Chapter 1

Physical Aspects of the Motion of a Viscous Fluid

This book deals with viscous fluids circulating in containers or in relative motion past solid bodies. Such flows will be analyzed by treating the fluid as a continuum and assuming that the classical laws, namely, the law of conservation of mass, Newton's second law applied to a continuum and the law of conservation of energy, hold. The laws lead to a system of partial differential equations for the field quantities like velocity and pressure which need to be solved for given boundary conditions. Various assumptions, both physical and mathematical, and all very reasonable, need to be made in the derivation of these equations. Here we will only outline the derivations and refer the reader to standard textbooks like Landau & Lifshitz (1959), Aris (1962), Batchelor (1967), and Pozrikidis (1997a) for more details. Although we will mostly be dealing with constant density Newtonian fluids, the derivations here will be for a general compressible fluid.

1.1 The Continuity, Navier–Stokes and Energy Equations

Figure 1.1 shows an arbitrary control volume $\tilde{V}$ through which the fluid is flowing. The surface of the control volume is denoted by $\tilde{S}$ and at any point on the surface a local surface area element is denoted by $d\tilde{S}$; the unit outward normal at the point is $\mathbf{n}$. Let $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ be a suitable Cartesian coordinate system and let the fluid velocity at a field point be denoted by $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$, where $\tilde{u}_1, \tilde{u}_2$ and $\tilde{u}_3$ are the components of the velocity. The tildes indicate that these quantities are dimensional; when we non-dimensionalize later, the tilde-less symbols will represent dimensionless quantities which will be used in the rest of the book. The procedure

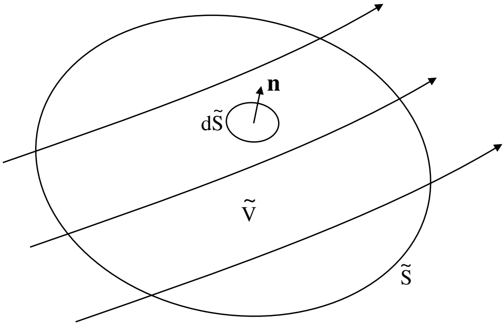

Fig. 1.1. An arbitrary control volume $\tilde{V}$ in the fluid flow bounded by the surface $\tilde{S}$. A typical element on the bounding surface is $d\tilde{S}$ with local unit outward normal $\mathbf{n}$.

for obtaining the necessary field equations is to apply the physical laws to the *moving* fluid in the *fixed* control volume, taking into account the flow through the bounding surface $\tilde{S}$ and the forces on it. Although we will, for simplicity, use Cartesian tensor notation we need nothing beyond standard vector analysis.

In particular we will use the summation convention that repeated indices imply summation over the index, e.g. $a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$. Here and throughout the book, a comma followed by an index in the subscript implies differentiation with respect to that index, e.g. $\psi_{,j} = \partial\psi/\partial x_j$ while $u_{i,j} = \partial u_i/\partial x_j$. Note that $u_{i,i} = u_{1,1} + u_{2,2} + u_{3,3} = \nabla \cdot \mathbf{u}$. The most important result from vector analysis that we will need is Green's theorem, or the divergence theorem, relating surface integrals to volume integrals. Let $\mathbf{u}(\mathbf{x})$ be a continuously differentiable vector field and let the boundary S of volume V be sufficiently smooth; let $\mathbf{n}$ be the unit outward normal on the boundary. Then the following relation holds

$$\int_S \mathbf{u} \cdot \mathbf{n} dS = \int_S u_i n_i dS = \int_V \nabla \cdot \mathbf{u} dV = \int_V u_{i,i}\, dV. \tag{1.1}$$

The divergence theorem (1.1) will be used to convert surface integrals to volume integrals.

We first ensure that mass is conserved in $\tilde{V}$. Let $\tilde{\rho}$ be the (dimensional) density of the fluid. Then the rate of increase of mass in the control volume is just $d(\int_{\tilde{V}} \tilde{\rho} d\tilde{V})/d\tilde{t} = \int_{\tilde{V}} \tilde{\rho}_{,t} d\tilde{V}$, since $\tilde{V}$ is a fixed volume. This increase in mass can only occur on account of the net fluid flowing in through the boundaries. The rate at which fluid is being transported through the surface element $d\tilde{S}$ into $\tilde{V}$ is $-\tilde{\rho}\tilde{u}_j n_j d\tilde{S}$, since $\mathbf{n}$ is the unit outward normal. Thus we conclude that to ensure that mass is conserved in $\tilde{V}$,

$$\int_{\tilde{V}} \frac{\partial \tilde{\rho}}{\partial \tilde{t}} d\tilde{V} = -\int_{\tilde{S}} \tilde{\rho}\tilde{u}_j n_j d\tilde{S} \quad \text{or} \quad \int_{\tilde{V}} \frac{\partial \tilde{\rho}}{\partial \tilde{t}} d\tilde{V} + \int_{\tilde{S}} \tilde{\rho}\tilde{u}_j n_j d\tilde{S} = 0. \tag{1.2}$$

If we now apply the divergence theorem (1.1) to the surface integral in (1.2), we obtain

$$\int_{\tilde{V}} \left\{ \frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}_j}(\tilde{\rho}\tilde{u}_j) \right\} d\tilde{V} = 0. \tag{1.3}$$

Now since the volume $\tilde{V}$ is arbitrary, (1.3) implies that the quantity in braces must vanish everywhere, or

$$\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}_j}(\tilde{\rho}\tilde{u}_j) = 0 \quad \text{or} \quad \frac{\partial \tilde{\rho}}{\partial t} + \tilde{\nabla} \cdot (\tilde{\rho}\tilde{\mathbf{u}}) = 0. \tag{1.4}$$

This is the continuity or mass conservation equation. Note that for a constant density fluid the first term will vanish and the equation will reduce to $\tilde{u}_{i,i} = \tilde{\nabla} \cdot (\tilde{\mathbf{u}}) = 0$.

