Chapter 1

Bounded Block Operator Matrices

A block operator matrix is a matrix the entries of which are linear operators. Every bounded linear operator can be written as a block operator matrix if the space in which it acts is decomposed in two or more components. In this chapter we present methods that allow us to use information on the entries in such a representation to investigate the spectral properties of the given operator. The key tool here is the quadratic numerical range or, more generally, the block numerical range. Our main results include a spectral inclusion theorem, an estimate of the resolvent in terms of the quadratic numerical range, factorization theorems for the Schur complements, and a theorem about angular operator representations of spectral invariant subspaces; the latter implies *e.g.* the existence of solutions of the corresponding Riccati equations and a block diagonalization. Many of the results are also of interest for partitioned matrices.

1.1 The quadratic numerical range

The numerical range is an important tool in the spectral analysis of bounded and unbounded linear operators in Hilbert spaces. We begin by collecting some if its useful properties (see *e.g.* [Hal82], [GR97], [Kat95], and [Ber62]).

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{A}$ be a bounded linear operator in $\mathcal{H}$. Then the numerical range of $\mathcal{A}$ is the set

$$W(\mathcal{A}) := \big\{ (\mathcal{A}\mathbf{x}, \mathbf{x}) : \mathbf{x} \in S_{\mathcal{H}} \big\}$$

where $S_{\mathcal{H}} := \{\mathbf{x} \in \mathcal{H} : \|\mathbf{x}\| = 1\}$ is the unit sphere in $\mathcal{H}$. By the well-known Toeplitz-Hausdorff theorem, the numerical range is a convex subset of $\mathbb{C}$ and it satisfies the so-called *spectral inclusion property*

$$\sigma_{\mathrm{p}}(\mathcal{A}) \subset W(\mathcal{A}), \quad \sigma(\mathcal{A}) \subset \overline{W(\mathcal{A})} \tag{1.1.1}$$

for the point spectrum $\sigma_{\mathrm{p}}(\mathcal{A})$ (or set of eigenvalues) and the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$; note that $W(\mathcal{A})$ is closed if $\dim \mathcal{H} < \infty$. Further, the resolvent of $\mathcal{A}$ can be estimated in terms of the distance to the numerical range,

$$\left\|(\mathcal{A}-\lambda)^{-1}\right\| \leq \frac{1}{\operatorname{dist}\big(\lambda, W(\mathcal{A})\big)}, \quad \lambda \notin \overline{W(\mathcal{A})}. \tag{1.1.2}$$

If a point $\lambda \in \overline{W(\mathcal{A})}$ is a corner of the numerical range (*i.e.* $W(\mathcal{A})$ lies in a sector with vertex λ and angle less than π), then $\lambda \in \sigma(\mathcal{A})$; if, in addition, $\lambda \in W(\mathcal{A})$, then $\lambda \in \sigma_{\mathrm{p}}(\mathcal{A})$. The estimate (1.1.2) implies that if $\lambda \in \sigma_{\mathrm{p}}(\mathcal{A})$ is a boundary point of $W(\mathcal{A})$, then there are no associated vectors at λ.

If the Hilbert space $\mathcal{H}$ is the product of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, then every bounded linear operator $\mathcal{A} \in L(\mathcal{H})$ has a block operator matrix representation

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.1.3}$$

with bounded linear operators $A \in L(\mathcal{H}_1)$, $B \in L(\mathcal{H}_2, \mathcal{H}_1)$, $C \in L(\mathcal{H}_1, \mathcal{H}_2)$, and $D \in L(\mathcal{H}_2)$. The following generalization of the numerical range of $\mathcal{A}$ takes into account the block structure (1.1.3) of $\mathcal{A}$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.

Definition 1.1.1 For $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$ we define the 2×2 matrix

$$\mathcal{A}_{f,g} := \begin{pmatrix} (Af, f) & (Bg, f) \\ (Cf, g) & (Dg, g) \end{pmatrix} \in M_2(\mathbb{C}). \tag{1.1.4}$$

Then the set

$$W^2(\mathcal{A}) \; := \bigcup_{f \in S_{\mathcal{H}_1}, g \in S_{\mathcal{H}_2}} \sigma_{\mathrm{p}}(\mathcal{A}_{f,g}) \tag{1.1.5}$$

is called the *quadratic numerical range* of $\mathcal{A}$ (with respect to the block operator matrix representation (1.1.3)).

For two different decompositions of the Hilbert space $\mathcal{H}$, the corresponding quadratic numerical ranges may differ considerably:

Example 1.1.2 The quadratic numerical ranges of the 4×4 matrix

$$\mathcal{A}_0 := \begin{pmatrix} -2 & -1 & 1 & 0 \\ -1 & -2 & 0 & 1 \\ -2 & -1 & 0 & -3\mathrm{i} \\ -1 & -2 & 3\mathrm{i} & 0 \end{pmatrix}$$

with respect to the two decompositions $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$ and $\mathbb{C}^4 = \mathbb{C}^3 \oplus \mathbb{C}^1$ are shown in Fig. 1.1; the black dots mark the eigenvalues of $\mathcal{A}_0$.

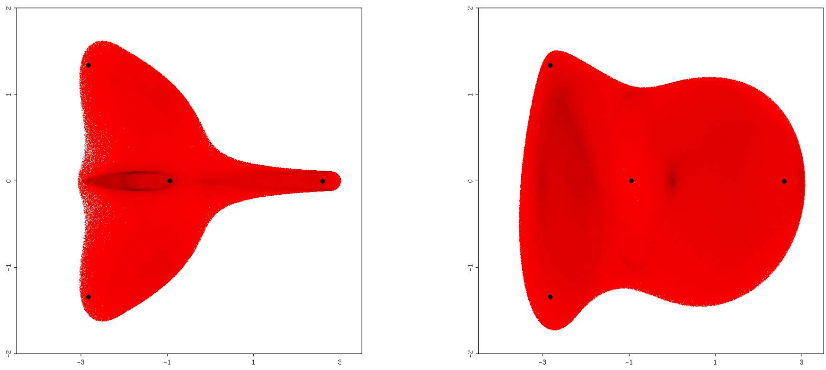

Figure 1.1 Quadratic numerical ranges of $\mathcal{A}_0$ for $\mathbb{C}^4=\mathbb{C}^2\oplus\mathbb{C}^2$ and $\mathbb{C}^4=\mathbb{C}^3\oplus\mathbb{C}^1$.

Sometimes it is more convenient to use an equivalent description of the quadratic numerical range which uses non-zero elements f, g that need not have norm one.

Proposition 1.1.3 *For $f\in\mathcal{H}_1$, $g\in\mathcal{H}_2$, f, $g\neq 0$, we define*

$$\mathcal{A}_{f,g} := \begin{pmatrix} \dfrac{(Af,f)}{\|f\|^2} & \dfrac{(Bg,f)}{\|f\|\,\|g\|} \\ \dfrac{(Cf,g)}{\|f\|\,\|g\|} & \dfrac{(Dg,g)}{\|g\|^2} \end{pmatrix} \in M_2(\mathbb{C}) \tag{1.1.6}$$

and

$$\Delta(f,g;\lambda) := \det\begin{pmatrix} (Af,f)-\lambda(f,f) & (Bg,f) \\ (Cf,g) & (Dg,g)-\lambda(g,g) \end{pmatrix}. \tag{1.1.7}$$

Then

$$\begin{aligned} W^2(\mathcal{A}) &= \bigcup_{\substack{f\in\mathcal{H}_1,g\in\mathcal{H}_2\\ f,g\neq 0}} \sigma_{\mathrm{p}}(\mathcal{A}_{f,g}) \\ &= \big\{\lambda\in\mathbb{C} : \exists\, f\in\mathcal{H}_1,\ g\in\mathcal{H}_2,\ f,g\neq 0\ \ \det\big(\mathcal{A}_{f,g}-\lambda\big)=0\big\} \\ &= \big\{\lambda\in\mathbb{C} : \exists\, f\in\mathcal{H}_1,\ g\in\mathcal{H}_2,\ f,g\neq 0\ \ \Delta(f,g;\lambda)=0\big\}. \end{aligned}$$

Proof. The claims are immediate if we observe that the definition of $\mathcal{A}_{f,g}$ in (1.1.6) coincides with the one in (1.1.4) if $f\in S_{\mathcal{H}_1}$, $g\in S_{\mathcal{H}_2}$, *i.e.* $\|f\|=\|g\|=1$, and that $\Delta(f,g;\lambda)=\|f\|^2\|g\|^2\det\big(\mathcal{A}_{f,g}-\lambda\big)$. □

In the special case that $W^2(\mathcal{A})$ is real, it can also be described by means of the formulae for the roots of the quadratic equation $\det\big(\mathcal{A}_{f,g}-\lambda\big)=0$.

In the following, we choose a branch of the square root such that $\sqrt{z} \geq 0$ if $z \geq 0$ and $\operatorname{Im}\sqrt{z} > 0$ if $z < 0$.

Corollary 1.1.4 *For $f \in \mathcal{H}_1$, $g \in \mathcal{H}_2$, $f, g \neq 0$, we define*

$$\operatorname{dis}_{\mathcal{A}}(f,g) := \left(\frac{(Af,f)}{\|f\|^2} - \frac{(Dg,g)}{\|g\|^2} \right)^2 + 4\frac{(Bg,f)(Cf,g)}{\|f\|^2\,\|g\|^2}$$

and, if $\operatorname{dis}_{\mathcal{A}}(f,g) \geq 0$, we set

$$\lambda_\pm\begin{pmatrix} f \\ g \end{pmatrix} := \frac{1}{2}\left(\frac{(Af,f)}{\|f\|^2} + \frac{(Dg,g)}{\|g\|^2} \pm \sqrt{\left(\frac{(Af,f)}{\|f\|^2} - \frac{(Dg,g)}{\|g\|^2} \right)^2 + 4\frac{(Bg,f)(Cf,g)}{\|f\|^2\,\|g\|^2}} \right).$$

Further we let

$$\Lambda_\pm(\mathcal{A}) := \left\{ \lambda_\pm\begin{pmatrix} f \\ g \end{pmatrix} : \ f \in \mathcal{H}_1, g \in \mathcal{H}_2, \ f, g \neq 0, \operatorname{dis}_{\mathcal{A}}(f,g) \geq 0 \right\}. \quad (1.1.8)$$

Then $W^2(\mathcal{A}) \subset \mathbb{R}$ if and only if $\operatorname{dis}_{\mathcal{A}}(f,g) \geq 0$ for all $f \in \mathcal{H}_1$, $g \in \mathcal{H}_2$, $f, g \neq 0$, and in this case

$$W^2(\mathcal{A}) = \Lambda_-(\mathcal{A}) \cup \Lambda_+(\mathcal{A}).$$

For convenience, we also use the notation $\lambda_\pm(f,g)$ in the following.

Proof. The claim is immediate from the fact that $\lambda_\pm(f,g)$ are the solutions of the quadratic equation

$$\lambda^2 - \lambda\left(\frac{(Af,f)}{\|f\|^2} + \frac{(Dg,g)}{\|g\|^2} \right) + \frac{(Af,f)(Dg,g)}{\|f\|^2\|g\|^2} - \frac{(Bg,f)(Cf,g)}{\|f\|^2\|g\|^2} = 0, \quad (1.1.9)$$

that is, of $\det\big(\mathcal{A}_{f,g} - \lambda\big) = 0$. □

Like the numerical range, the quadratic numerical range of a bounded block operator matrix $\mathcal{A}$ is a bounded subset of $\mathbb{C}$,

$$W^2(\mathcal{A}) \subset \big\{ \lambda \in \mathbb{C} : \ |\lambda| \leq \|\mathcal{A}\| \big\},$$

and it is closed if $\dim \mathcal{H} < \infty$. In contrast to the numerical range, it consists of at most two (connected) components. This follows from the fact that the set of all matrices $\mathcal{A}_{f,g}$, $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$, is connected and from a continuity argument for the eigenvalues of matrices (see [Kat95, Theorem II.5.14] and [Wag00]). If, for example, $\mathcal{A}$ is upper or lower block triangular, then $W^2(\mathcal{A}) = W(A) \cup W(D)$. Hence the quadratic numerical range is, in general, not convex; the following example shows that even its components need not be so (see Fig. 1.2).

