

Introduction

This book is primarily concerned with the dynamics of slow viscous or low Reynolds number flows. It is traditional and convenient to divide fluid flows into two classes. Flows in which a fluid of infinite extent flows past solid or liquid bodies are known as external flows. The second class consists of flows involving fluids confined by the solid boundaries of containers, usually of finite extent; these are known as internal flows. There are flows, of course, which exhibit features of both classes but that need not concern us here. Almost all the treatises and books dealing with slow viscous flows, for example, Langlois (1964), Happel & Brenner (1965), Kim & Karilla (1991), and Pozrikidis (1992), deal almost entirely with external flows. There appear to be two reasons for this. Important applications like the problems associated with sedimentation, with suspensions, and with the motion of self-propelling microorganisms do indeed fall into the class of external flows. The other reason is that since external flows are of infinite extent the outer boundary conditions are usually only decay conditions. These conditions are easier to handle especially when using singularity methods, the methods of choice for external flows. Internal flows are more difficult to handle analytically since a full set of conditions have also to be satisfied on the solid boundaries of the container. Although we will also deal with a special class of external flows, the main emphasis in this book is on internal flows which have so far only been dealt with in the journal literature. And we wish to give a somewhat complete and connected account of these flows which now have important applications.

As the subtitle might indicate, there is a secondary theme to this book. The method of eigenfunction expansions, whose history goes back to Fourier, is a classical method of solving linear boundary value problems.

The idea is to generate sets of exact solutions to the field equations, called eigenfunctions, which also exactly satisfy the boundary conditions on a part of the boundary. Associated with the eigenfunctions are scalars called eigenvalues. These sets of eigenfunctions are then combined in the form of infinite series whose real coefficients are determined such that the conditions on the rest of the boundary are satisfied, in principle, exactly. The classical procedure, for example, with the use of Fourier series, depends crucially on the use of the orthogonality property of the eigenfunctions: each coefficient in the series can be determined independently of the others. This property follows automatically for self-adjoint operators with certain canonical boundary conditions. Almost invariably, the eigenvalues and eigenfunctions are real valued, also a natural consequence for canonical self-adjoint problems. There are virtually no books that deal with non-canonical problems that lead to complex eigenfunction expansions — ones where the eigenvalues and eigenfunctions are complex valued and where a convenient orthogonality property is not available. In this book we will deal primarily with such expansions involving complex eigenfunctions. In the process the book also attempts to show how this powerful analytical tool can be used to solve practical problems that arise in engineering and science.

Consider the flow of a fluid of density ρ and viscosity μ . Say the flow involves a velocity scale U and a length scale L . For example, U could be the speed of a body in the fluid or the speed of sliding of one of the boundaries; L could be a body dimension or a characteristic length of the container. Then the Reynolds number Re , which is a measure of the relative importance of inertial forces in the fluid to the viscous forces, is defined by $Re = \rho UL/\mu$. Slow viscous flows or low Reynolds number flows are ones in which the inertial forces are small compared to the viscous forces, or ones in which Re is small in some sense. The most important consequence is that we can in the equations governing the fluid motion, the Navier-Stokes equations, legitimately drop the nonlinear inertial terms. We are then left with linear equations, ones that are normally considered to be easy to solve. A further consequence is that superposition of solutions is valid in slow viscous flows. This important property will be used repeatedly in this book.

It is a strange and interesting fact that although classical treatises like Morse & Feshbach (1953) may give the impression that all linear problems have already been solved, problems involving low Reynolds number flows continue to prove to be stubbornly resistant to complete resolution. To this day we are not really sure how a flow behaves in the neighborhood of a three-dimensional corner. Until very recently there was no analytical solution to something as apparently simple as the flow in a three-dimensional lid

driven container. There *are* linear problems that are difficult to resolve. As interesting is the fact that slow viscous flows often exhibit physical features that can hardly be guessed at, that are counter intuitive.