We need next to apply Newton's second law to the fluid in the control volume $\tilde{V}$. Let us assume that a body force field $\tilde{\mathbf{F}}(\tilde{\mathbf{x}})$ per unit mass acts on the fluid. We take Newton's second law to imply that the rate of increase of momentum in $\tilde{V}$ must equal the sum of the rate at which momentum is flowing in through the boundary $\tilde{S}$ and the total force acting on the contents of $\tilde{V}$. The rate of increase of the $\tilde{x}_i$-component of momentum in $\tilde{V}$ is given by $d\{\int_{\tilde{V}} \tilde{\rho}\tilde{u}_i d\tilde{V}\}/d\tilde{t} = \int_{\tilde{V}} (\tilde{\rho}\tilde{u}_i)_{,\tilde{t}} d\tilde{V}$. The rate at which $\tilde{x}_i$-component of momentum flows into $\tilde{V}$ through the boundary $\tilde{S}$ is equal to $-\int_{\tilde{S}} (\tilde{\rho}\tilde{u}_j n_j)\tilde{u}_i d\tilde{S}$; the minus sign is again due to the fact that $\mathbf{n}$ is the outward normal. The total force acting on the volume of fluid is the sum of the body force acting on the volume and the net effect of the stresses acting on the fluid over the whole surface $\tilde{S}$. The net body force component is just $\int_{\tilde{V}} \tilde{\rho}\tilde{F}_i d\tilde{V}$. Let $\tilde{\sigma}_{ij}$ be the stress tensor in the fluid. By definition, if $d\tilde{S}$ is an infinitesimal surface element with outward normal $\mathbf{n}$, the $\tilde{x}_i$th component of force on the element, due to the internal stresses, is given by $\tilde{\sigma}_{ji} n_j d\tilde{S}$. Thus the net $\tilde{x}_i$th component of force on $\tilde{V}$ due to the surface stresses on the boundary is equal to $\int_{\tilde{S}} \tilde{\sigma}_{ji} n_j d\tilde{S}$.[1] Putting everything together, Newton's second law takes the form

$$\int_{\tilde{V}} \frac{\partial}{\partial \tilde{t}}(\tilde{\rho}\tilde{u}_i) d\tilde{V} = -\int_{\tilde{S}} (\tilde{\rho}\tilde{u}_j n_j)\tilde{u}_i d\tilde{S} + \int_{\tilde{V}} \tilde{\rho}\tilde{F}_i d\tilde{V} + \int_{\tilde{S}} \tilde{\sigma}_{ji} n_j d\tilde{S}. \tag{1.5}$$

[1] Here $\tilde{\sigma}_{ji}$ is the ith component of force per unit area exerted across a plane surface normal to the jth direction. Some authors use the symbol $\tilde{\sigma}_{ij}$ for this quantity. However, this causes no problems since the stress tensor is symmetric.

Now applying the divergence theorem (1.1) to the two surface integrals and rearranging we have

$$\int_{\tilde{V}} \left[\frac{\partial}{\partial \tilde{t}}(\tilde{\rho}\tilde{u}_i) + \frac{\partial}{\partial \tilde{x}_j}(\tilde{\rho}\tilde{u}_i\tilde{u}_j) - \frac{\partial \tilde{\sigma}_{ji}}{\partial \tilde{x}_j} - \tilde{\rho}\tilde{F}_i \right] d\tilde{V} = 0. \tag{1.6}$$

Once again, since $\tilde{V}$ is arbitrary, the expression within the square brackets has to vanish; this leads, with the use of the continuity equation (1.4), to the momentum equation

$$\tilde{\rho}\left\{ \frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} \right\} = \frac{\partial \tilde{\sigma}_{ji}}{\partial \tilde{x}_j} + \tilde{\rho}\tilde{F}_i. \tag{1.7}$$

So far, no assumption has been made about the fluid, about the constitutive relation applicable to it. From now on we will assume that we are dealing with a *Newtonian* fluid which satisfies the constitutive relation

$$\tilde{\sigma}_{ij} = -\tilde{p}\delta_{ij} + \mu(\tilde{u}_{i,j} + \tilde{u}_{j,i}) + \left(\zeta - \frac{2}{3}\mu \right) \delta_{ij}\tilde{u}_{k,k}, \tag{1.8}$$

where μ and ζ, the coefficient of viscosity and the second coefficient of viscosity respectively, are material properties which we will assume to be constant. Note that $\tilde{p}$ is the pressure in the fluid while δ_{ij} is the Kronecker delta, i.e. $\delta_{ij} = 1$ if $i = j$, otherwise it is 0. If (1.8) is substituted into (1.7), we obtain the *Navier–Stokes equation*

$$\tilde{\rho}\left\{ \frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} \right\} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \mu \left(\frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j^2} + \frac{\partial^2 \tilde{u}_j}{\partial \tilde{x}_j \partial \tilde{x}_i} \right) + \left(\zeta - \frac{2}{3}\mu \right) \frac{\partial^2 \tilde{u}_k}{\partial \tilde{x}_i \partial \tilde{x}_k} + \tilde{\rho}\tilde{F}_i. \tag{1.9}$$

This is the momentum equation or the equation of motion for a general Newtonian fluid. Suppose that the velocity field is solenoidal, i.e. $\nabla.\tilde{u} = \tilde{u}_{i,i} = 0$, as it is for an incompressible fluid; in such a case the Navier–Stokes equation takes the simpler form

$$\tilde{\rho}\left\{ \frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} \right\} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \mu \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j^2} + \tilde{\rho}\tilde{F}_i. \tag{1.10}$$