Example 1.1.5 Consider the 4×4 matrices

$$\mathcal{A}_1 := \left(\begin{array}{cc|cc} 1 & 0 & 1 & \mathrm{i} \\ 0 & 1 & 0 & 1 \\ \hline \mathrm{i} & 0 & 0 & 0 \\ 1 & \mathrm{i} & 0 & 0 \end{array}\right), \quad \mathcal{A}_2 := \left(\begin{array}{cc|cc} 2 & \mathrm{i} & 1 & 3+\mathrm{i} \\ \mathrm{i} & 2 & 3+\mathrm{i} & 1 \\ \hline 1 & 3+\mathrm{i} & -2 & \mathrm{i} \\ 3+\mathrm{i} & 1 & \mathrm{i} & -2 \end{array}\right)$$

with respect to the decomposition $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$. Figure 1.2 shows that in both cases the quadratic numerical range consists of two disjoint non-convex components.

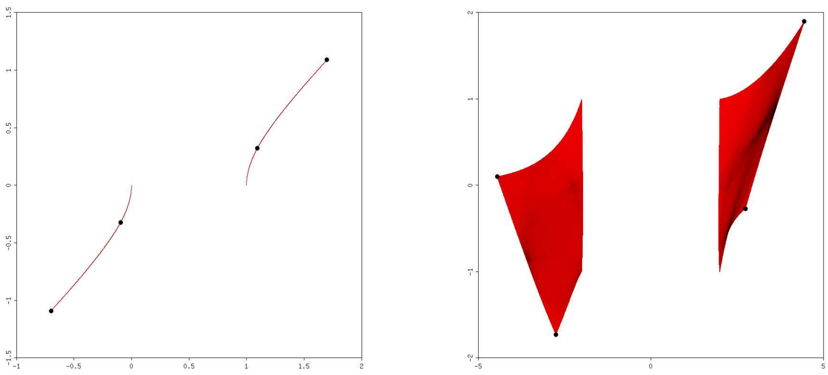

Figure 1.2 Quadratic numerical ranges of $\mathcal{A}_1$ and $\mathcal{A}_2$.

Remark 1.1.6 The fact that all matrices $\mathcal{A}_{f,g}$, $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$, have two different eigenvalues does not imply that $W^2(\mathcal{A})$ consists of two disjoint components.

In fact, there exist self-adjoint block operator matrices $\mathcal{A}$ such that for all $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$ the two eigenvalues $\lambda_+(f,g)$, $\lambda_-(f,g)$ of the matrix $\mathcal{A}_{f,g}$ are different, that is, $\lambda_-(f,g) < \lambda_+(f,g)$, but there exist $f, f' \in S_{\mathcal{H}_1}$, $g, g' \in S_{\mathcal{H}_2}$ such that $\lambda_-(f,g) = \lambda_+(f',g')$. To this end, consider a self-adjoint block operator matrix $\mathcal{A}$ as in (1.1.3) with $\mathcal{H}_1 = \mathcal{H}_2$, $\dim \mathcal{H}_1 \geq 2$, $C = B^*$, and self-adjoint operators A, D such that $\min W(A) = \max W(D) = \beta$. Assume further that β is a simple eigenvalue of A and D with a common eigenvector $f_0 \in \mathcal{H}_1$, $\|f_0\| = 1$, and $(Bf_0, f_0) \neq 0$. Then it is easy to see that for $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$

$$\lambda_+(f,g) - \lambda_-(f,g) = 2\sqrt{\left(\frac{(Af,f) - (Dg,g)}{2}\right)^2 + |(Bg,f)|^2} > 0.$$

On the other hand, if we choose $f_0' \in \mathcal{H}_1$, $g_0' \in \mathcal{H}_1$ so that $\|f_0'\| = \|g_0'\| = 1$, $(Bf_0, f_0') = 0$ and $(B^* f_0, g_0') = 0$, then, by the definition of $\lambda_\pm$ in Corollary 1.1.4, we have $\lambda_+(f_0, g_0') = \lambda_-(f_0', f_0) = \beta$.

The following elementary properties of the quadratic numerical range with respect to certain transformations of the block operator matrix $\mathcal{A}$ are easy to check.

Proposition 1.1.7 *We have*

i) $W^2(\alpha\mathcal{A} + \beta) = \alpha W^2(\mathcal{A}) + \beta$ *for* $\alpha, \beta \in \mathbb{C}$,
ii) $W^2(U^{-1}\mathcal{A}U) = W^2(\mathcal{A})$ *for* $U = \operatorname{diag}\big(U_1, U_2\big)$, $U_1 \in L(\mathcal{H}_1)$, $U_2 \in L(\mathcal{H}_2)$ *unitary.*

Proof. Claim i) follows from the fact that $(\alpha\mathcal{A} + \beta)_{f,g} = \alpha\mathcal{A}_{f,g} + \beta$ for $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$. The assertion in ii) is a consequence of the equivalence $(f\ \ g)^{\mathrm{t}} \in S_{\mathcal{H}_1} \oplus S_{\mathcal{H}_2} \iff (U_1 f\ \ U_2 g)^{\mathrm{t}} \in S_{\mathcal{H}_1} \oplus S_{\mathcal{H}_2}$ and of the relation $(U^{-1}\mathcal{A}U)_{f,g} = \mathcal{A}_{U_1 f, U_2 g}$ for $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$. □

The first non-trivial property of the quadratic numerical range is that it is contained in the numerical range.

Theorem 1.1.8 $W^2(\mathcal{A}) \subset W(\mathcal{A})$.

Proof. Let $\lambda_0 \in W^2(\mathcal{A})$. Then, by definition (1.1.5), there exist $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$, and $(\alpha_1\ \ \alpha_2)^{\mathrm{t}} \in \mathbb{C}^2$, $|\alpha_1|^2 + |\alpha_2|^2 = 1$, such that

$$\mathcal{A}_{f,g}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \lambda_0\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}.$$

Taking the scalar product with $(\alpha_1\ \ \alpha_2)^{\mathrm{t}}$ and observing the definition of $\mathcal{A}_{f,g}$ in (1.1.4), we obtain

$$\left(\mathcal{A}\begin{pmatrix}\alpha_1 f\\ \alpha_2 g\end{pmatrix}, \begin{pmatrix}\alpha_1 f\\ \alpha_2 g\end{pmatrix}\right) = \lambda_0.$$

Since $\|\alpha_1 f\|^2 + \|\alpha_2 g\|^2 = 1$, this implies that $\lambda_0 \in W(\mathcal{A})$. □

Another feature of the quadratic numerical range is that the numerical ranges of the diagonal elements $W(A)$ and $W(D)$ are contained in $W^2(\mathcal{A})$ if the dimensions of $\mathcal{H}_1$ and $\mathcal{H}_2$ are at least two, more exactly:

Theorem 1.1.9 *We have*

i) $\dim \mathcal{H}_2 \geq 2 \implies W(A) \subset W^2(\mathcal{A})$,
ii) $\dim \mathcal{H}_1 \geq 2 \implies W(D) \subset W^2(\mathcal{A})$.

Proof. Let $f \in S_{\mathcal{H}_1}$ be arbitrary. If $\dim \mathcal{H}_2 \geq 2$, then there exists a $g \in S_{\mathcal{H}_2}$ such that $(Cf, g) = 0$. Thus

$$\mathcal{A}_{f,g} = \begin{pmatrix} (Af, f) & (Bg, f) \\ 0 & (Dg, g) \end{pmatrix}$$

and hence $(Af, f) \in \sigma_{\mathrm{p}}(\mathcal{A}_{f,g}) \subset W^2(\mathcal{A})$. The proof for $W(D)$ is similar. □

Corollary 1.1.10 *Suppose that* $\dim \mathcal{H}_1 \geq 2$ *and* $\dim \mathcal{H}_2 \geq 2$.

i) *If* $W^2(\mathcal{A})$ *consists of two disjoint components,* $W^2(\mathcal{A}) = \mathcal{F}_1 \,\dot{\cup}\, \mathcal{F}_2$*, they can be enumerated such that*

$$W(A) \subset \mathcal{F}_1, \quad W(D) \subset \mathcal{F}_2.$$

ii) *If* $W(A) \cap W(D) \neq \emptyset$*, then* $W^2(\mathcal{A})$ *consists of only one component.*

Proof. By the assumptions on the dimensions of $\mathcal{H}_1$ and $\mathcal{H}_2$, there exist $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$ with $(Cf, g) = 0$. The eigenvalues of the corresponding matrix $\mathcal{A}_{f,g}$ are (Af, f) and (Dg, g); they belong to different components of $W^2(\mathcal{A})$ if the latter consists of two disjoint components. Theorem 1.1.9 and the fact that the numerical ranges $W(A)$ and $W(D)$ are connected (even convex) now imply claims i) and ii). □

The inclusions in Theorem 1.1.9 need not be true if $\dim \mathcal{H}_1 = 1$ or $\dim \mathcal{H}_2 = 1$:

Example 1.1.11 Consider the 4×4 matrix from Example 1.1.2 with respect to the decomposition $\mathbb{C}^4 = \mathbb{C}^3 \oplus \mathbb{C}^1$,

$$\mathcal{A}_0 := \left(\begin{array}{ccc|c} -2 & -1 & 1 & 0 \\ -1 & -2 & 0 & 1 \\ -2 & -1 & 0 & -3\mathrm{i} \\ \hline -1 & -2 & 3\mathrm{i} & 0 \end{array}\right).$$

Figure 1.3 illustrates that the numerical range of the left upper corner of $\mathcal{A}_0$ is not contained in $W^2(\mathcal{A})$.

The property that $W^2(\mathcal{A})$ (or even its closure $\overline{W^2(\mathcal{A})}$) consists of two disjoint components will be of particular interest in the following sections. In this respect, the following results are useful.

Proposition 1.1.12 *If* $\overline{W(A)} \cap \overline{W(D)} = \emptyset$ *and*

$$2\sqrt{\|B\|\,\|C\|} < \operatorname{dist}\big(W(A), W(D)\big),$$

then $\overline{W^2(\mathcal{A})}$ *consists of two disjoint components.*

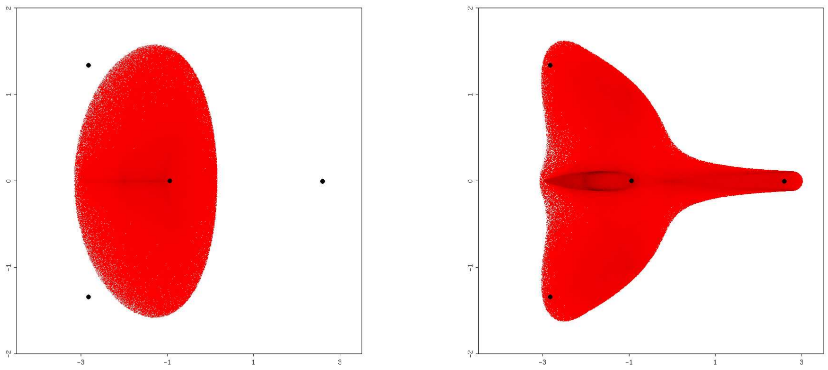

Figure 1.3 Numerical range of left upper corner and quadratic numerical range of $\mathcal{A}_0$.