Consider the two flow fields that are sketched in Fig. 11. Panel (a) of the figure is a sketch of streamlines in planar viscous flow past a corner of angle 2α . How will the flow look like as we approach the corner? The most natural guess is that it will look much like it is in the figure. It is a surprising fact that, actually, the field will consist, in general, of an infinite sequence of eddies of decreasing size and intensity as the corner is approached (Moffatt 1964a). Panel (b) is a sketch of flow past a cut-out in the solid wall past which a liquid is flowing. What will the field in the cavity look like as a function of the parameters of the flow? First consider slow viscous flow in which the Reynolds number $Re = \rho UL/\mu$ is small. The flow being laminar, it might appear that the streamlines might appear as in the sketch no matter how large the depth H is. Another possibility is that at some depth there may be flow separation resulting in a second eddy. A little more thought makes clear that it is impossible to guess on simple physical intuition what the flow field will look like as H increases. In actual fact beyond a fairly small H/L a second eddy is formed; with further increases in the ratio, more eddies are formed until for $H/L \rightarrow \infty$ the number of weaker and weaker eddies increases indefinitely. A more difficult question: what if Re is not small? What if the flow is turbulent? The answers to these questions were found only in the last decade or so using the methods suggested here.

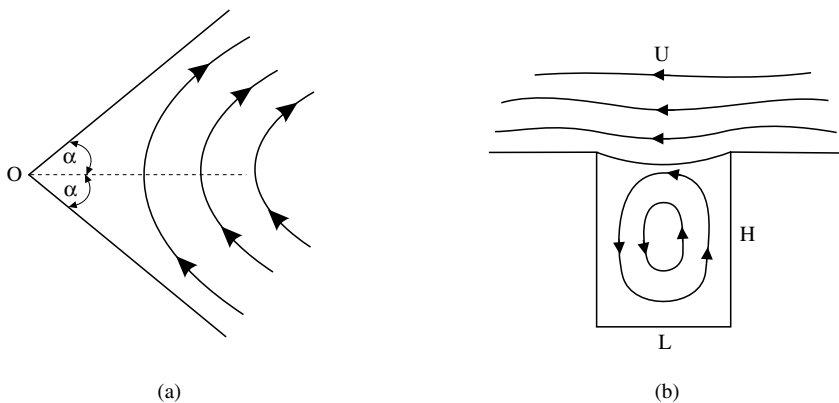


Fig. 11. (a) Planar flow past a corner of angle 2α . (b) Viscous flow past a cut-out in the container wall.

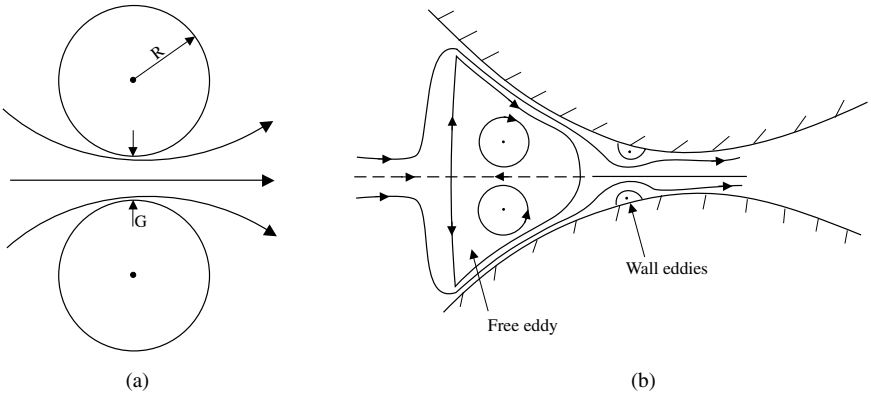


Fig. I2. Viscous flow past two cylinders with a gap between them. (a) Sketch of the field when the gap is large. (b) Field when the gap is sufficiently small.

Another class of intriguing low Re flow fields is sketched in Fig. I2. Here we have streaming planar flow past a pair of cylinders with a gap G between them. When the gap is sufficiently large the fluid flows through the gap without any separation from the cylinder surfaces. If G/R is now reduced, interesting things begin to happen below some critical value of the ratio. Figure I2(b) shows that at some stage there is a comparatively large, symmetrical free eddy essentially blocking the gap followed by one wall eddy each on the cylinder walls. Note that the through-flow fluid has to meander through the small gap between the free eddy and the cylinder walls and between the two wall bound eddies. As the gap is reduced, more and more wall bound eddies are formed with greater meandering of the through-flow. An axisymmetric version of a related field is shown in Fig. I3 where the

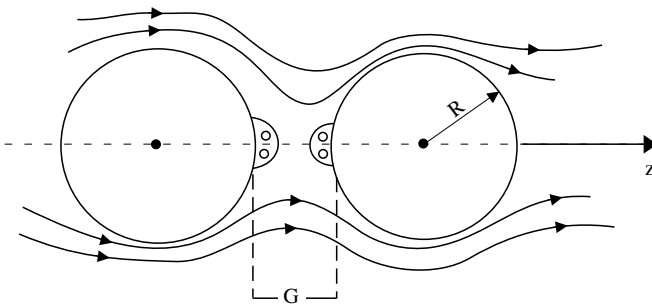


Fig. I3. Axisymmetric flow past two spheres whose centers are aligned with the flow.