We now apply the first law of thermodynamics to the contents of the control volume. The rate of increase of energy inside $\tilde{V}$ is equal to the sum of (1) the rate at which energy flows into $\tilde{V}$ through the boundary $\tilde{S}$, (2) the rate at which heat flows into $\tilde{V}$ through the boundary $\tilde{S}$ and (3) the rate at which work is done on $\tilde{V}$. The rate of increase of energy in $\tilde{V}$ is $(d/d\tilde{t})\{\int_{\tilde{V}} \tilde{\rho}(\tilde{e} + \frac{1}{2}\tilde{u}_i\tilde{u}_i)d\tilde{V}\} = \int_{\tilde{V}}(\partial/\partial\tilde{t})\{\tilde{\rho}(\tilde{e} + \frac{1}{2}\tilde{u}_j\tilde{u}_j)\}d\tilde{V}$, where $\tilde{e}$ is the internal energy of the fluid per unit mass. The rate at which energy flows into $\tilde{V}$ through $\tilde{S}$ equals $-\int_{\tilde{S}}(\tilde{e} + \frac{1}{2}\tilde{u}_k\tilde{u}_k)\tilde{\rho}\tilde{u}_j n_j d\tilde{S}$. The rate at which heat flows in through the boundary is $-\int_{\tilde{S}} \tilde{q}_j n_j d\tilde{S}$ where $\tilde{\mathbf{q}}$ is the heat flux vector. Since both the body force field and the surface stresses at the boundaries can do work, the net rate at which work is done on V is equal to $\int_{\tilde{S}} \tilde{\sigma}_{ij}\tilde{u}_j n_i d\tilde{S} + \int_{\tilde{V}} \tilde{\rho}\tilde{F}_j\tilde{u}_j d\tilde{V}$. The overall balance requires that

$$\int_{\tilde{V}} \frac{\partial}{\partial \tilde{t}} \left\{\tilde{\rho}\left(\tilde{e} + \frac{1}{2}\tilde{u}_j\tilde{u}_j\right)\right\} d\tilde{V} = -\int_{\tilde{S}} \left(\tilde{e} + \frac{1}{2}\tilde{u}_k\tilde{u}_k\right)\tilde{\rho}\tilde{u}_j n_j d\tilde{S} - \int_{\tilde{S}} \tilde{q}_j n_j d\tilde{S} + \int_{\tilde{S}} \tilde{\sigma}_{ij}\tilde{u}_j n_i d\tilde{S} + \int_{\tilde{V}} \tilde{\rho}\tilde{F}_j\tilde{u}_j d\tilde{V}. \quad (1.11)$$

If we now apply the divergence theorem to the surface integrals, we are led, in view of the fact that $\tilde{V}$ is arbitrary, to

$$\frac{\partial}{\partial \tilde{t}}\left\{\tilde{\rho}\left(\tilde{e} + \frac{1}{2}|\tilde{\mathbf{u}}|^2\right)\right\} + \frac{\partial}{\partial \tilde{x}_j}\left\{\tilde{\rho}\tilde{u}_j\left(\tilde{e} + \frac{1}{2}|\tilde{\mathbf{u}}|^2\right)\right\} = \frac{\partial}{\partial \tilde{x}_j}(\tilde{\sigma}_{jk}\tilde{u}_k) - \frac{\partial \tilde{q}_j}{\partial \tilde{x}_j} + \tilde{\rho}\tilde{F}_j\tilde{u}_j. \quad (1.12)$$

This can be simplifed by the use of the momentum equation. If we take the dot product of (1.7) with $\tilde{\mathbf{u}}$ and use it, along with (1.4), in (1.12), the energy equation takes the form

$$\tilde{\rho}\left\{\frac{\partial \tilde{e}}{\partial \tilde{t}} + \tilde{u}_j\frac{\partial \tilde{e}}{\partial \tilde{x}_j}\right\} = \tilde{\sigma}_{ij}\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} - \frac{\partial \tilde{q}_j}{\partial \tilde{x}_j}. \quad (1.13)$$

We will now assume that the fluid has a constant specific heat at constant volume c_v and that heat conduction in it satisfies Fourier's law with a thermal conductivity k_c; then

$$\tilde{e} = c_v\tilde{T}, \quad \tilde{\mathbf{q}} = -k\tilde{\nabla}\tilde{T} \quad \text{or} \quad \tilde{q}_j = -k_c\frac{\partial \tilde{T}}{\partial \tilde{x}_j}. \quad (1.14\text{a,b})$$

If (1.8) and (1.14) are substituted into (1.13), the energy equation takes the form

$$\tilde{\rho}c_v\frac{\tilde{D}\tilde{T}}{\tilde{D}\tilde{t}} = -\tilde{p}\frac{\partial \tilde{u}_i}{\partial \tilde{x}_i} + \frac{\partial}{\partial \tilde{x}_j}\left(k_c\frac{\partial \tilde{T}}{\partial \tilde{x}_j}\right) + \left[\left(\zeta - \frac{2}{3}\mu\right)\left(\frac{\partial \tilde{u}_i}{\partial \tilde{x}_i}\right)^2 + \mu\left(\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} + \frac{\partial \tilde{u}_j}{\partial \tilde{x}_i}\right)\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j}\right], \quad (1.15)$$

where the term in square brackets is the dissipation in the fluid. In (1.15), we have used the standard notation

$$\frac{\tilde{D}}{\tilde{D}\tilde{t}} = \left(\frac{\partial}{\partial\tilde{t}} + \tilde{u}_j\frac{\partial}{\partial\tilde{x}_j}\right), \tag{1.16}$$

for the material derivative, the right side of (1.16).

Finally, to complete the system of equations we need an equation of state. In general, the equation of state will be an equation of the form

$$\tilde{\rho} = \tilde{\rho}(\tilde{p}, \tilde{T}). \tag{1.17}$$

In summary, for a Newtonian fluid with constant properties satisfying the Fourier law of heat conduction, we have the six scalar field Eqs. (1.4), (1.9), (1.15) and (1.17) for the six unknown field quantities $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{\rho}, \tilde{p}$ and $\tilde{T}$.