Proof. Set $\beta := \operatorname{dist}\big(W(A), W(D)\big)$ and assume that λ belongs to the line that separates the convex sets $W(A)$ and $W(D)$ and has distance $\beta/2$ to both of them. Then, for all $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$,

$$\begin{aligned}\big|\det(\mathcal{A}_{f,g} - \lambda)\big| &= \big|\big(\lambda - (Af,f)\big)\big(\lambda - (Dg,g)\big) - (Bg,f)(Cf,g)\big| \\ &\geq \big|\lambda - |(Af,f)|\big|\,\big|\lambda - |(Dg,g)|\big| - \|B\|\,\|C\| \\ &\geq \frac{\beta^2}{4} - \|B\|\,\|C\| > 0,\end{aligned}$$

which shows that $\lambda \notin W^2(\mathcal{A})$. □

The numerical range $W(\mathcal{A})$ of a bounded linear operator $\mathcal{A}$ is real if and only if $\mathcal{A}$ is self-adjoint. For the quadratic numerical range, we only have the following property.

Proposition 1.1.13 *If $\mathcal{A}^*$ denotes the adjoint of $\mathcal{A}$, then*

i) $W^2(\mathcal{A}^*) = \big\{\overline{\lambda} \in \mathbb{C} : \lambda \in W^2(\mathcal{A})\big\} =: W^2(\mathcal{A})^*$,
ii) $\mathcal{A} = \mathcal{A}^* \implies W^2(\mathcal{A}) \subset \mathbb{R}$.

Proof. Assertion i) follows from $(\mathcal{A}_{f,g})^* = (\mathcal{A}^*)_{f,g}$ for $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$; claim ii) is obvious since in this case all matrices $\mathcal{A}_{f,g}$ are symmetric. □

If the quadratic numerical range is real, then, in the generic case, only self-adjointness with respect to a possibly indefinite inner product holds.

The corresponding notion of $\mathcal{J}$-self-adjointness plays a role in a number of other subsections and also in the next chapter on unbounded block operator matrices; therefore we give the definition for the unbounded case here.

Definition 1.1.14 Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and let $\mathcal{J} \in L(\mathcal{H})$ have the corresponding block operator representation

$$\mathcal{J} := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{1.1.10}$$

A densely defined linear operator $\mathcal{A}$ in $\mathcal{H}$ is called *$\mathcal{J}$-self-adjoint* if $\mathcal{J}\mathcal{A}$ is self-adjoint in $\mathcal{H}$; it is called *$\mathcal{J}$-symmetric* if $\mathcal{J}\mathcal{A}$ is symmetric in $\mathcal{H}$.

Clearly, every bounded $\mathcal{J}$-symmetric operator is $\mathcal{J}$-self-adjoint. If we define the indefinite inner product $[\cdot,\cdot] := (\mathcal{J}\cdot,\cdot)$ on $\mathcal{H}$, then $\mathcal{A} \in L(\mathcal{H})$ is $\mathcal{J}$-self-adjoint if and only if

$$[\mathcal{A}\mathbf{x},\mathbf{y}] = [\mathbf{x},\mathcal{A}\mathbf{y}], \quad \mathbf{x},\mathbf{y} \in \mathcal{H}.$$

The Hilbert space $\mathcal{H}$ equipped with the indefinite inner product $[\cdot,\cdot]$ is a Krein space; every $\mathcal{J}$-self-adjoint operator is a self-adjoint operator in this Krein space. For the definition of Krein spaces and properties of linear operators therein we refer to [Bog74], [AI89], [Lan82]. We only mention that the spectrum of a $\mathcal{J}$-self-adjoint operator is symmetric to $\mathbb{R}$.

Obviously, if $\mathcal{A} \in L(\mathcal{H})$ has a block operator representation (1.1.3), then

$$\begin{aligned} &\mathcal{A} \text{ is self-adjoint} &&\Longleftrightarrow && A = A^*, \quad D = D^*, \quad C = B^*, \\ &\mathcal{A} \text{ is } \mathcal{J}\text{-self-adjoint} &&\Longleftrightarrow && A = A^*, \quad D = D^*, \quad C = -B^*. \end{aligned}$$

Theorem 1.1.15 *Let either* $\dim \mathcal{H}_1 \geq 2$ *or* $\dim \mathcal{H}_2 \geq 2$. *If* $W^2(\mathcal{A}) \subset \mathbb{R}$, *then* $A = A^*$, $D = D^*$, *and* $\mathcal{A}$ *is either block triangular* (*i.e.* $B = 0$ *or* $C = 0$) *or there exists a* $\gamma \in \mathbb{R}$, $\gamma \neq 0$, *such that*

$$\mathcal{A} = \begin{pmatrix} A & B \\ \gamma B^* & D \end{pmatrix};$$

in the latter case, $\mathcal{A}$ *is similar to the block operator matrix*

$$\widetilde{\mathcal{A}} = \begin{pmatrix} A & \widetilde{B} \\ (\operatorname{sign}\gamma)\, \widetilde{B}^* & D \end{pmatrix}, \quad \widetilde{B} := \sqrt{|\gamma|}\, B;$$

$\widetilde{\mathcal{A}}$ *is self-adjoint in* $\mathcal{H}$ *if* $\operatorname{sign}\gamma = 1$ *and* $\mathcal{J}$*-self-adjoint if* $\operatorname{sign}\gamma = -1$.

In the proof of Theorem 1.1.15 we use the following lemma; in view of the next chapter, we formulate it for unbounded operators.

Lemma 1.1.16 *If* B *and* C *are closed densely defined linear operators from* $\mathcal{H}_2$ *to* $\mathcal{H}_1$ *and from* $\mathcal{H}_1$ *to* $\mathcal{H}_2$, *respectively, such that*

$$(By,x)(Cx,y) \in \mathbb{R} \quad \text{for all } \ x \in \mathcal{D}(C),\ y \in \mathcal{D}(B), \tag{1.1.11}$$

then $B = 0$, $C = 0$, *or* $C \subset \gamma B^*$ *with* $\gamma \in \mathbb{R}$ ($C = \gamma B^*$ *if* B, C *are bounded*).

Proof. If $x \in \mathcal{D}(C)$, $y \in \mathcal{D}(B)$ are such that $(By, x) \neq 0$, then condition (1.1.11) implies that

$$\frac{(Cx,y)}{(x,By)} = \frac{(By,x)(Cx,y)}{(By,x)(x,By)} \in \mathbb{R}. \tag{1.1.12}$$

Assume that $B \neq 0$. Then, because C is densely defined, there exist elements $x_0 \in \mathcal{D}(C)$, $y_0 \in \mathcal{D}(B)$ such that $(x_0, By_0) \neq 0$. For $u \in \mathcal{D}(C)$, $v \in \mathcal{D}(B)$, we consider the function

$$\begin{aligned} f_{u,v}(z) :=& \frac{\big(C(x_0+zu), y_0 + \overline{z}\,v\big)}{\big(x_0+zu, B(y_0+\overline{z}\,v)\big)} \\ =& \frac{(Cx_0,y_0) + z\big((Cx_0,v)+(Cu,y_0)\big) + z^2(Cu,v)}{(x_0,By_0) + z\big((x_0,Bv)+(u,By_0)\big) + z^2(u,Bv)}, \quad z \in \mathbb{C}. \end{aligned}$$

Since $(x_0, By_0) \neq 0$, the denominator is not identically zero and hence the function $f_{u,v}$ is rational in $\mathbb{C}$ with at most two poles, say ζ_1, ζ_2. Because of (1.1.12), it is real on its domain of holomorphy and thus constant there: $f_{u,v}(z) = f_{u,v}(0) = (Cx_0, y_0)/(x_0, By_0) =: \gamma \in \mathbb{R}$, or

$$\begin{aligned} &(Cx_0,y_0) + z\big((Cx_0,v)+(Cu,y_0)\big) + z^2(Cu,v) \\ &= \gamma\,\big((x_0,By_0) + z\big((x_0,Bv)+(u,By_0)\big) + z^2(u,Bv)\big) \end{aligned}$$

for $z \in \mathbb{C} \setminus \{\zeta_1, \zeta_2\}$. Comparing coefficients, we find $(Cu, v) = \gamma(u, Bv)$ for all $u \in \mathcal{D}(C)$, $v \in \mathcal{D}(B)$ and hence $\gamma B \subset C^*$ or, taking adjoints, $C \subset \gamma B^*$. The last claim is obvious. □

***Proof of Theorem* 1.1.15.** Without loss of generality, let $\dim \mathcal{H}_2 \geq 2$. Then, by Theorem 1.1.9, $W(A) \subset W^2(\mathcal{A}) \subset \mathbb{R}$ and hence A is self-adjoint. This and the equality

$$(Af,f) + (Dg,g) = \lambda_+\begin{pmatrix} f \\ g \end{pmatrix} + \lambda_-\begin{pmatrix} f \\ g \end{pmatrix} \in \mathbb{R}, \quad f \in S_{\mathcal{H}_1},\ g \in S_{\mathcal{H}_2},$$

show that D is self-adjoint as well. Since

$$\det \mathcal{A}_{f,g} = (Af,f)(Dg,g) - (Bg,f)(Cf,g) = \lambda_+\begin{pmatrix} f \\ g \end{pmatrix}\lambda_-\begin{pmatrix} f \\ g \end{pmatrix} \in \mathbb{R}$$

for all $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$, we have $(Bg, f)(Cf, g) \in \mathbb{R}$ for all $f \in \mathcal{H}_1$, $g \in \mathcal{H}_2$. Now Lemma 1.1.16 yields the second claim. The last assertion about the similarity of $\mathcal{A}$ follows from the identity

$$\mathcal{A} = \begin{pmatrix} I & 0 \\ 0 & \sqrt{|\gamma|} \end{pmatrix} \begin{pmatrix} A & \widetilde{B} \\ (\operatorname{sign}\gamma)\,\widetilde{B}^* & D \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \sqrt{|\gamma|}^{\,-1} \end{pmatrix}.$$

Obviously, $\widetilde{\mathcal{A}}$ is self-adjoint in $\mathcal{H}$ if $\operatorname{sign}\gamma = 1$; if $\operatorname{sign}\gamma = -1$, then $\mathcal{J}\widetilde{\mathcal{A}}$ is self-adjoint in $\mathcal{H}$ since

$$\mathcal{J}\widetilde{\mathcal{A}} = \begin{pmatrix} A & \widetilde{B} \\ \widetilde{B}^* & -D \end{pmatrix}.$$

□

1.2 Special classes of block operator matrices

A major advantage of the quadratic numerical range is that it reflects symmetries and other properties of the entries of a block operator matrix. Some of the results obtained here also play a role in the unbounded case considered in the next chapter. Therefore special emphasis is placed on structures occurring in applications *e.g.* from mathematical physics or systems theory.

Theorem 1.2.1 *Let the block operator matrix $\mathcal{A}$ be of the form*

$$\mathcal{A} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}.$$

For $\omega \in [0, \pi)$, define the sector $\Sigma_\omega := \{re^{i\phi} : r \geq 0, |\phi| \leq \omega\}$. If there exist α, $\delta > 0$ and angles $\varphi, \vartheta \in [0, \pi/2]$ such that

$$W(D) \subset \{z \in -\Sigma_\varphi : \operatorname{Re} z \leq -\delta\}, \quad W(A) \subset \{z \in \Sigma_\vartheta : \operatorname{Re} z \geq \alpha\}$$

and $\theta := \max\{\varphi, \vartheta\}$, then

$$W^2(\mathcal{A}) \subset \{z \in -\Sigma_\theta : \operatorname{Re} z \leq -\delta\} \cup \{z \in \Sigma_\theta : \operatorname{Re} z \geq \alpha\}$$

consists of two components separated by the strip $\{z \in \mathbb{C} : -\delta < \operatorname{Re} z < \alpha\}$.

For the proof of this theorem we use the following elementary lemma for the eigenvalues of 2×2 matrices (see [LT98, Lemma 3.1]).