flow is past two equal spheres of radii R with a gap G between them. The line connecting the centers is parallel to the free stream. If the spheres are far apart there is, as one would expect, no flow separation on either sphere. When the spheres are brought closer, changes in flow topology occur. The first bifurcation occurs when $G/R \approx 1.57$, when wall eddies form in the gap as shown in the figure. As the gap is reduced further at $G/R \approx 1.22$, the wall eddies touch to form a ring vortex in the gap; with further reductions in the gap width, more and more ring vortices are formed. Although the exact solution for this problem was obtained by Stimson & Jeffery (1926), who also deduced the forces acting on the spheres, they do not appear to have known of the intricate structure of the flow field which was hidden in their complicated formulae. It required the more detailed numerical and analytical work, 50 years later, of Davis, O'Neill, Dorrepaal & Ranger (1976) to determine this structure.

A three-dimensional example is afforded by natural convection. Say a cylinder of radius R and depth H is filled with liquid and heated at the bottom non-uniformly. Suitably non-dimensionalized, the sidewall and lid are at zero temperature while the temperature on the bottom wall is proportional to $\cos \phi$, where ϕ is the azimuthal angle. In Fig. I4, $\phi = 0$ is in

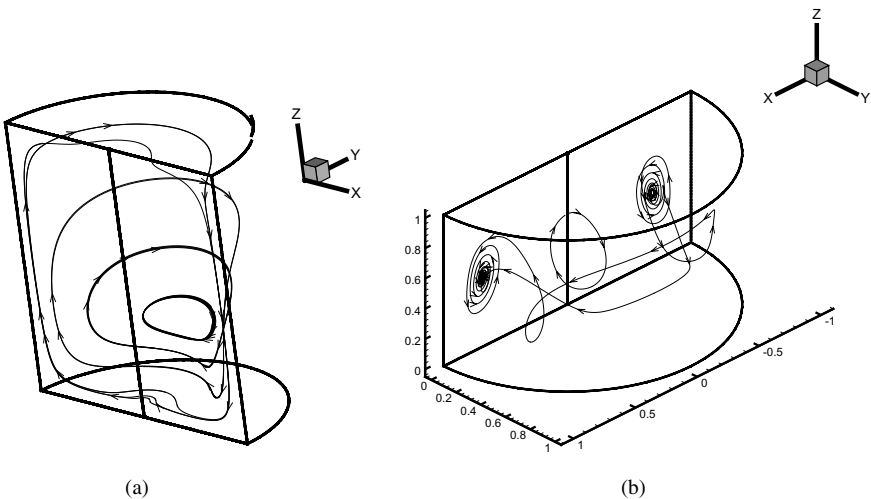


Fig. I4. Three-dimensional streamlines in natural convection in a cylinder full of liquid. In each case the bottom wall is heated non-uniformly, with the dimensionless temperature proportional to $\cos \phi$, the azimuthal angle; the sidewall and the top are at zero temperature. (a) $H/R = 2$, (b) $H/R = 1$.

the direction of positive x and so the right half of the bottom in frame (a) is warm and the left half cold with respect to the other walls. Gravity acts downwards in the negative z -direction. A reasonable physical argument suggests that in Fig. I4(a) there will be a generally clockwise circulatory motion in the fluid. And indeed this is what is seen in this figure which is for the case where $H/R = 2$; and a similar pattern is found when this aspect ratio is larger. But when the aspect ratio is lowered, at some critical value there is a striking change in the topology of the flow field. The field when $H/R = 1$ is shown in Fig. I4(b); note that the direction of view is the opposite of that in frame (a). Whereas the streamlines seen in the latter form distorted closed curves in the clockwise circulatory motion, in frame (b) the most striking feature is the existence of foci with streamlines going from one focus to the other; there are closed streamlines but these are restricted to the central portion of the cylinder.