We will be mainly concerned with constant density fluids, or ones which are very nearly so, for which

$$\tilde{\rho}(\tilde{\mathbf{x}}, \tilde{t}) = \text{constant} = \tilde{\rho}_0 \tag{1.18}$$

is either exact or a very good approximation. We use the phrase 'constant density' since some authors (e.g. Batchelor 1967) use the term 'incompressible fluid' for a field in which $(\tilde{D}/\tilde{D}\tilde{t})\tilde{\rho} = 0$. In any case, for a constant density fluid the field equations simplify considerably to the following set:

$$\tilde{\nabla}\cdot\tilde{\mathbf{u}} = 0 \quad \text{or} \quad \frac{\partial\tilde{u}_i}{\partial\tilde{x}_i} = 0, \tag{1.19a}$$

$$\tilde{\rho}\frac{\tilde{D}\tilde{u}_i}{\tilde{D}\tilde{t}} = -\frac{\partial\tilde{p}}{\partial\tilde{x}_i} + \mu\frac{\partial^2\tilde{u}_i}{\partial\tilde{x}_j^2} + \tilde{\rho}\tilde{F}_i, \tag{1.19b}$$

$$\tilde{\rho}c_v\frac{\tilde{D}\tilde{T}}{\tilde{D}\tilde{t}} = \frac{\partial}{\partial\tilde{x}_j}\left(k_c\frac{\partial\tilde{T}}{\partial\tilde{x}_j}\right) + \mu\left(\frac{\partial\tilde{u}_i}{\partial\tilde{x}_j} + \frac{\partial\tilde{u}_j}{\partial\tilde{x}_i}\right)\frac{\partial\tilde{u}_i}{\partial\tilde{x}_j}, \tag{1.19c}$$

$$\tilde{\rho} = \tilde{\rho}_0. \tag{1.19d}$$

Note that the material constants μ, ζ and k_c are all dimensional. Another point to be noted is that for a constant density fluid satisfying (1.18), the momentum equations (1.19b) uncouple from the energy equation (1.19c). This means that the velocity and pressure fields can be determined first and then be used to solve for the temperature field from (1.19c).[2]

[2]Strictly, (1.18) is not an equation of state at all, since given $\tilde{\rho}$ and $\tilde{T}$ the hydrostatic pressure is not uniquely determined. But this subtlety need not concern us here. See Langlois (1964) for a nice discussion.

1.2 The Boundary Conditions

When we analyze physical problems using the field equations of Sec. 1.1, we will need to provide suitable boundary conditions for the dependent field variables. There are three types of boundaries that we need to consider: (1) where the fluid is in contact with a solid, (2) where the fluid is unbounded, and (3) where the fluid is in contact with another fluid. The question of the physically correct boundary conditions is in fact a deep and historically rich one; we will, however, have to refer the interested reader to the literature for the details and only state below the conditions that we will be using.

For a viscous liquid at an impermeable, solid boundary, the *non-penetration* and *no-slip* conditions hold. In other words, at such a boundary the velocity of the fluid $\tilde{\mathbf{u}}$ must equal the velocity of the solid boundary $\tilde{\mathbf{u}}_B$; we will refer to this boundary condition as the 'no-slip' condition. In particular, at a stationary solid boundary the fluid must be at rest. For a constant density fluid these conditions would be sufficient to determine the velocity field. However, conditions are also necessary for the temperature field. At a solid boundary one can, for instance, either prescribe the temperature of the fluid at the boundary or the heat flux.[3]

In situations where the field is unbounded, we will assume uniform conditions to hold in the far field. The boundary conditions will then require that the field variables tend to their far field values asymptotically.

The conditions at a fluid–fluid interface require a little more care. First of all we need to ensure that the interface remains an interface during any motion. This purely kinematical condition requires that the component of velocity normal to the interface be continuous across it. Next, if we can assume that thermodynamic and mechanical equilibrium hold at the interface, both the temperature and velocity must be continuous across the interface. As for the stresses, let us first rewrite the constitutive relation (1.8) in the form

$$\tilde{\sigma}_{ij} = -\tilde{p}\delta_{ij} + 2\mu\tilde{e}_{ij} + \left(\zeta - \frac{2}{3}\mu\right)\delta_{ij}\tilde{\Delta}, \tag{1.20}$$

[3]There is a subtle point here. Although one can only prescribe either the temperature or the heat flux at the boundary, both the temperature and heat flux have, if we assume thermodynamic equilibrium or near thermodynamic equilibrium at the interface, to be continuous across the solid–liquid interface. Obviously, if the thermal conductivities are different on the two sides, the temperature gradient will not be continuous.

where $\tilde{e}_{ij} = \frac{1}{2}(\tilde{u}_{i,j} + \tilde{u}_{j,i})$ is the rate of strain tensor and $\tilde{\Delta} = \tilde{u}_{i,i}$ is the dilatation. A schematic of a typical interface between two fluids is sketched in Fig. 1.2, where $\mathbf{n}$ is the unit normal going from fluid 1 to fluid 2. At any point on the interface the surface forces tangential to the boundary have to be continuous. If $\mathbf{t}$ is a vector tangential to the interface, then

$$\tilde{\sigma}_{ij}^{(1)} n_i t_j = \tilde{\sigma}_{ij}^{(2)} n_i t_j \implies \mu^{(1)} \tilde{e}_{ij}^{(1)} n_i t_j = \mu^{(2)} \tilde{e}_{ij}^{(2)} n_i t_j. \tag{1.21}$$

In this connection, the earlier footnote on the definition of the stress tensor is pertinent. Note that (1.21) will, in general, lead to two scalar boundary conditions on the interface. On the other hand, the surface forces normal to the boundary have to account for the surface energy of the interface. If the surface tension of the interface is σ_s, the Young–Laplace equation relates the jump in the normal stresses to the mean curvature $\tilde{\kappa} = \pm \mid \tilde{\nabla} \cdot \mathbf{n} \mid$,[4]

$$\begin{aligned}
&\tilde{\sigma}_{ij}^{(2)} n_i - \tilde{\sigma}_{ij}^{(1)} n_i \\
&\quad = -\sigma_s \tilde{\kappa} n_i \implies (\tilde{p}^{(1)} - \tilde{p}^{(2)}) n_i + 2(\mu^{(2)} \tilde{e}_{ij}^{(2)} - \mu^{(1)} \tilde{e}_{ij}^{(1)}) n_j \\
&\qquad + \left\{ \left(\zeta^{(2)} - \frac{2}{3}\mu^{(2)} \right) \tilde{\Delta}^{(2)} - \left(\zeta^{(1)} - \frac{2}{3}\mu^{(1)} \right) \tilde{\Delta}^{(1)} \right\} n_i \\
&\quad = -\sigma_s \tilde{\kappa} n_i.
\end{aligned} \tag{1.22}$$

This has to hold at every point on the interface. Here, the curvature is taken to be positive if the local center of curvature lies on the side of the interface to which $\mathbf{n}$ points. Thus the curvature at the point shown in Fig. 1.2 is negative.