Lemma 1.2.2 *Let $a, b, c, d \in \mathbb{C}$ be complex numbers with $\operatorname{Re} d < 0 < \operatorname{Re} a$ and $bc \geq 0$. Then the matrix*

$$\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has eigenvalues λ_1, λ_2 such that:

i) $\operatorname{Re}\lambda_2 \leq \operatorname{Re} d < 0 < \operatorname{Re} a \leq \operatorname{Re}\lambda_1$,
ii) $\min\{\operatorname{Im} a, \operatorname{Im} d\} \leq \operatorname{Im}\lambda_1, \quad \operatorname{Im}\lambda_2 \leq \max\{\operatorname{Im} a, \operatorname{Im} d\}$,
iii) $\lambda_1, -\lambda_2 \in \{z \in \mathbb{C} : |\arg z| \leq \max\{|\arg a|, \pi - |\arg d|\}\}$.

Proof. We suppose that $\operatorname{Im} a \geq 0$ (otherwise we consider A^*) and

$$\arg a \geq \pi - |\arg d| \tag{1.2.1}$$

(otherwise we start from d instead of a in the following). Assumption (1.2.1) implies that

$$\left|\frac{\operatorname{Im}(a-d)}{\operatorname{Re}(a-d)}\right| \le \tan(\arg a). \tag{1.2.2}$$

The eigenvalues λ_1, λ_2 satisfy the equation

$$(a-\lambda)(d-\lambda) - t = 0, \quad t := bc \ge 0.$$

We consider them as functions $\lambda_{1,2}$ of t and write

$$\lambda_{1,2}(t) - \frac{a+d}{2} = \pm\sqrt{\frac{(a-d)^2}{4} + t}, \quad t \ge 0. \tag{1.2.3}$$

Now we decompose $\lambda_i(t) =: x_i(t) + \mathrm{i}y_i(t)$, $i = 1, 2$, and $(a+d)/2 =: \beta + \mathrm{i}\gamma$ into real and imaginary parts. Squaring equation (1.2.3) and taking real and imaginary parts, we see that $x_1(t), y_1(t)$ and $x_2(t), y_2(t)$ satisfy the relations

$$\big(x(t)-\beta\big)^2 - \big(y(t)-\gamma\big)^2 = \frac{1}{4}\operatorname{Re}(a-d)^2 + t, \tag{1.2.4}$$

$$\big(x(t)-\beta\big)\big(y(t)-\gamma\big) = \frac{1}{8}\operatorname{Im}(a-d)^2. \tag{1.2.5}$$

The last equation shows that the eigenvalues $\lambda_1(t), \lambda_2(t)$ lie on a hyperbola with centre $\beta + \mathrm{i}\gamma = (a+d)/2$ and asymptotes $\operatorname{Im} z = \gamma$ and $\operatorname{Re} z = \beta$ parallel to the real and imaginary axis, the right hand branch passing through a and the left hand branch through d. From the identity (1.2.4) it follows that for $0 \le t \le \infty$ the eigenvalues $\lambda_1(t)$ fill the part of the right hand branch which extends from a to $\infty + \mathrm{i}\gamma$, and the eigenvalues $\lambda_2(t)$ fill the part of the left hand branch from d to $-\infty + \mathrm{i}\gamma$. This implies i) and ii). In order to prove iii), it is sufficient to show that the derivatives of the hyperbola at d and at a are in modulus less than $\tan(\arg a)$. For example, for the derivative at d, it follows from (1.2.5) that

$$\frac{\dot{y}(0)}{\dot{x}(0)} = -\frac{y(0)-\gamma}{x(0)-\beta} = -\frac{\operatorname{Im} d - \frac{1}{2}\operatorname{Im}(a+d)}{\operatorname{Re} d - \frac{1}{2}\operatorname{Re}(a+d)} = -\frac{\operatorname{Im}(d-a)}{\operatorname{Re}(d-a)},$$

which is in modulus less than $\tan(\arg a)$ by (1.2.2). □

***Proof of Theorem* 1.2.1.** All assertions follow by applying Lemma 1.2.2 to the 2×2 matrices $\mathcal{A}_{f,g}$ defined in (1.1.4) for $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$. □

For self-adjoint block operator matrices, the following corollary is obvious from Theorem 1.2.1.

Corollary 1.2.3 *Let $\mathcal{A} = \mathcal{A}^*$ and suppose that*

$$\sup W(D) < \inf W(A).$$

Then $W^2(\mathcal{A}) = \Lambda_-(\mathcal{A}) \,\dot\cup\, \Lambda_+(\mathcal{A})$ consists of two components satisfying

$$\sup \Lambda_-(\mathcal{A}) \le \sup W(D) < \inf W(A) \le \inf \Lambda_+(\mathcal{A}).$$

In the following proposition we generalize this estimate to non-separated diagonal entries and we derive two-sided estimates for the outer end-points $\inf \Lambda_-(\mathcal{A})$ and $\sup \Lambda_+(\mathcal{A})$ of the quadratic numerical range (see [KMM07]).

Proposition 1.2.4 *If $\mathcal{A} = \mathcal{A}^*$, then the quadratic numerical range $W^2(\mathcal{A}) = \Lambda_-(\mathcal{A}) \cup \Lambda_+(\mathcal{A})$ satisfies the estimates*

$$\inf \Lambda_+(\mathcal{A}) \ge \max\bigl\{\inf W(A),\, \inf W(D)\bigr\},$$
$$\sup \Lambda_-(\mathcal{A}) \le \min\bigl\{\sup W(A),\, \sup W(D)\bigr\},$$

and

$$\min\bigl\{\inf W(A), \inf W(D)\bigr\} - \delta_B^- \le \inf \Lambda_-(\mathcal{A}) \le \min\bigl\{\inf W(A), \inf W(D)\bigr\},$$
$$\max\bigl\{\sup W(A), \sup W(D)\bigr\} \le \sup \Lambda_+(\mathcal{A}) \le \max\bigl\{\sup W(A), \sup W(D)\bigr\} + \delta_B^+,$$

where

$$\delta_B^- := \|B\| \tan\left(\frac{1}{2}\arctan \frac{2\|B\|}{|\inf W(A) - \inf W(D)|}\right),$$
$$\delta_B^+ := \|B\| \tan\left(\frac{1}{2}\arctan \frac{2\|B\|}{|\sup W(A) - \sup W(D)|}\right);$$

if $\inf W(A) = \inf W(D)$ or $\sup W(A) = \sup W(D)$, we set $\arctan \infty := \pi/2$.

Proof. Since $\mathcal{A} = \mathcal{A}^*$, we have $C = B^*$. Then the definition of λ_+ in Corollary 1.1.4 shows that, for $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$,

$$\lambda_+\begin{pmatrix} f \\ g \end{pmatrix} = \frac{(Af,f)+(Dg,g)}{2} + \sqrt{\left(\frac{(Af,f)-(Dg,g)}{2}\right)^2 + |(Bg,f)|^2} \tag{1.2.6}$$
$$\ge \frac{(Af,f)+(Dg,g)}{2} + \left|\frac{(Af,f)-(Dg,g)}{2}\right|$$
$$= \max\bigl\{(Af,f), (Dg,g)\bigr\}.$$

From this estimate we obtain

$$\inf \Lambda_+(\mathcal{A}) \ge \max\bigl\{\inf W(A),\, \inf W(D)\bigr\},$$
$$\sup \Lambda_+(\mathcal{A}) \ge \max\bigl\{\sup W(A),\, \sup W(D)\bigr\}.$$

The proof of the second inequality and of the right part of the third inequality is analogous.

For the proof of the remaining inequalities, we observe that the solutions (1.2.6) of the quadratic equations $\det(\mathcal{A}_{f,g} - \lambda) = 0$ defining the quadratic numerical range can also be written in the form

$$\lambda_-\begin{pmatrix} f \\ g \end{pmatrix} = \min\{(Af,f),(Dg,g)\} - |(Bg,f)| \tan\left(\frac{1}{2}\arctan\frac{2|(Bg,f)|}{|(Af,f)-(Dg,g)|}\right),$$
$$\lambda_+\begin{pmatrix} f \\ g \end{pmatrix} = \max\{(Af,f),(Dg,g)\} + |(Bg,f)| \tan\left(\frac{1}{2}\arctan\frac{2|(Bg,f)|}{|(Af,f)-(Dg,g)|}\right).$$

Without loss of generality, we assume that $\inf W(A) \geq \inf W(D)$; otherwise we reverse the components in the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Suppose that $(Af,f) \geq (Dg,g)$; then

$$\lambda_-\begin{pmatrix} f \\ g \end{pmatrix} = (Dg,g) - |(Bg,f)| \tan\left(\frac{1}{2}\arctan\frac{2|(Bg,f)|}{(Af,f)-(Dg,g)}\right).$$

We define the auxiliary function

$$h(t) := t - |(Bg,f)| \tan\left(\frac{1}{2}\arctan\frac{2|(Bg,f)|}{(Af,f)-t}\right), \quad t \in \mathbb{R}.$$

It is easy to see that h is strictly monotonically increasing (with a jump of height $2|(Bg,f)|$ at the singularity (Af,f)); in fact, we have $h' \geq 1/2$. Hence

$$\begin{aligned}\lambda_-\begin{pmatrix} f \\ g \end{pmatrix} &\geq \inf W(D) - |(Bg,f)| \tan\left(\frac{1}{2}\arctan\frac{2|(Bg,f)|}{(Af,f)-\inf W(D)}\right) \\ &\geq \inf W(D) - \|B\| \tan\left(\frac{1}{2}\arctan\frac{2\|B\|}{\inf W(A)-\inf W(D)}\right)\end{aligned}$$

if $(Af,f) \geq (Dg,g)$. If $(Af,f) < (Dg,g)$, then we have the estimates $(Af,f) \geq \inf W(A) \geq \inf W(D)$ and $(Dg,g) > (Af,f) \geq \inf W(A)$. Thus, in the same way as above, we obtain

$$\begin{aligned}\lambda_-\begin{pmatrix} f \\ g \end{pmatrix} &= (Af,f) - |(Bg,f)| \tan\left(\frac{1}{2}\arctan\frac{2|(Bg,f)|}{(Dg,g)-(Af,f)}\right) \\ &\geq \inf W(D) - |(Bg,f)| \tan\left(\frac{1}{2}\arctan\frac{2|(Bg,f)|}{(Dg,g)-\inf W(D)}\right) \\ &\geq \inf W(D) - \|B\| \tan\left(\frac{1}{2}\arctan\frac{2\|B\|}{\inf W(A)-\inf W(D)}\right).\end{aligned}$$

The estimate for λ_+ is proved analogously. □

In the following, for a self-adjoint operator T in a Hilbert space $\mathcal{H}$ and a subinterval $I \subset \mathbb{R}$, we denote by $E_T(I)$ the spectral projection and by $\mathcal{L}_I(T) = E_T(I)\mathcal{H}$ the spectral subspace, respectively, corresponding to I.

Remark 1.2.5 Suppose that $\mathcal{A} = \mathcal{A}^*$ and $\inf W(A) \neq \inf W(D)$ with

$$\begin{aligned} \dim \mathcal{L}_{(-\infty, \inf W(A)]}(D) \geq 2 \quad &\text{if} \quad \inf W(D) < \inf W(A), \\ \dim \mathcal{L}_{(-\infty, \inf W(D)]}(A) \geq 2 \quad &\text{if} \quad \inf W(A) < \inf W(D). \end{aligned}$$

Then

$$\inf \Lambda_+(\mathcal{A}) = \max\bigl\{\inf W(A), \inf W(D)\bigr\}.$$

In general, the strict inequality $\inf \Lambda_+(\mathcal{A}) > \max\bigl\{\inf W(A), \inf W(D)\bigr\}$ may occur. Analogous statements hold for $\sup \Lambda_-(\mathcal{A})$.