These examples, and others that we will come across, show that we often come across unexpected changes in the topology of the flow field when some parameter, say an aspect ratio or a velocity ratio, goes through a critical value. And this is what makes the whole field of low Reynolds number flows so interesting. The somewhat unpredictable nature of low Reynolds number flow phenomena was noted a long time ago. In a paper dealing with the biharmonic equation, which governs Stokes flow in the plane, Jeffery (1920) noted that the area seemed to be “a branch of mathematical physics in which knowledge comes by the patient accumulation of special solutions rather than by the establishment of great general propositions.” The above examples also illustrate another aspect of the area. Neither pure analysis nor computations alone are likely to lead to advances in the field: this is really an area where analysis, asymptotic methods, computations and experiments all play complementary roles in leading to understanding.

There are many industrial, technological and scientific problems where low Reynolds number flow principles find application. For the well known applications to determining the terminal velocities of small particles, more generally to problems of sedimentation, to lubrication theory and to flow through porous media, we refer the reader to the works cited above and to the textbook by Batchelor (1967). Here we will just describe a few applications that have arisen more recently and which concern internal flows.

It is the uniformity of structure that makes single crystals so useful in technological applications. This property permits the transmission in a crystal, without scattering, of acoustic and electromagnetic waves and

charged particles. Thus in any crystal growth process it is important to try to maintain crystal quality. Many crystals are formed by the solidification of a melt and there are a variety of such techniques (Hurle 1994). In the Czochralski technique, shown schematically in Fig. I5(a), the material to be crystallized is melted in a crucible. A pull rod, with a seed crystal at its lower end, is inserted into the melt and the melt temperature adjusted until a meniscus is supported. It is then slowly rotated and lifted to produce a crystal of the required diameter. As shown in the figure, convection will normally exist in the melt and this is affected by density gradients, the pulling rate, rotation rate, etc. Another method of crystal growth, the floating zone technique, is schematically represented in Fig. I5(b). Here, in order to avoid the contaminating effects of a crucible, the molten zone is established between a feed rod of polycrystalline material and a crystal rod; the melting can be done by radio-frequency induction heating or by electron beam melting, for example, with the melt zone traversing the feed rod. In panel (b), which is supposed to represent a low gravity situation, the principal convection driving mechanism is the surface tension gradient along the free surface of the melt. There are a number of other crystal growth techniques in each of which natural, forced and thermocapillary convection either singly or jointly bear on the process. Thus, among the

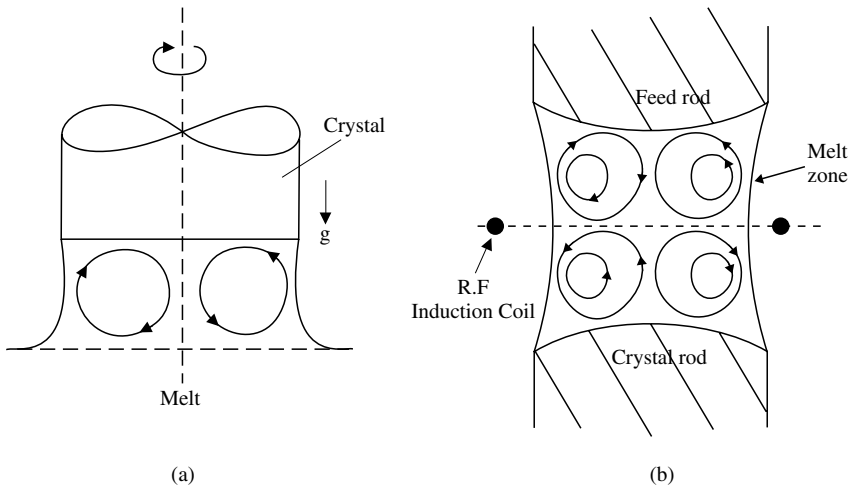


Fig. I5. Thermal convection currents in crystal growth processes. (a) A schematic of the Czochralski technique. (b) Schematic of the floating zone method in low gravity conditions.

various factors which play important roles in crystal growth, convection in the melt crucially affects crystal quality. For a range of parameters these effects can be usefully studied using the slow viscous flow model. There are means by which one can consider controlling convection in a liquid, for example, by the motion of a boundary, or by the use of magnetic fields (Hof, Juel & Mullin 2003, Aleksandrova & Molokov 2004). These effects can also be analyzed in the framework of low Reynolds number flows.

