Preface

Block operator matrices are matrices the entries of which are linear operators between Banach or Hilbert spaces. They arise in various areas of mathematics and its applications: in systems theory as Hamiltonians (see [CZ95]), in the discretization of partial differential equations as large partitioned matrices due to sparsity patterns (see [SAD$^+$00]), in saddle point problems in non-linear analysis (see [BGL05]), in evolution problems as linearizations of second order Cauchy problems (see [EN00]), and as linear operators describing coupled systems of partial differential equations. Such systems occur widely in mathematical physics, *e.g.* in fluid mechanics (see [Cha61]), magnetohydrodynamics (see [Lif89]), and quantum mechanics (see [Tha92]). In all these applications, the spectral properties of the corresponding block operator matrices are of vital importance as they govern for instance the time evolution and hence the stability of the underlying physical systems.

The aim of this book is to present a wide panorama of methods to investigate the spectral properties of block operator matrices. Particular emphasis is placed on classes of block operator matrices to which standard operator theoretical methods do not readily apply: non-self-adjoint block operator matrices, block operator matrices with unbounded entries, non-semi-bounded block operator matrices, and classes of block operator matrices arising in mathematical physics. The main topics include:

- localization of the spectrum and investigation of its structure,
- description of the essential spectrum,
- characterization and estimates of eigenvalues,
- block diagonalization and invariant subspaces,
- solutions of algebraic Riccati equations,
- applications to concrete problems from mathematical physics.

We shall address these problems using the particular block structure of the operators and the properties of their operator entries. The methods employed come from a number of different areas:

- the theory of linear operators (numerical ranges, perturbation theory),
- classical functional analysis (fixed point theorems),
- complex analysis (analytic operator functions, factorization theorems).

The book gives an account on recent research on the spectral theory of block operator matrices and its applications in mathematical physics. It contains results which were published roughly during the last 15 years. A number of theorems, however, was found while this book was written and are still in the process of being published. The three chapters may be of interest for different groups of readers: Chapter 1 deals exclusively with bounded block operator matrices and contains methods and results which are of interest even in the matrix case. Chapter 2, which may be read independently of Chapter 1, is focused on unbounded block operator matrices and is particularly suited for applications to differential operators. Chapter 3 contains applications of the results of Chapter 2 to various spectral problems from mathematical physics.

No particular focus is placed on block operator matrices arising in systems theory and in evolution equations. Although some of the methods presented may be applicable, it seems to be impossible for a single author and a single book to cover also the vast range of results in these two important areas; nevertheless, certain points of intersection will be mentioned.

This book could not have be written without the contributions and help of many people. Sincere thanks go to my coauthors and friends Vadim Adamjan, Margarita Kraus, Heinz Langer, Matthias Langer, Alexander Markus, Marco Marletta, and Volodya Matsaev; my teacher Reinhard Mennicken; my former PhD students Markus Wagenhofer (with special thanks for providing the beautiful figures and the cover), Monika Winklmeier, and Christian Wyss; my present PhD students Jean-Claude Cuenin and Jan Nesemann; my colleague Alexander Motovilov; Deutsche Forschungsgemeinschaft (DFG, Germany) and Engineering and Physical Sciences Research Council (EPSRC, UK) for their most valuable funding; and, finally, to Zhang Ji and Jessie Tan from World Scientific for all their support and patience.

Bern, July 2008 Christiane Tretter