Proof. Let $\inf W(D) < \inf W(A)$ and $\dim \mathcal{L}_{(-\infty, \inf W(A)]}(D) > 1$. Since $\inf W(A) \in \sigma(A)$, there exists a sequence $(x_n)_1^\infty \subset \mathcal{D}(A)$, $\|x_n\| = 1$, such that $\bigl\|\bigl(A - \inf W(A)\bigr)x_n\bigr\| \to 0$, $n \to \infty$. Due to the dimension condition, for each $n \in \mathbb{N}$ there exists $y_n \in \mathcal{L}_{(-\infty, \inf W(A)]}(D)$, $\|y_n\| = 1$, such that $(B^* x_n, y_n) = 0$. Then $(Dy_n, y_n) \leq (Ax_n, x_n)$ and hence, by (1.2.6),

$$\lambda_+\begin{pmatrix} x_n \\ y_n \end{pmatrix} = (Ax_n, x_n) \longrightarrow \inf W(A), \quad n \to \infty.$$

This implies $\inf W(A) \in \overline{\Lambda_+(\mathcal{A})}$. Together with Proposition 1.2.4, the first assertion follows. An example for strict inequality is furnished by the matrix

$$\mathcal{A} := \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \end{array}\right);$$

here $\max\bigl\{\inf W(A), \inf W(D)\bigr\} = \inf W(A) = -1$ and

$$\min \Lambda_+(\mathcal{A}) = \lambda_+\begin{pmatrix} e_2 \\ e_2 \end{pmatrix} = -\frac{3}{2} + \frac{1}{2}\sqrt{5} > -1 \quad \text{with} \quad e_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \qquad \square$$

Next we consider block operator matrices for which $C = -B^*$ and, more specifically, $\mathcal{J}$-self-adjoint block operator matrices (see Definition 1.1.14). We estimate their quadratic numerical range in the case when the off-diagonal element B is sufficiently small. In the $\mathcal{J}$-self-adjoint case, the quadratic numerical range is real for small B, but it may become complex if B is sufficiently large; more detailed estimates are given in Proposition 1.3.9.

Proposition 1.2.6 *Let the block operator matrix* $\mathcal{A}$ *be of the form*

$$\mathcal{A} = \begin{pmatrix} A & B \\ -B^* & D \end{pmatrix}$$

and define

$$a_- := \inf \operatorname{Re} W(A), \quad a_+ := \sup \operatorname{Re} W(A),$$
$$d_- := \inf \operatorname{Re} W(D), \quad d_+ := \sup \operatorname{Re} W(D).$$

Then the quadratic numerical range of $\mathcal{A}$ *satisfies the following estimates:*

i) $\min\{a_-, d_-\} \le \operatorname{Re} W^2(\mathcal{A}) \le \max\{a_+, d_+\}$.

ii) *If* $d_+ < a_-$ *and* $\|B\| < (a_- - d_+)/2$, *then* $W^2(\mathcal{A}) = \Lambda_-(\mathcal{A}) \dot\cup \Lambda_+(\mathcal{A})$ *consists of two components satisfying*

$$\operatorname{Re} \Lambda_-(\mathcal{A}) \le d_+ + \|B\| < a_- - \|B\| \le \operatorname{Re} \Lambda_+(\mathcal{A}).$$

iii) *If* $A = A^*$, $D = D^*$, *then* $W^2(\mathcal{A})$ *is symmetric to* $\mathbb{R}$, $|\operatorname{Im} W^2(\mathcal{A})| \le \|B\|$; *if, in addition,* $d_+ < a_-$, *then*

$$\|B\| \le (a_- - d_+)/2 \implies W^2(\mathcal{A}) \subset \mathbb{R},$$
$$\|B\| > (a_- - d_+)/2 \implies \left|\operatorname{Im} W^2(\mathcal{A})\right| \le \sqrt{\|B\|^2 - \frac{(a_- - d_+)^2}{4}}.$$

The case $a_+ < d_-$ *in* ii) *and* iii) *is analogous.*

For the proof of Proposition 1.2.6, we use the following simple lemma (see [Tre08, Lemma 5.1 ii)]).

Lemma 1.2.7 *Let* $a, b, c, d \in \mathbb{C}$ *be complex numbers with* $\operatorname{Re} d < \operatorname{Re} a$ *and* $bc \le 0$. *Then the matrix*

$$\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has eigenvalues λ_1, λ_2 *such that*

i) $\operatorname{Re} d \le \operatorname{Re} \lambda_2 \le \operatorname{Re} \lambda_1 \le \operatorname{Re} a$,

ii) $\operatorname{Re} \lambda_2 \le \operatorname{Re} d + \sqrt{|bc|} < \operatorname{Re} a - \sqrt{|bc|} \le \operatorname{Re} \lambda_1$ *if* $\sqrt{|bc|} < (\operatorname{Re} a - \operatorname{Re} d)/2$, *and* $\lambda_1,\ \lambda_2 \in \mathbb{R}$ *if, in addition,* $a,\ d \in \mathbb{R}$,

iii) $\operatorname{Re} \lambda_1 = \operatorname{Re} \lambda_2 = (a + d)/2$, $|\operatorname{Im} \lambda_1| = |\operatorname{Im} \lambda_2| = \sqrt{|bc| - (a - d)^2/4}$ *if* $\sqrt{|bc|} \ge (a - d)/2$ *and* $a,\ d \in \mathbb{R}$.

Proof. i) If $\operatorname{Re} \lambda < \operatorname{Re} d$ $(< \operatorname{Re} a)$ or $\operatorname{Re} \lambda > \operatorname{Re} a$ $(> \operatorname{Re} d)$, then the eigenvalue equation $(a - \lambda)(d - \lambda) = bc \le 0$ cannot hold. In fact, decomposing all numbers therein into real and imaginary parts, one can show that $\operatorname{Im} a$–$\operatorname{Im} \lambda$ and $\operatorname{Im} d - \operatorname{Im} \lambda$ have different signs and $\operatorname{Re}\big((a - \lambda)(d - \lambda)\big) > 0$.

ii) If $\operatorname{Re} d + \sqrt{|bc|} < \operatorname{Re}\lambda < \operatorname{Re} a - \sqrt{|bc|}$, then

$$\left|\det(\mathrm{A}-\lambda)\right| \geq |a-\lambda|\,|d-\lambda| - |bc| \geq |\operatorname{Re} a - \operatorname{Re}\lambda|\,|\operatorname{Re} d - \operatorname{Re}\lambda| - |bc| > 0,$$

hence λ is not an eigenvalue of A. The relation $\operatorname{Re}\lambda_1 + \operatorname{Re}\lambda_2 = \operatorname{Re} a + \operatorname{Re} d$ excludes the possibility that *e.g.* $\operatorname{Re}\lambda_1, \operatorname{Re}\lambda_2 \leq \operatorname{Re} d + \sqrt{|bc|}$.

The claims in ii) for a, $d \in \mathbb{R}$ and in iii) are immediate from the formula

$$\lambda_{1/2} = \frac{a+d}{2} \pm \sqrt{\frac{(a-d)^2}{4} + bc}\,. \qquad \square$$

***Proof of Proposition* 1.2.6.** If $A = A^*$, $D = D^*$, then $\overline{\det(\mathcal{A}_{f,g} - \lambda)} = \det(\mathcal{A}_{f,g} - \overline{\lambda})$ for $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$, which implies $W^2(\mathcal{A}) = W^2(\mathcal{A})^*$ and hence the first claim in i). All other claims follow by applying Lemma 1.2.7 to the 2×2 matrices $\mathcal{A}_{f,g}$ defined in (1.1.4) for $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$. $\square$

Proposition 1.2.8 *Let $\mathcal{H}_1 = \mathcal{H}_2$ and suppose that*

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & A^* \end{pmatrix}$$

is such that either $B = B^*$ and $C = C^*$ or $B = -B^*$ and $C = -C^*$. Then $W^2(\mathcal{A})$ is symmetric to $\mathbb{R}$.

Proof. For $f, g \in S_{\mathcal{H}_1}$ and $\lambda \in \mathbb{C}$, it is easy to see that

$$\begin{aligned} \det\big((\mathcal{A}^*)_{g,f} - \lambda\big) &= \det \begin{pmatrix} (A^*g,g)-\lambda & (C^*f,g) \\ (B^*g,f) & (Af,f)-\lambda \end{pmatrix} \\ &= \det \begin{pmatrix} (Af,f)-\lambda & \pm(Bg,f) \\ \pm(Cf,g) & (A^*g,g)-\lambda \end{pmatrix} = \det\big(\mathcal{A}_{f,g} - \lambda\big), \end{aligned}$$

which implies that $W^2(\mathcal{A}) = W^2(\mathcal{A}^*)$. On the other hand, by Proposition 1.1.13 i), we have $W^2(\mathcal{A}^*) = W^2(\mathcal{A})^*$ and hence $W^2(\mathcal{A}) = W^2(\mathcal{A})^*$. $\square$

Proposition 1.2.9 *Let $\mathcal{H}_1 = \mathcal{H}_2$ and suppose that*

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}$$

is such that either $B = B^*$ and $C = C^*$ or $B = -B^*$ and $C = -C^*$. Then $W^2(\mathcal{A})$ is symmetric to $\mathrm{i}\mathbb{R}$.

Proof. The assertion follows since $\mathrm{i}\mathcal{A}$ satisfies the assumptions of Proposition 1.2.8 and $W^2(\mathrm{i}\mathcal{A}) = \mathrm{i}W^2(\mathcal{A})$ by Proposition 1.1.7 i). $\square$

1.3 Spectral inclusion

The most important feature of the quadratic numerical range is that, like the numerical range, it has the spectral inclusion property (see (1.1.1)). Since the quadratic numerical range is always contained in the numerical range (see Theorem 1.1.8), it furnishes a possibly tighter spectral enclosure.

Theorem 1.3.1 $\sigma_{\rm p}(\mathcal{A}) \subset W^2(\mathcal{A}),\ \sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A})}.$

For the proof of Theorem 1.3.1 we need a simple lemma about 2×2-matrices, which we prove for the convenience of the reader.

Lemma 1.3.2 *If for $\mathcal{M} \in M_2(\mathbb{C})$ there exists a vector $x \in \mathbb{C}^2$ such that*

$$\|x\| = 1 \quad \text{and} \quad \|\mathcal{M}x\| < \varepsilon, \tag{1.3.1}$$

then $\operatorname{dist}\big(0, \sigma(\mathcal{M})\big) \le \sqrt{\|\mathcal{M}\|\,\varepsilon}$.