Mixing processes are of great importance in a range of industries from food processing to the manufacture of paints. What do we mean by ‘mixing’ in a fluid? If we inject a small blob of tracer material in some portion of a fluid in motion, good mixing implies that in a short period of some suitably defined time the tracer will be dispersed throughout the fluid. One would normally expect that good mixing can only take place in a turbulent flow. It therefore comes as a surprise that this is not necessary — good mixing can take place even in slow laminar flows, provided that they are suitably unsteady. The mixing is said to take place by chaotic advection (Ottino 1990). Figure I6(a) is a sketch of a container full of liquid where both the top lid and the bottom wall can be oscillated independently of one another. If the motion is of the top lid alone the mixing is poor — this is because to a good approximation, i.e. in the quasi-steady approximation, the streamlines are steady (even though the velocity is not!) and a particle on a streamline stays on that streamline throughout its motion. The situation is quite different when both boundaries are allowed to move. Now the streamlines are truly unsteady and the instantaneous pattern shown in Fig. I6(a) will a short while later look very different. This means that a small particle can jump from one streamline to another and in general this motion will appear to be chaotic, which is what is wanted.

A small blob of dye, consisting of many ‘particles’ is also shown in frame (a) of the figure; in suitably unsteady laminar motion different particles will be advected differently and in a short while the ‘blob’ could be dispersed throughout the fluid. How soon can the blob be well dispersed? Where should the blob be injected so that it will indeed disperse? Are all locations suitable? These are some of the important questions that arise and whose answers can be of industrial importance. Figure I6(b) is what is called a Poincaré map which contains some of this information. Using a low Reynolds number calculational procedure, the movement of a *single point* was traced through 10,000 periods for a given lid and bottom movement protocol. Each point in the map marks the position of the point at the end of a period. The figure shows that although the particle moves throughout

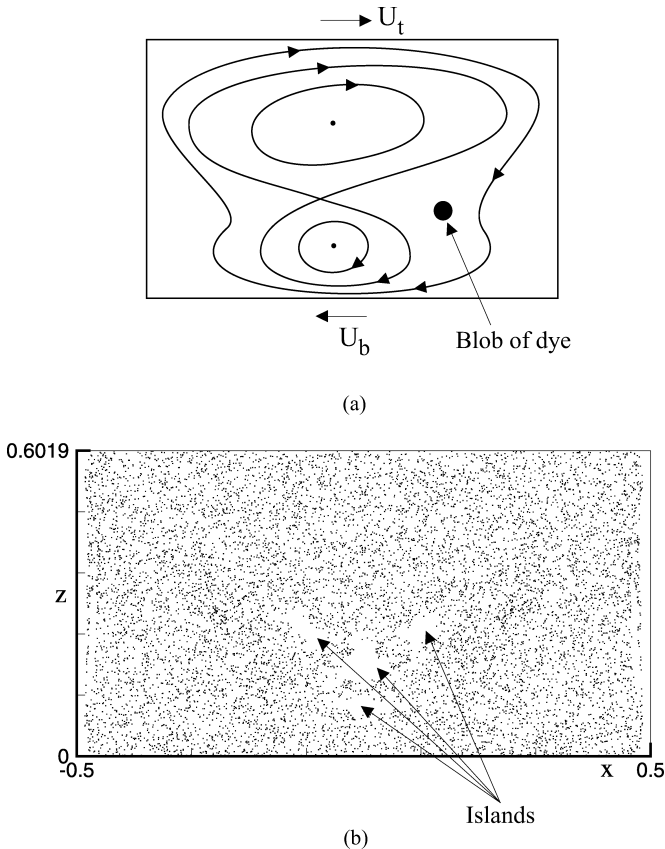


Fig. I6. Mixing in a double lid driven container. (a) Schematic of the field in the container at some instant. (b) Poincaré map obtained by iterating a single initial position over 10,000 periods.

the interior of the container, there are islands, marked with arrows, where it never appears to go. Clearly, if a blob were to be released in these regions the mixing would be poor. This example shows how useful information in problems dealing with mixing by chaotic advection can be obtained from efficient slow viscous flow computations.