The only case of a fluid–fluid interface that we will be considering is a gas–liquid interface above a liquid of constant density. In this case we note that, since the density and viscosity of a gas are very much smaller than those of a liquid, it is a good approximation to ignore all motions in the gas and to take the stress in it as due to a uniform static pressure $\tilde{p}_g$ alone. In this case, (1.21) and (1.22) become the *free surface* conditions

$$\tilde{e}_{ij} n_i t_j = 0, \tag{1.23a}$$

$$(\tilde{p} - \tilde{p}_g) n_i - 2\mu \tilde{e}_{ij} n_j = -\sigma_s \tilde{\kappa} n_i, \tag{1.23b}$$

for the liquid alone, where $\tilde{p}, \tilde{e}_{ij}$ and μ are quantities pertaining to the liquid. In this useful approximation the motion in the gas is ignored.

[4]One should be careful to note that different authors have definitions of the mean curvature that differ by a factor of 2. A compensating factor of 2 will then appear in the Young–Laplace equation to make everything consistent.

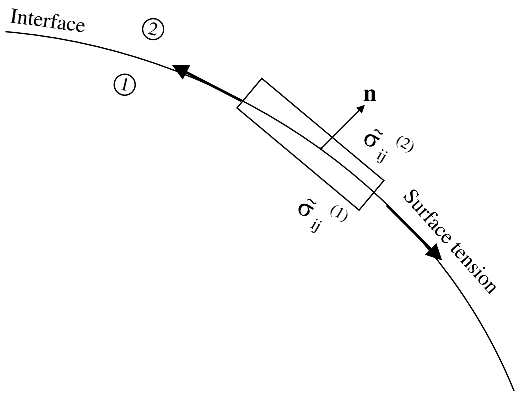

Fig. 1.2. The conditions at the interface between two fluids, fluids 1 and 2. The unit normal to the interface and the mean curvature of the interface at a point are $\mathbf{n}$ and $\tilde{\kappa}$ respectively.

1.3 Non-Dimensionalization: The Reynolds and Other Dimensionless Numbers

We will now shed some of the baggage that we have had to carry so far, particularly the tildes. Unless explicitly stated otherwise, we will assume that the fluid is of constant density $\tilde{\rho}_0$. The field equations are then the set of Eqs. (1.19a–d). To simplify matters let all the fluid properties be constant. We will now assume, in any given physical situation of interest, that there exists a natural length scale $\tilde{L}$, a natural velocity scale $\tilde{U}$, and a natural temperature scale $\tilde{T}_0$. We now define dimensionless independent and dependent variables as follows

$$\tilde{\mathbf{x}} = \tilde{L}\mathbf{x}, \tag{1.24a}$$

$$\tilde{t} = \frac{\tilde{L}}{\tilde{U}}t, \tag{1.24b}$$

$$\tilde{\mathbf{u}}(\tilde{\mathbf{x}}, \tilde{t}) = \tilde{U}\mathbf{u}(\mathbf{x}, t), \tag{1.24c}$$

$$\tilde{p}(\tilde{\mathbf{x}}, \tilde{t}) = \frac{\mu\tilde{U}}{\tilde{L}}p(\mathbf{x}, t), \tag{1.24d}$$

$$\tilde{\mathbf{F}}(\tilde{\mathbf{x}}, \tilde{t}) = \frac{\tilde{U}^2}{\tilde{L}}\mathbf{F}(\mathbf{x}, t), \tag{1.24e}$$

$$\tilde{T}(\tilde{\mathbf{x}}, \tilde{t}) = \tilde{T}_0 T(\mathbf{x}, t). \tag{1.24f}$$

Note that the tilde-less quantities $\mathbf{x}, t, \mathbf{u},\ p, \mathbf{F}$ and T are all dimensionless. If these forms are substituted into (1.19), we obtain the field equations in dimensionless form

$$\nabla \cdot \mathbf{u} = 0 \quad \text{or} \quad \frac{\partial u_i}{\partial x_i} = 0, \tag{1.25a}$$

$$\frac{Du_i}{Dt} = -\frac{1}{Re}\frac{\partial p}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 u_i}{\partial x_j^2} + F_i \quad \text{or} \quad \frac{D\mathbf{u}}{Dt} = -\frac{1}{Re}\nabla p + \frac{1}{Re}\nabla^2 \mathbf{u} + \mathbf{F}, \tag{1.25b}$$

$$\frac{DT}{Dt} = \frac{1}{Pr\,Re}\nabla^2 T + \frac{Ec}{Re}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j}, \tag{1.25c}$$

where

$$Re = \frac{\tilde{\rho}_0 \tilde{U} \tilde{L}}{\mu}, \quad Pr = \frac{\mu c_v}{k_c}, \quad Ec = \frac{\tilde{U}^2}{c_v \tilde{T}_0}, \tag{1.26a, b, c}$$

are dimensionless numbers which parameterize the flow. Re, the Reynolds number, is a measure of the relative importance of inertial forces to viscous forces in the field. Note that it appears in both the momentum and energy equations. On the other hand, Pr, a Prandtl number, and Ec, the Eckert number, appear only in the energy equation.

A matter of considerable importance for us is that for a constant density fluid, the one that we shall for the most part be considering, the momentum and energy equations uncouple. As can be seen from (1.25b), the temperature does not appear in this equation. The implication is that (1.25a) and (1.25b), the continuity and momentum equations, alone can be solved for $\mathbf{u}$ and p independently of (1.25c) which can then be subsequently solved for the temperature T. Thus in most of our work we will only be considering the velocity and pressure fields with the understanding that the temperature field can be determined later, if needed.