Proof. Only the case that the matrix $\mathcal{M}$ is invertible has to be considered. Then the inverse matrix $\mathcal{M}^{-1}$ can be written as

$$\mathcal{M}^{-1} = \frac{1}{\det \mathcal{M}} (J^{\rm t} \mathcal{M} J)^{\rm t}$$

with $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus

$$\|\mathcal{M}^{-1}\| = \frac{\|\mathcal{M}\|}{|\det \mathcal{M}|} = \frac{\|\mathcal{M}\|}{|\lambda_1 \lambda_2|}, \tag{1.3.2}$$

where λ_1, λ_2 are the eigenvalues of $\mathcal{M}$. The assumption (1.3.1) implies that

$$\|\mathcal{M}^{-1}\| > \varepsilon^{-1}. \tag{1.3.3}$$

From (1.3.2) and (1.3.3) we obtain $\min\big\{|\lambda_1|, |\lambda_2|\big\} \le \sqrt{\|\mathcal{M}\|\,\varepsilon}$. □

In the following, for a bounded or unbounded linear operator T in $\mathcal{H}$, we define its *approximate point spectrum* $\sigma_{\rm app}(T)$ as

$$\sigma_{\rm app}(T) := \big\{\lambda \in \mathbb{C} : \exists\, (\mathbf{x}_n)_1^\infty \subset \mathcal{D}(T),\, \|\mathbf{x}_n\| = 1,\ (T-\lambda)\mathbf{x}_n \to 0,\, n \to \infty\big\}. \tag{1.3.4}$$

***Proof of Theorem* 1.3.1.** First we consider $\lambda \in \sigma_{\rm p}(\mathcal{A})$. Then there exists a nonzero element $(f\ g)^{\rm t} \in \mathcal{H}$ such that

$$(A-\lambda)f + Bg = 0,$$
$$Cf + (D-\lambda)g = 0.$$

We write $f = \|f\|\,\widehat{f}$, $g = \|g\|\,\widehat{g}$ with elements $\widehat{f} \in S_{\mathcal{H}_1}$, $\widehat{g} \in S_{\mathcal{H}_2}$ (here, if *e.g.* $f = 0$, then $\widehat{f}$ can be chosen arbitrarily). It follows that

$$\big(Af, \widehat{f}\,\big) - \lambda\big(f, \widehat{f}\,\big) + \big(Bg, \widehat{f}\,\big) = 0,$$
$$\big(Cf, \widehat{g}\big) + \big(Dg, \widehat{g}\big) - \lambda\big(g, \widehat{g}\big) = 0,$$

and, consequently,

$$\mathcal{A}_{\widehat{f},\widehat{g}} \begin{pmatrix} \|f\| \\ \|g\| \end{pmatrix} = \lambda \begin{pmatrix} \|f\| \\ \|g\| \end{pmatrix}. \tag{1.3.5}$$

Hence $\lambda \in \sigma_{\mathrm{p}}(\mathcal{A}_{\widehat{f},\widehat{g}}) \subset W^2(\mathcal{A})$.

If $\lambda \in \sigma(\mathcal{A}) \setminus \sigma_{\mathrm{p}}(\mathcal{A})$, then $\overline{\lambda} \in \sigma_{\mathrm{p}}(\mathcal{A}^*)$ or $\lambda \in \sigma_{\mathrm{app}}(\mathcal{A})$. If $\overline{\lambda} \in \sigma_{\mathrm{p}}(\mathcal{A}^*)$, then, according to what was shown above, we have $\overline{\lambda} \in W^2(\mathcal{A}^*)$ and hence $\lambda \in W^2(\mathcal{A})$ by Proposition 1.1.13 i). If $\lambda \in \sigma_{\mathrm{app}}(\mathcal{A})$, then there exists a sequence of elements $(f_n \; g_n)^{\mathrm{t}} \in \mathcal{H}$, $n = 1, 2, \dots$, such that

$$\|f_n\|^2 + \|g_n\|^2 = 1, \qquad \left\| \mathcal{A} \begin{pmatrix} f_n \\ g_n \end{pmatrix} - \lambda \begin{pmatrix} f_n \\ g_n \end{pmatrix} \right\| \longrightarrow 0, \quad n \to \infty.$$

Then, with $\widehat{f}_n \in S_{\mathcal{H}_1}$, $\widehat{g}_n \in S_{\mathcal{H}_2}$ as in the first part of the proof, we obtain

$$\mathcal{A}_{\widehat{f}_n,\widehat{g}_n} \begin{pmatrix} \|f_n\| \\ \|g_n\| \end{pmatrix} - \lambda \begin{pmatrix} \|f_n\| \\ \|g_n\| \end{pmatrix} \longrightarrow 0, \quad n \to \infty.$$

Since $\|\mathcal{A}_{\widehat{f}_n,\widehat{g}_n}\| \le \|\mathcal{A}\|$, $n \in \mathbb{N}$, we have $\mathrm{dist}\big(\lambda, \sigma_{\mathrm{p}}(\mathcal{A}_{\widehat{f}_n,\widehat{g}_n})\big) \to 0$ for $n \to \infty$ by Lemma 1.3.2. Thus $\lambda \in \overline{\bigcup_{n\in\mathbb{N}} \sigma_{\mathrm{p}}(\mathcal{A}_{\widehat{f}_n,\widehat{g}_n})} \subset \overline{W^2(\mathcal{A})}$. □

Altogether, in Theorem 1.3.1 and Theorem 1.1.8, we have shown that

$$\sigma_{\mathrm{p}}(\mathcal{A}) \subset W^2(\mathcal{A}) \subset W(\mathcal{A}), \quad \sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A})} \subset \overline{W(\mathcal{A})}.$$

Therefore, and because of its non-convexity, the quadratic numerical range $W^2(\mathcal{A})$ may give better information about the localization of the spectrum $\sigma(\mathcal{A})$ than the numerical range $W(\mathcal{A})$.

Example 1.3.3 Consider the 4×4 matrix

$$\mathcal{A}_3 := \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -2 & -1 & \mathrm{i} & 5\mathrm{i} \\ -1 & -2 & -5\mathrm{i} & \mathrm{i} \end{array} \right)$$

with respect to the decomposition $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$. Figure 1.4 shows its numerical range, quadratic numerical range, and the four different eigenvalues marked by black dots, two in each component of $W^2(\mathcal{A}_3)$.

If $\mathcal{A}$ is self-adjoint, Theorems 1.3.1 and 1.1.9 allow to characterize the quadratic numerical range more explicitly.

Proposition 1.3.4 *Suppose that* $\mathcal{A} = \mathcal{A}^*$ *and* $\dim \mathcal{H}_1 \ge 2$, $\dim \mathcal{H}_2 \ge 2$.

i) *If* $\overline{W(A)} \cap \overline{W(D)} \neq \emptyset$, *then* $\overline{W^2(\mathcal{A})}$ *is the single interval*

$$\overline{W^2(\mathcal{A})} = \big[\min \sigma(\mathcal{A}), \max \sigma(\mathcal{A})\big] = \overline{W(\mathcal{A})}. \tag{1.3.6}$$

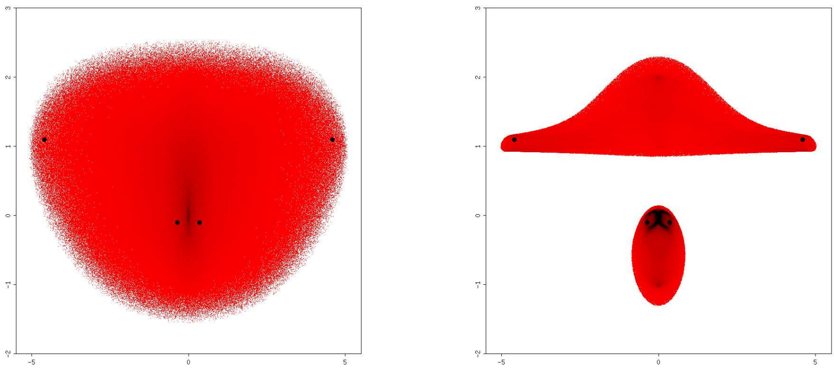

Figure 1.4 Numerical range, quadratic numerical range, and eigenvalues of $\mathcal{A}_3$.

ii) *If* $\overline{W(A)} \cap \overline{W(D)} = \emptyset$, *then* $\overline{W^2(\mathcal{A})}$ *consists of two disjoint intervals*

$$\overline{W^2(\mathcal{A})} = \big[\min \sigma(\mathcal{A}), d\big] \,\dot\cup\, \big[a, \max \sigma(\mathcal{A})\big] \tag{1.3.7}$$

where

$$d := \min\big\{\sup W(A),\ \sup W(D)\big\}, \tag{1.3.8}$$

$$a := \max\big\{\inf W(A),\ \inf W(D)\big\}. \tag{1.3.9}$$

Proof. Theorem 1.1.8 implies that

$$\overline{W^2(\mathcal{A})} \subset \overline{W(\mathcal{A})} = \big[\min \sigma(\mathcal{A}), \max \sigma(\mathcal{A})\big].$$

Since $\mathcal{A}$ is self-adjoint and $W^2(\mathcal{A})$ satisfies the spectral inclusion property by Theorem 1.3.1, we have

$$\min \sigma(\mathcal{A}),\ \max \sigma(\mathcal{A}) \in \sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A})}.$$

Because $\overline{W^2(\mathcal{A})}$ consists of at most two connected sets, it follows that it is either of the form (1.3.6) or of the form (1.3.7).

Without loss of generality, we may assume that $\inf W(D) \leq \inf W(A)$; otherwise we reverse the enumeration of the components in $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Then $\overline{W(A)} \cap \overline{W(D)} = \emptyset$ means that $\sup W(D) < \inf W(A)$. By Proposition 1.2.4, we conclude that

$$\sup \Lambda_-(\mathcal{A}) \leq \sup W(D) < \inf W(A) \leq \inf \Lambda_+(\mathcal{A}).$$

Since $\dim \mathcal{H}_1 \geq 2$, Theorem 1.1.9 applies and shows that $\sup W(D) \in \overline{W(D)} \subset \overline{W^2(\mathcal{A})} = \overline{\Lambda_-(\mathcal{A})} \,\dot\cup\, \overline{\Lambda_+(\mathcal{A})}$. Hence $d = \sup \Lambda_-(\mathcal{A}) = \sup W(D)$.

In the same way, it follows that $a = \inf \Lambda_+(\mathcal{A}) = \inf W(A)$ if $\dim \mathcal{H}_2 \geq 2$. This proves ii).

In order to show i), suppose to the contrary that $\overline{W^2(\mathcal{A})}$ consists of two disjoint intervals. Then, by Corollary 1.1.10 i), one of them contains $\overline{W(A)}$ and the other one $\overline{W(D)}$ so that $\overline{W(A)} \cap \overline{W(D)} = \emptyset$, a contradiction. $\square$

Remark 1.3.5 If $\mathcal{A} = \mathcal{A}^*$ and $\dim \mathcal{H}_1 = 1$, we only obtain "$\leq$" in (1.3.8); analogously, if $\dim \mathcal{H}_2 = 1$, we only obtain "$\geq$" in (1.3.9).

Note that if $\dim \mathcal{H}_1 = 1$ and $\dim \mathcal{H}_2 = 1$, then $\overline{W^2(\mathcal{A})} = W^2(\mathcal{A}) = \sigma_{\rm p}(\mathcal{A})$ consists of the eigenvalues of $\mathcal{A}$.

In the sequel, we present two theorems on the classical problem of perturbation of spectra of bounded self-adjoint operators (see [Dav63], [Dav65], [DK70]). The key tool is the spectral inclusion theorem for the quadratic numerical range, combined with the estimates given for it in Section 1.2.

First we consider arbitrary bounded self-adjoint operators subject to perturbations that are off-diagonal with respect to a certain decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of the underlying Hilbert space (see [KMM07, Lemma 1.1], [LT98, Theorem 3.2]); the case that the spectrum of the unperturbed operator splits into two parts separated by a point is crucial in the following.

Theorem 1.3.6 *If $\mathcal{A} = \mathcal{A}^*$, then $\sigma(\mathcal{A})$ satisfies the following estimates.*

i) *Define $\delta_B^{\pm}$ as in Proposition 1.2.4. Then*

$$\min\bigl\{\min \sigma(A), \min \sigma(D)\bigr\} - \delta_B^- \leq \min \sigma(\mathcal{A}) \leq \min\bigl\{\min \sigma(A), \min \sigma(D)\bigr\},$$
$$\max\bigl\{\max \sigma(A), \max \sigma(D)\bigr\} \leq \max \sigma(\mathcal{A}) \leq \max\bigl\{\max \sigma(A), \max \sigma(D)\bigr\} + \delta_B^+.$$

ii) *If $\max \sigma(D) < \min \sigma(A)$, then*

$$\sigma(\mathcal{A}) \cap \bigl(\max \sigma(D), \min \sigma(A)\bigr) = \emptyset$$

independently of the norm of B.

Figure 1.5 Enclosures for the spectra of $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$ and of $\left(\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right)$.