As a final example we consider coating flows. Coating flows arise in a number of industries, including those involved in the production of continuous metal ribbons of microcrystalline material, in the production of photographic film and high grade paper (see Aidun, Triantafillopoulos & Benson 1991 and the references therein). It is usually important in these

applications to have uniform coating thickness without patches and streaks. It turns out that the fluid dynamic characteristics of the coater can seriously affect coating quality. Figure I7(a) is a sketch of a ‘two-roll coater’ whose purpose is to coat the ‘web’ with the coating fluid which is in a pan. The lower roller picks up the coating fluid and transfers it to the lower surface of the web which moves with the upper roller. This transfer is effected through the ‘nip’ which is a circulating body of fluid between the rollers and is fed by the lower roller. The eddy structure in the nip greatly affects the

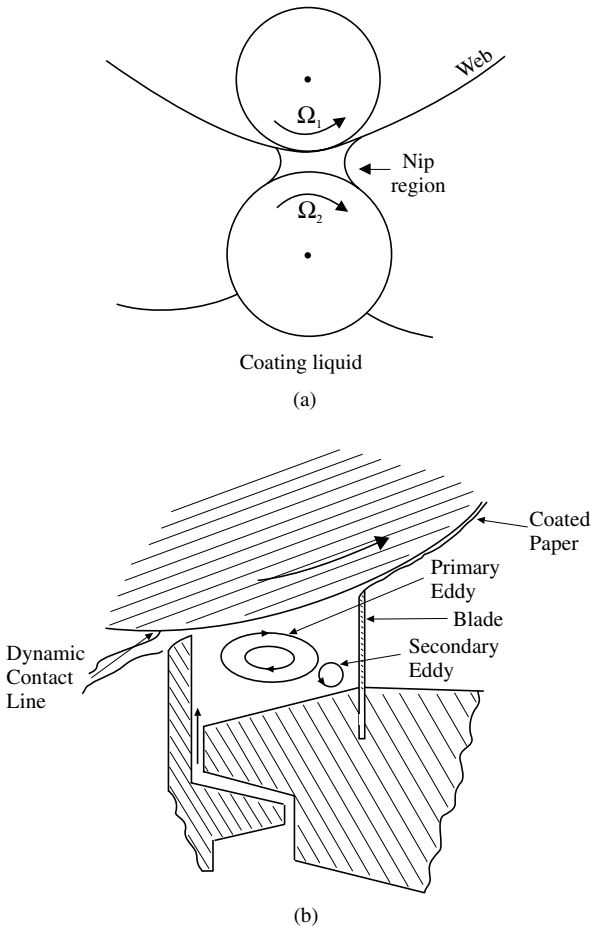


Fig. I7. The role of eddy structure in coating flows. (a) Schematic of a two-roll coater. (b) Schematic of a short-dwell coater.

coating performance of the coater. There are two main coating regimes: (a) the inlet-flooded or classical regime and (b) the inlet-starved or meniscus regime (see Gaskell, Savage, Summers & Thompson 1995 and their citations). Contrary to the assumptions of early practitioners, the two regimes have very different characteristics. In the meniscus regime the field consists of two eddies with the transfer of the fluid to the web by a meandering or snake-like path. A different type of coater, a ‘short-dwell coater’ is shown in Fig. I7(b). It is important in such coaters to be able to increase machine speed while reducing coat weight; but this has to be done while maintaining uniformity in thickness. Instability of the viscous layer near the dynamic contact line and the eddy structure in the pond can affect the coating quality. Although the Reynolds number here is normally somewhat above that of the slow flow regime, low Reynolds number studies can help to establish the qualitative picture, even if the quantitative data may be somewhat inaccurate.

We conclude this introduction with a short description of the structure of the book and suggestions as to how it might be used. For the reasons mentioned in the preface, the book has been written with a number of different goals and different types of readers in mind. As a consequence there are repetitions of some formulae, an unevenness in level and some imbalance in the lengths of the chapters. The book has been divided into five parts with the necessary physical and mathematical background being given in the two chapters of Part 1. Planar flows are dealt with in Part 2, three-dimensional flows in Part 3, unsteady flows in Part 4 and external flows in Part 5. An attempt has been made to keep the parts independent of one another so that the specialist can read any one of them without reference to the others. However, two chapters are special. The long Chap. 3 gives a great deal of detail about the eigenfunction method used in conjunction with the least squares technique, detail that is not repeated elsewhere. So a user of a later part may have to refer back to Chap. 3 for these details regarding, for example, accuracy and convergence. Chapter 5 is the only chapter to give details of the newly discovered embedding method to solve problems involving complex geometries. Thus a reader needing to deal with complex geometries, say in three-dimensional flows or external flows, would have to refer back to this chapter. A semester course on low Reynold number flows could comprise Chaps. 1, 3, 5 or 1–3 and two or three chapters from the others depending on the interests of the students. A specialist would best go to the chapter of his or her interest, say convection or mixing, and only refer to an earlier chapter, say Chap. 2 or 3, if needed.