The body force field $\tilde{\mathbf{F}}$ will normally also involve a dimensionless group which will depend on the phenomenon present. For example, if a constant gravitational field is in the z-direction, $\tilde{\mathbf{F}} = g\mathbf{e}_z$ and so from (1.24e) $\mathbf{F} = g\tilde{L}/\tilde{U}^2\mathbf{e}_z$. The dimensionless group $Fr = \tilde{U}^2/g\tilde{L}$, the Froude number, is a measure of the importance of buoyancy in the field.

There is no uniqueness in the choice made in (1.24) in defining the dimensionless variables. Other choices can be made and, indeed, when we deal with mixed convection in Chap. 7, we will make a different choice, more suitable to our needs there. The choice of the most suitable non-dimensionalizing scales will often be dictated by the physical situation being studied. A point to be noted is that non-dimensionalization of the boundary conditions may also yield other dimensionless parameters that need to be considered. For example, when considering flow in a finite container, an aspect ratio or ratio of length scales will naturally arise.

Finally, the main advantage, apart from simplifying the notation, of working with dimensionless variables is that one can clearly identify the minimum set of parameters on which the field depends; this set will be much smaller than the one associated with the dimensional variables. We will throughout work with dimensionless quantities.

1.4 Slow Viscous or Low Reynolds Number Flows: The Stokes Equation

We will be concerned with fluid flows in which viscous forces predominate over the inertial forces. Physically this would imply in some sense high viscosity and/or low fluid speeds. This notion can be made precise if we examine the system of dimensionless field equations (1.25a, b) which we rewrite as follows:

$$\nabla \cdot \mathbf{u} = 0, \tag{1.27a}$$

$$Re\,\frac{D\mathbf{u}}{Dt} = -\nabla p + \nabla^2 \mathbf{u} + Re\,\mathbf{F}. \tag{1.27b}$$

First consider steady flows, ones in which the field quantities are independent of time. Clearly, the limit that we are interested in is one in which $Re \to 0$ and so a plausible approximation for the momentum equation appears to be

$$\mathbf{0} = -\nabla p + \nabla^2 \mathbf{u} + Re\,\mathbf{F}. \tag{1.28}$$

This is the Stokes equation where inertial forces are negligible and the viscous forces are balanced by the pressure and body forces. Often, $\mathbf{F}$ is absent or can be absorbed in the pressure, in which case the last term in (1.28) will be absent. We will at times call (1.28) and the continuity equation 'the Stokes system' or 'the Stokes equations'.

There are certain oscillatory flows, for example, when the lid of a container full of liquid oscillates in its own plane, where an unsteady form of the Stokes equation applies. Assume that $\mathbf{F}$ is absent or has been absorbed in the pressure. Let a frequency $\tilde{\Omega}$ and a velocity scale $\tilde{U}$ be associated with the motion; for example, $\tilde{\Omega}$ could be the frequency of the lid motion and $\tilde{U}$ its maximum linear speed. Then choose $\tilde{L}$ as the length scale, $\tilde{\Omega}^{-1}$ the time scale, $\tilde{U}$ the velocity scale and $\mu\tilde{U}/\tilde{L}$ the pressure scale. Substituting these into the momentum equation (1.19b), we obtain the dimensionless equation

$$\left[Re_{os}\frac{\partial u_i}{\partial t} + Re_{tr}\,u_j\frac{\partial u_i}{\partial x_j}\right] = -\frac{\partial p}{\partial x_i} + \frac{\partial^2 u_i}{\partial x_j^2}, \tag{1.29}$$

where

$$Re_{tr} = \frac{\tilde{\rho}_0\tilde{U}\tilde{L}}{\mu}, \quad Re_{os} = \frac{\tilde{\rho}_0\tilde{\Omega}\tilde{L}^2}{\mu} \qquad \text{(1.30a, b)}$$

are a translational Reynolds number and an oscillatory Reynolds number respectively. Now if $Re_{tr} \to 0$, irrespective of the value of Re_{os}, the limiting form of (1.29) becomes

$$Re_{os}\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} + \frac{\partial^2 u_i}{\partial x_j^2}. \tag{1.31}$$

Thus in this limit, it is meaningful to drop the nonlinear convective terms while retaining the linear unsteady term $u_{i,t}$. For a detailed discussion of the Stokes system, see Happel & Brenner (1965).

It is conceptually meaningful to consider the Stokes approximation as the leading approximation for the Navier–Stokes equation in the limit $Re \to 0$. One can then consider the possibility of obtaining the next terms in such an asymptotic analysis for $Re \to 0$. Van Dyke (1964) and the references cited therein may be consulted for the details of this classical view point. There is, however, a serious difficulty with such an asymptotic analysis when one considers unbounded, exterior flows past bodies: the Stokes approximation is not a uniformly valid approximation for large distances from a body. One can show that far from a body, no matter how small the Reynolds number, the inertial terms in the momentum equation cannot be neglected. One needs another approximation, the Oseen approximation, in the far field. For uniform flow in the x-direction, the Oseen equation may be written

$$Re\ \mathbf{u}_{,x} = -\nabla p + \nabla^2\mathbf{u}. \tag{1.32}$$

The derivation of this equation is left to Chap. 18 when we will have to deal with external flows.

The situation then is that we will use the Stokes system for the internal flows that we will consider; for these the Stokes equation is a valid approximation over the whole field. For external flows, where the Stokes equation is invalid in the far field, we will use the Oseen equation (1.32) which can be applied in a uniform manner over the whole field.

1.5 The Pressure and the Vorticity

It was seen in Sec. 1.1 that for a constant density fluid, the velocity and pressure fields uncouple from the temperature field and can, in principle, be solved independently of the temperature field. Moreover, as was discussed in Sec. 1.2, unless fluid–fluid interfaces are present, the boundary conditions involve the velocity alone; there are no boundary conditions on the pressure. Except in the case of two-dimensional Stokes flows, however, there is no further simplification: the velocity and pressure have to be solved together.