Proof. By Theorem 1.3.1 it is sufficient to prove that $W^2(\mathcal{A})$ satisfies the estimates claimed in i) and ii). This was proved in Proposition 1.2.4

for i) (observe that $\inf W(A) = \min\sigma(A)$, $\sup W(A) = \max\sigma(A)$, and analogously for D) and in Corollary 1.2.3 for ii). □

Next we consider the case that the spectra of the diagonal entries A and D are disjoint, *i.e.* their distance $\delta_{A,D}$ is positive. Classical perturbation theory yields that the spectrum of the perturbed operator $\mathcal{A}$ remains separated into two disjoint parts as long as $\|B\| < \delta_{A,D}/2$ (see [Kat95, Theorem V.4.10]). By means of Theorem 1.3.6, we are able to improve this result and derive an optimal bound on $\|B\|$ (see [KMM07, Theorem 1.3]).

Theorem 1.3.7 *Let* $\mathcal{A} = \mathcal{A}^*$, $\delta_{A,D} := \operatorname{dist}\big(\sigma(A),\sigma(D)\big) > 0$, *and set*

$$\delta_B := \|B\| \tan\left(\frac{1}{2}\arctan\frac{2\|B\|}{\delta_{A,D}}\right).$$

i) *Then*

$$\sigma(\mathcal{A}) \subset \big\{\lambda\in\mathbb{R} : \operatorname{dist}\big(\lambda, \sigma(A)\,\dot\cup\,\sigma(D)\big) \le \delta_B\big\}.$$

ii) *If* $\|B\| < \dfrac{\sqrt{3}}{2}\,\delta_{A,D}$, *then* $\delta_B < \dfrac{1}{2}\delta_{A,D}$ *and* $\sigma(\mathcal{A}) = \sigma_1\,\dot\cup\,\sigma_2$, $\sigma_1,\sigma_2 \neq \emptyset$, *with*

$$\sigma_1 \subset \big\{\lambda\in\mathbb{R} : \operatorname{dist}\big(\lambda,\sigma(A)\big)\le \delta_B\big\} \subset \big\{\lambda\in\mathbb{R} : \operatorname{dist}\big(\lambda,\sigma(A)\big)< \delta_{A,D}/2\big\},$$
$$\sigma_2 \subset \big\{\lambda\in\mathbb{R} : \operatorname{dist}\big(\lambda,\sigma(D)\big)\le \delta_B\big\} \subset \big\{\lambda\in\mathbb{R} : \operatorname{dist}\big(\lambda,\sigma(D)\big)< \delta_{A,D}/2\big\}.$$

iii) *If* $\big(\operatorname{conv}\sigma(A)\big)\cap\sigma(D) = \emptyset$ *and* $\|B\| < \sqrt{2}\,\delta_{A,D}$, *then* $\delta_B < \delta_{A,D}$ *and* $\sigma(\mathcal{A}) = \sigma_1\,\dot\cup\,\sigma_2$, $\sigma_1,\sigma_2 \neq \emptyset$, *with*

$$\sigma_1 \subset \big\{\lambda\in\mathbb{R} : \operatorname{dist}\big(\lambda,\sigma(A)\big)\le \delta_B\big\} \subset \big\{\lambda\in\mathbb{R} : \operatorname{dist}\big(\lambda,\sigma(A)\big)< \delta_{A,D}\big\},$$
$$\sigma_2 \subset \big\{\lambda\in\mathbb{R} : \operatorname{dist}\big(\lambda,\sigma(A)\big)\ge \delta_{A,D}\big\}.$$

Proof. i) Let $\lambda\in\mathbb{R}$ be such that $\operatorname{dist}\big(\lambda,\sigma(A)\,\dot\cup\,\sigma(D)\big) > \delta_B$ and set $I_- := (-\infty,\lambda)$, $I_+ := (\lambda,\infty)$. If we write

$$\mathcal{A} = \mathcal{T} + \mathcal{S}, \quad \mathcal{T} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad \mathcal{S} := \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix},$$

then $\lambda\notin\sigma(\mathcal{T})$ and hence $\mathcal{H} = \mathcal{L}_{I_-}(\mathcal{T}) \oplus \mathcal{L}_{I_+}(\mathcal{T})$. If we denote $P_\pm := E_{\mathcal{T}}(I_\pm)$, then $\mathcal{L}_{I_\pm}(\mathcal{T}) = P_\pm\mathcal{H}$ and $P_-\mathcal{T}P_+ = 0$. Hence, with respect to the decomposition $\mathcal{H} = \mathcal{L}_{I_-}(\mathcal{T}) \oplus \mathcal{L}_{I_+}(\mathcal{T})$, the operator $\mathcal{A}$ can be written as

$$\mathcal{A} = \begin{pmatrix} P_-\mathcal{A}P_- & P_-\mathcal{A}P_+ \\ (P_-\mathcal{A}P_+)^* & P_+\mathcal{A}P_+ \end{pmatrix} = \begin{pmatrix} P_-\mathcal{A}P_- & P_-\mathcal{S}P_+ \\ (P_-\mathcal{S}P_+)^* & P_+\mathcal{A}P_+ \end{pmatrix}. \qquad (1.3.10)$$

If we further decompose $\mathcal{L}_{I_\pm}(\mathcal{T}) = \mathcal{L}_{I_\pm}(A) \oplus \mathcal{L}_{I_\pm}(D)$, then the diagonal elements in (1.3.10) have block operator representations

$$P_- \mathcal{A} P_- = \begin{pmatrix} A_- & B_- \\ B_-^* & D_- \end{pmatrix}, \quad P_+ \mathcal{A} P_+ = \begin{pmatrix} A_+ & B_+ \\ B_+^* & D_+ \end{pmatrix}$$

with

$$A_\pm := E_A(I_\pm) A E_A(I_\pm), \; B_\pm := E_A(I_\pm) B E_D(I_\pm), \; D_\pm := E_D(I_\pm) D E_D(I_\pm).$$

Now Theorem 1.3.6 i), applied to the block operator matrices $P_- \mathcal{A} P_-$ and $P_+ \mathcal{A} P_+$, shows that

$$\max \sigma(P_- \mathcal{A} P_-) \le \max\bigl\{\max \sigma(A_-), \max \sigma(D_-)\bigr\} + \delta_{B_-}^+, \tag{1.3.11}$$

$$\min \sigma(P_+ \mathcal{A} P_+) \ge \min\bigl\{\min \sigma(A_+), \min \sigma(D_+)\bigr\} - \delta_{B_+}^- \tag{1.3.12}$$

where

$$\delta_{B_-}^+ := \|B_-\| \tan\left(\frac{1}{2} \arctan \frac{2\,\|B_-\|}{|\sup W(A_-) - \sup W(D_-)|}\right),$$

$$\delta_{B_+}^- := \|B_+\| \tan\left(\frac{1}{2} \arctan \frac{2\,\|B_+\|}{|\inf W(A_+) - \inf W(D_+)|}\right).$$

Obviously, $\sigma(A_\pm) \subset \sigma(A)$, $\sigma(D_\pm) \subset \sigma(D)$ so that

$$\begin{aligned} \bigl|\sup W(A_-) - \sup W(D_-)\bigr| &= \bigl|\max \sigma(A_-) - \max \sigma(D_-)\bigr| \\ &\ge \operatorname{dist}\bigl(\sigma(A_-), \max \sigma(D_-)\bigr) \\ &\ge \operatorname{dist}\bigl(\sigma(A), \max \sigma(D)\bigr) = \delta_{A,D}; \end{aligned}$$

analogously, we see that $\bigl|\inf W(A_+) - \inf W(D_+)\bigr| \ge \delta_{A,D}$. Since the function

$$h(t) := t \tan\left(\frac{1}{2} \arctan(2t)\right), \quad t \in [0, \infty), \tag{1.3.13}$$

is strictly monotonically increasing and $\|B_\pm\| \le \|B\|$, we conclude that $\delta_{B_-}^+ \le \delta_B$ and $\delta_{B_+}^- \le \delta_B$. Furthermore, we have $\operatorname{dist}\bigl(\lambda, \sigma(A) \dot\cup \sigma(D)\bigr) > \delta_B$ by assumption and

$$\max\bigl\{\max \sigma(A_-), \max \sigma(D_-)\bigr\}, \min\bigl\{\min \sigma(A_+), \min \sigma(D_+)\bigr\} \in \sigma(A) \dot\cup \sigma(D).$$

This and the inequalities (1.3.11), (1.3.12) imply that

$$\max \sigma(P_- \mathcal{A} P_-) < \lambda < \min \sigma(P_+ \mathcal{A} P_+).$$

Applying Theorem 1.3.6 ii) to $\mathcal{A}$ with respect to the block operator matrix representation (1.3.10), we conclude that

$$\lambda \in \bigl((\max \sigma(P_- \mathcal{A} P_-), \min \sigma(P_+ \mathcal{A} P_+)\bigr) \subset \rho(\mathcal{A}).$$

For the proof of ii) and iii), we note that for the function h defined in (1.3.13), we have $h(\sqrt{3}/2) = 1/2$, $h(\sqrt{2}) = 1$. Hence

$$\|B\| < \frac{\sqrt{3}}{2}\, \delta_{A,D} \implies \delta_B < \frac{\delta_{A,D}}{2}, \quad \|B\| < \sqrt{2}\, \delta_{A,D} \implies \delta_B < \delta_{A,D}. \tag{1.3.14}$$

Then ii) is immediate from i) and the first implication in (1.3.14). For the proof of iii), we observe that for $\sigma(\mathcal{A}) \cap \big(\operatorname{conv}\sigma(A)\big)$, the claim follows from i). For $\sigma(\mathcal{A}) \cap \big(\mathbb{R} \setminus \big(\operatorname{conv}\sigma(A)\big)\big)$, the claim follows if we show that

$$\big(\max\sigma(A) + \delta_B, \max\sigma(A) + \delta_{A,D}\big) \subset \rho(\mathcal{A}), \tag{1.3.15}$$

$$\big(\min\sigma(A) - \delta_{A,D}, \min\sigma(A) - \delta_B\big) \subset \rho(\mathcal{A}). \tag{1.3.16}$$

We prove (1.3.15); the proof of (1.3.16) is similar. We let $\lambda = \max\sigma(A) + \delta_B$ and proceed as in the proof of i). Then $E_A(I_+) = 0$ and thus

$$P_+\mathcal{A}P_+ = \begin{pmatrix} 0 & 0 \\ 0 & D_+ \end{pmatrix}.$$

Using analogous estimates as in the proof of i) and the second implication in (1.3.14), we arrive at

$$\begin{aligned} \max\sigma(P_-\mathcal{A}P_-) < \lambda = \max\sigma(A) + \delta_B < \max\sigma(A) + \delta_{A,D} \\ \le \min\sigma(D_+) = \min\sigma(P_+\mathcal{A}P_+). \end{aligned}$$

Now Theorem 1.3.6 ii) shows that (1.3.15) holds. □

The following example illustrating Theorem 1.3.7 shows that the norm bounds therein are sharp.

Example 1.3.8 Consider the family of 3×3 matrices

$$\mathcal{A}_\varepsilon := \left(\begin{array}{c|cc} 0 & \sqrt{2}\,\varepsilon & 0 \\ \hline \sqrt{2}\,\varepsilon & -1 & 0 \\ 0 & 0 & 1 \end{array}\right), \quad 0 \le \varepsilon \le 1,$$

with respect to the decomposition $\mathbb{C}^3 = \mathbb{C} \oplus \mathbb{C}^2$. Then the spectra of the diagonal elements, $\{0\}$ and $\{-1, 1\}$, are disjoint and have distance 1. The eigenvalues of $\mathcal{A}_\varepsilon$ are given by

$$\lambda_1^\varepsilon = -\frac{1}{2} - \sqrt{2\,\varepsilon^2 + \frac{1}{4}}, \quad \lambda_2^\varepsilon = -\frac{1}{2} + \sqrt{2\,\varepsilon^2 + \frac{1}{4}}, \quad \lambda_3^\varepsilon = 1.$$

If $\varepsilon < \sqrt{3}/(2\sqrt{2})$, then the norm of the off-diagonal entry satisfies the assumption in Theorem 1.3.7 ii) which yields the inclusions

$$\sigma_1 = \{\lambda_2^\varepsilon\} \subset \big(-1/2, 1/2\big), \quad \sigma_2 = \{\lambda_1^\varepsilon, \lambda_3^\varepsilon\} \subset \big(-3/2, -1/2\big) \cup \big(1/2, 3/2\big);$$

if $\varepsilon = \sqrt{3}/(2\sqrt{2})$ and hence the norm of the off-diagonal entry reaches the critical value of the norm bound, then $\lambda_2^\varepsilon = 1/2$ and $\lambda_1^\varepsilon = -3/2$ reach the boundaries of the above inclusion intervals.

