a comprehensive numerical material concerning the Stark effect of atomic hydrogen has been obtained. It is presented in Chapter 8, where positions and half-widths for various levels are compared with corresponding results reported by other authors.