Although the velocity and pressure are coupled, there is a certain characteristic of the pressure field in Stokes flow, even unsteady Stokes flow, that must be noted. If one takes the divergence of (1.28) or (1.31), assuming that $\mathbf{F}$ is absent or has been absorbed into p, one concludes

$$\frac{\partial^2 p}{\partial x_i \partial x_i} = 0 \quad \text{or} \quad \nabla^2 p = 0, \tag{1.33}$$

or that $p(x, y, z)$ is a harmonic function in both steady and unsteady Stokes flow; note that continuity has been used here. It turns out that this special feature of the pressure cannot easily be used in the analysis of Stokes flows, principally because the boundary conditions are usually on the velocity.

There is another dependent variable that is harmonic in Stokes flows, the vorticity $\tilde{\boldsymbol{\omega}}(x, y, z)$. By definition, the vorticity in any fluid flow is the curl of the velocity vector

$$\tilde{\omega}_i = \epsilon_{ijk} \frac{\partial \tilde{u}_k}{\partial \tilde{x}_j} \quad \text{or} \quad \tilde{\boldsymbol{\omega}} = \tilde{\nabla} \times \tilde{\mathbf{u}}, \tag{1.34}$$

where ϵ_{ijk}, the alternating symbol, is 0 if two or three of the subscripts have the same value, is 1 if the subscripts are an even permutation of $1, 2$ and 3, and -1 if an odd permutation. Although the vorticity is not a fundamental field variable or a quantity that is easily measured, it plays an important conceptual role in the analysis of high Reynolds number flows and in certain schemes to compute them. One can obtain an equation for the transport of vorticity by taking the curl of the momentum equation (Batchelor 1967)

$$\frac{\tilde{D}\tilde{\boldsymbol{\omega}}}{\tilde{D}\tilde{t}} = (\tilde{\boldsymbol{\omega}} \cdot \tilde{\nabla})\tilde{\mathbf{u}} + \frac{\mu}{\rho}\tilde{\nabla}^2\tilde{\boldsymbol{\omega}}, \tag{1.35}$$

where $\tilde{D}/\tilde{D}\tilde{t}$ is the material derivative defined by (1.16). This equation is useful in the analysis of high Reynolds number flows, especially turbulent flows where the vortex stretching term, the first term on the right, plays a unique role. Vorticity is important even in laminar high Reynolds number flows because, normally, vorticity can only be generated at solid surfaces and then be shed into wakes. Thus at high Reynolds numbers the vorticity, which is primarily confined to narrow layers near solid surfaces or in narrow wakes behind bodies, is much more compact than the velocity field itself. This fact can at times be exploited; certainly it is conceptually useful.

In low Reynolds number flows, however, the situation is quite different. If (1.35) is linearized for $Re \to 0$, it is immediately obvious that the vorticity will satisfy an unsteady diffusion equation which will reduce to Laplace's equation in the steady case. In fact, taking the curl of the dimensionless unsteady Stokes equation (1.31), we find

$$\frac{\partial \omega_i}{\partial t} = \frac{1}{Re_{os}}\frac{\partial^2 \omega_i}{\partial x_j^2}, \tag{1.36}$$

which for steady flows reduces to

$$\frac{\partial^2 \omega_i}{\partial x_j^2} = 0 \quad \text{or} \quad \nabla^2 \boldsymbol{\omega} = \mathbf{0}. \tag{1.37}$$

Thus the vorticity is also harmonic in steady Stokes flow. A consequence is that at low Reynolds numbers, the vorticity is no more compact than the velocity and so it does not seem to play an especially helpful role in the analysis of such flows. We will only occasionally examine the vorticity in the fields that we will study.

The fact that both the pressure and the vorticity are harmonic in Stokes flow opens up the possibility that they may play complementary roles in planar flows. Consider a steady, plane, incompressible low Reynolds number flow governed by (1.28) with the body force absorbed in the pressure. Let the velocity field be given by $\mathbf{u}(x, y, z) = (u(x, z), 0, w(x, z))$, where u and w are the x- and z-components of velocity and where all field variables are independent of the y-coordinate. In this case the the vorticity has only one component, the y-component, and we can write $\boldsymbol{\omega} = (0, \omega(x, z), 0)$ with

$$\omega = u_{,z} - w_{,x}. \tag{1.38}$$

From this we deduce

$$\omega_{,x} = u_{,zx} - w_{,xx} = -\nabla^2 w = -p_{,z}, \tag{1.39a}$$

$$\omega_{,z} = u_{,zz} - w_{,xz} = \nabla^2 u = p_{,x}, \tag{1.39b}$$

where the continuity equation has been used in both (a) and (b). But (1.39a, b) show that p and ω satisfy the Cauchy–Riemann equations and hence are conjugate harmonic functions. In other words, they can be regarded as the real and imaginary parts of an analytic function of the complex variable $t = x + \mathrm{i}z$. This leads to the possibility of using the powerful tools of complex analysis to solve problems of Stokes flows. Unfortunately, the normal no-slip conditions are on the velocity, not on the pressure or vorticity and this fact restricts the use of complex variable methods to a small, very special class of problems.

1.6 Streamlines and the Stream Function

Although the equations governing slow viscous flows are linear, they are not only difficult to solve in any generality, the fields that they yield are at times so intricate that they challenge normal physical intuition. In these circumstances, especially when dealing with three-dimensional flows, it proves to be a great help to visualize the flow fields. This is where the concept of the *streamline* is indispensible. Informally, we can think of a streamline in the flow field as being a line such that at every point on it, the local tangent to the streamline points in the direction of the local velocity vector [Fig. 1.3(a)]. A more precise definition is that streamlines are trajectories of the velocity field, i.e. are solutions of the set of three simultaneous differential equations,

$$\frac{d\mathbf{x}}{ds} = \mathbf{u}(\mathbf{x}, t) \quad \text{or} \quad \frac{dx_i}{ds} = u_i(\mathbf{x}, t), \tag{1.40}$$

where s is a parameter along the streamline. It is important to note that streamlines are defined in both three-dimensional and unsteady flows. If the flow is not steady in time, it is normal to talk of the streamlines being 'unsteady streamlines'.

The reason that streamlines are useful objects is that they give us a physical feel for what the flow field is doing. In a steady flow, the streamlines are the lines along which small particles or dye, released in the fluid, would move. They can indicate where the fluid is accelerating or decelerating, where eddies or recirculating regions are formed, where flow separation

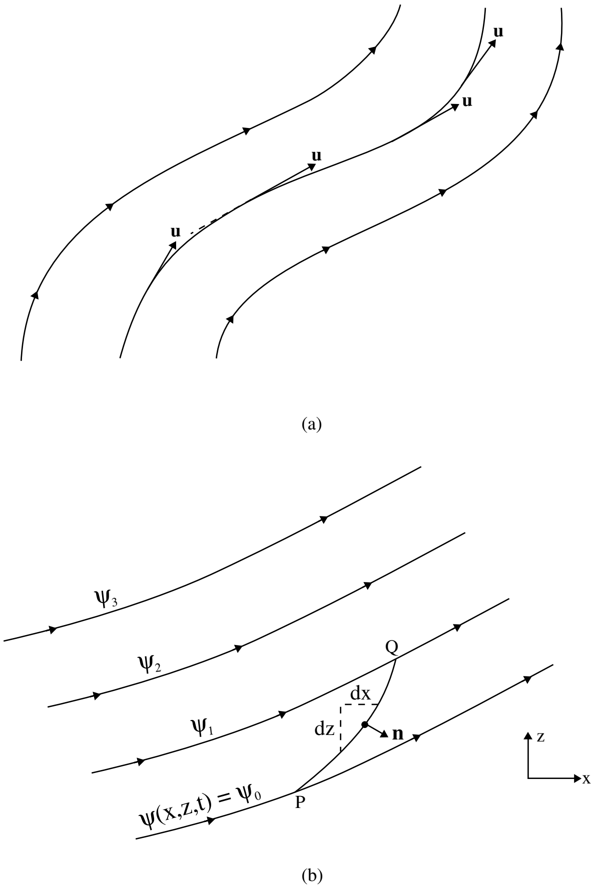

Fig. 1.3. (a) Streamlines in a general three-dimensional flow field. At every point on a streamline, the local tangent is in the direction of the local velocity vector $\mathbf{u}(\mathbf{x}, t)$. (b) The stream function $\psi(x, z, t)$ in planar flow. The instantaneous volumetric flux per unit depth between two streamlines $\psi_1(x, z, t)$ and $\psi_0(x, z, t)$ is just $\psi_1 - \psi_0$. Note that ψ is constant on each streamline.

occurs, etc. Often, a single glance at the streamline pattern of a field tells us more than a formula or a set of numerical values.

In two-dimensional flows, streamlines are associated with a field quantity, the *stream function* ψ. Consider a planar, constant density flow where the velocity field $\mathbf{u} = (u(x, z, t), w(x, z, t))$ is independent of the transverse y-coordinate. Then the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{1.41}$$

suggests the existence of a scalar, the stream function $\psi(x, z, t)$ such that

$$u(x, z, t) = \frac{\partial \psi}{\partial z}, \quad w(x, z, t) = -\frac{\partial \psi}{\partial x}. \tag{1.42}$$

Note that with the velocity related in this way, the continuity equation (1.41) is automatically satisfied. There are two advantages to working with the stream function ψ, rather than with the velocity components (u, w). First, one need only deal with a single scalar rather than with a vector field. Second, it can be shown that the lines of constant ψ are the streamlines. As a consequence the streamlines in a planar flow can be determined by just finding the lines on which ψ is constant.

Let $\psi(x, z, t) = \psi_0$ be a line on which the stream function is constant at some instant of time. At a point (x, z) on this line, a normal at that point is given by $(\psi_{,x}, \psi_{,z})$. Therefore, a vector tangent to the line is $(\psi_{,z}, -\psi_{,x})$. But this is just (u, w); thus the line $\psi(x, z, t) = \psi_0$ is an instantaneous streamline at time t. There is another important property of the stream function which is illustrated in Fig. 1.3(b). Consider $\mathcal{V}$ the volumetric flux, per unit depth in the y-direction, across any curve PQ connecting the point P on $\psi = \psi_0$ to the point Q on $\psi = \psi_1$. $\mathcal{V}$ is the volumetric flow per unit time per unit depth into the page across the curve PQ. Then, if $\mathbf{n}$ is the unit normal to the curve at any point,

$$\begin{aligned} \mathcal{V} &= \int_P^Q \mathbf{u} \cdot \mathbf{n} ds = \int_P^Q [u dz - w dx] = \int_P^Q [\psi_{,z} dz + \psi_{,x} dx] \\ &= \int_P^Q d\psi = \psi_Q - \psi_P = \psi_1 - \psi_0. \end{aligned} \tag{1.43}$$

Thus the volumetric flow between any two points is just equal to the difference between the two stream function values. And as must be, the volumetric flow between any two streamlines is constant.

These features are very helpful in the analysis of planar flows. Similar features exist for any two-dimensional flow. For example, for axially symmetric flows the continuity equation is automatically satisfied by the use of the *Stokes stream function* Ψ. In cylindrical polar coordinates the velocity field is given by

$$u_z = \frac{1}{r}\frac{\partial \Psi}{\partial r}, \quad u_r = -\frac{1}{r}\frac{\partial \Psi}{\partial z}, \tag{1.44}$$

while in spherical polar coordinates the velocity is given by

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial \Psi}{\partial \theta}, \quad u_\theta = -\frac{1}{r \sin\theta}\frac{\partial \Psi}{\partial r}. \tag{1.45}$$

Although the Stokes stream function is useful in dealing with problems of axial symmetry, the $1/r$ factors in the above formulae make the physical interpretation of the stream function less immediate.

In three-dimensional flows we will have no choice but to deal with the full vector velocity field. This naturally makes the analysis much more involved. Moreover, there are phenomena in three-dimensional flows which have no two-dimensional counterparts.