Since $\operatorname{conv}\{0\} \cap \{-1, 1\} = \emptyset$, Theorem 1.3.7 iii) applies as well. If $\varepsilon < 1$, then the norm of the off-diagonal entry satisfies the assumption in Theorem 1.3.7 iii) which yields the inclusions

$$\sigma_1 = \{\lambda_2^\varepsilon\} \subset \big(-1,1\big), \quad \sigma_2 = \{\lambda_1^\varepsilon, \lambda_3^\varepsilon\} \subset \big(-\infty,-1\big] \cup \big[1,\infty\big);$$

if $\varepsilon = 1$, then the norm of the off-diagonal entry reaches the critical value and we have $\lambda_2^\varepsilon = 1 = \lambda_3^\varepsilon$ and hence σ_1 and σ_2 are no longer disjoint.

If $\mathcal{A}$ is $\mathcal{J}$-self-adjoint, the spectral inclusion by the quadratic numerical range yields the following estimate for $\sigma(\mathcal{A})$ (see [LLMT05, Theorem 2.1] and [Tre08, Theorem 5.4]).

Proposition 1.3.9 *Let the block operator matrix $\mathcal{A}$ be of the form*

$$\mathcal{A} = \begin{pmatrix} A & B \\ -B^* & D \end{pmatrix}$$

with $A = A^*$, $D = D^*$ and define

$$\begin{aligned} a_- &:= \inf W(A), \quad & a_+ &:= \sup W(A), \\ d_- &:= \inf W(D), \quad & d_+ &:= \sup W(D). \end{aligned}$$

Then the spectrum of $\mathcal{A}$, which is symmetric to $\mathbb{R}$, satisfies the following estimates:

i) $\sigma(\mathcal{A}) \cap \mathbb{R} \subset \overline{\operatorname{conv}\big(W(A) \cup W(D)\big)} = \big[\min\{a_-, d_-\}, \max\{a_+, d_+\}\big]$.

ii) $\sigma(\mathcal{A}) \setminus \mathbb{R} \subset \left\{ z \in \mathbb{C} \colon \dfrac{a_- + d_-}{2} \le \operatorname{Re} z \le \dfrac{a_+ + d_+}{2},\ |\operatorname{Im} z| \le \|B\| \right\}$.

iii) *If* $\delta := \operatorname{dist}\big(W(A), W(D)\big) = \min\big\{a_- - d_+, d_- - a_+\big\} > 0$, *then*

$$\begin{aligned} \|B\| \le \delta/2 &\implies \sigma(\mathcal{A}) \subset \mathbb{R}, \\ \|B\| > \delta/2 &\implies \sigma(\mathcal{A}) \setminus \mathbb{R} \subset \left\{ z \in \mathbb{C} : |\operatorname{Im} z| \le \sqrt{\|B\|^2 - \delta^2/4} \right\}. \end{aligned}$$

Proof. Since the block operator matrix $\mathcal{A}$ is $\mathcal{J}$-self-adjoint (see Definition 1.1.14), the symmetry of $\sigma(\mathcal{A})$ to $\mathbb{R}$ is clear. All claims in i), ii), and iii) follow from the spectral inclusion in Theorem 1.3.1 and from the estimates for the quadratic numerical range in Proposition 1.2.6. □

Example 1.3.10 The 4×4 matrix

$$\mathcal{A}_4 := \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ \hline 0 & -2 & -1 & 0 \\ -1 & -3 & 0 & 0 \end{array}\right)$$

satisfies the assumptions of Proposition 1.3.9 with $a_- = 0$, $a_+ = 1$ and $d_- = -1$, $d_+ = 0$; the inclusion in ii) therein yields that the non-real part of $W^2(\mathcal{A}_4)$ is confined to the strip $-1/2 \le \operatorname{Re} z \le 1/2$ (see Fig. 1.6) .

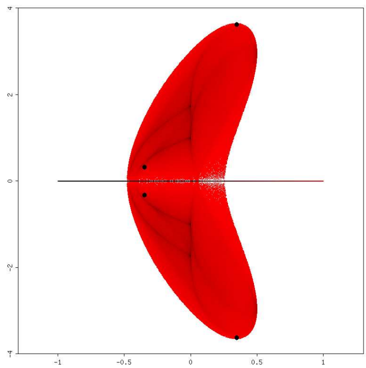

Figure 1.6 Quadratic numerical range of $\mathcal{A}_4$.

1.4 Estimates of the resolvent

The norm of the resolvent $(\mathcal{A}-\lambda)^{-1}$ of a bounded linear operator $\mathcal{A}$ can be estimated in terms of the numerical range as (see [Kat95, Theorem V.3.2])

$$\|(\mathcal{A}-\lambda)^{-1}\| \leq \frac{1}{\operatorname{dist}\big(\lambda, W(\mathcal{A})\big)}, \quad \lambda \notin \overline{W(\mathcal{A})}.$$

The quadratic numerical range yields an analogous estimate in which the distance of λ to $W^2(\mathcal{A})$ enters quadratically, not linearly.

Theorem 1.4.1 *The resolvent of $\mathcal{A}$ admits the estimate*

$$\|(\mathcal{A}-\lambda)^{-1}\| \leq \frac{\|\mathcal{A}\|+|\lambda|}{\operatorname{dist}\big(\lambda, W^2(\mathcal{A})\big)^2}, \quad \lambda \notin \overline{W^2(\mathcal{A})}. \tag{1.4.1}$$

In the proof of this theorem we use the following lemma.

Lemma 1.4.2 *If there exists a $\delta > 0$ such that for all $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$*

$$\|\mathcal{A}_{f,g}\,\alpha\| \geq \delta\,\|\alpha\|, \quad \alpha \in \mathbb{C}^2, \tag{1.4.2}$$

then

$$\|\mathcal{A}\mathbf{x}\| \geq \delta\,\|\mathbf{x}\|, \quad \mathbf{x} \in \mathcal{H}. \tag{1.4.3}$$

Proof. Let $\mathbf{x} \in \mathcal{H}$. Then $\mathbf{x} = (\alpha_1 f\ \alpha_2 g)^{\mathrm{t}}$ with elements $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$ and $\alpha_1, \alpha_2 \in \mathbb{C}$. For $\alpha = (\alpha_1\ \alpha_2)^{\mathrm{t}} \in \mathbb{C}^2$ we have

$$\mathcal{A}_{f,g}\alpha = \begin{pmatrix} (Af,f)\alpha_1 + (Bg,f)\alpha_2 \\ (Cf,g)\alpha_1 + (Dg,g)\alpha_2 \end{pmatrix} = \begin{pmatrix} \big(A(\alpha_1 f) + B(\alpha_2 g), f\big) \\ \big(C(\alpha_1 f) + D(\alpha_2 g), g\big) \end{pmatrix}$$

and hence

$$\begin{aligned}\|\mathcal{A}_{f,g}\alpha\|^2 &= \big|\big(A(\alpha_1 f)+B(\alpha_2 g), f\big)\big|^2+\big|\big(C(\alpha_1 f)+D(\alpha_2 g), g\big)\big|^2 \\ &\le \|A(\alpha_1 f)+B(\alpha_2 g)\|^2+\|C(\alpha_1 f)+D(\alpha_2 g)\|^2=\|\mathcal{A}\mathbf{x}\|^2.\end{aligned} \tag{1.4.4}$$

Since $\|\mathbf{x}\|^2=|\alpha_1|^2+|\alpha_2|^2=\|\alpha\|^2$, (1.4.3) follows from (1.4.2) and (1.4.4). □

***Proof of Theorem* 1.4.1.** Let $\lambda \notin \overline{W^2(\mathcal{A})}$. Then the relation (1.3.2) implies that, for $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$,

$$\big\|(\mathcal{A}_{f,g}-\lambda)^{-1}\big\| = \frac{\|\mathcal{A}_{f,g}-\lambda\|}{|\det(\mathcal{A}_{f,g}-\lambda)|} = \frac{\|\mathcal{A}_{f,g}-\lambda\|}{\big|\lambda-\lambda_1\binom{f}{g}\big|\big|\lambda-\lambda_2\binom{f}{g}\big|}$$

where $\lambda_{1,2}\binom{f}{g} \in W^2(\mathcal{A})$ are the eigenvalues of $\mathcal{A}_{f,g}$. Since $\|\mathcal{A}_{f,g}\| \le \|\mathcal{A}\|$ and $\big|\lambda-\lambda_{1,2}\binom{f}{g}\big| \ge \operatorname{dist}\big(\lambda, W^2(\mathcal{A})\big)$, we find that, for all $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$,

$$\big\|(\mathcal{A}_{f,g}-\lambda)^{-1}\big\| \le \frac{\|\mathcal{A}-\lambda\|}{\operatorname{dist}\big(\lambda, W^2(\mathcal{A})\big)^2} \le \frac{\|\mathcal{A}\|+|\lambda|}{\operatorname{dist}\big(\lambda, W^2(\mathcal{A})\big)^2}.$$

According to Lemma 1.4.2, this gives

$$\|(\mathcal{A}-\lambda)\mathbf{x}\| \ge \frac{\operatorname{dist}\big(\lambda, W^2(\mathcal{A})\big)^2}{\|\mathcal{A}\|+|\lambda|}\|\mathbf{x}\|, \quad \mathbf{x} \in \mathcal{H}.$$

Since $\lambda \notin \overline{W^2(\mathcal{A})}$ implies $\lambda \in \rho(\mathcal{A})$ by Theorem 1.3.1, (1.4.1) follows. □

The following example shows that, in general, the resolvent estimate in Theorem 1.4.1 cannot be improved.

Example 1.4.3 Let $A=C=D=0$ and $B \in L(\mathcal{H}_2,\mathcal{H}_1)$, $B \not\equiv 0$. Then

$$\mathcal{A} = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \quad W^2(\mathcal{A}) = \{0\},$$

and, for $\lambda \notin \overline{W^2(\mathcal{A})} = \{0\}$,

$$\big\|(\mathcal{A}-\lambda)^{-1}\big\| = \frac{1}{|\lambda|^2}\left\|\begin{pmatrix} -\lambda & -B \\ 0 & -\lambda \end{pmatrix}\right\| = \frac{1}{|\lambda|^2}\big(\|B\|+|\lambda|\big) = \frac{1}{|\lambda|^2}\big(\|\mathcal{A}\|+|\lambda|\big).$$

In a similar way as Theorem 1.4.1, the next two theorems can be proved.

Theorem 1.4.4 *Suppose that there exists a subset $\mathcal{F} \subset W^2(\mathcal{A})$ such that for all $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$ the matrix $\mathcal{A}_{f,g}$ has at most one eigenvalue in $\mathcal{F}$. Then, for all $\lambda \notin \overline{W^2(\mathcal{A})}$ such that $\operatorname{dist}\big(\lambda, W^2(\mathcal{A}) \setminus \mathcal{F}\big) \ge \delta$ with some $\delta > 0$, there exists a constant $\gamma(\delta) > 0$ (independent of λ) such that*

$$\big\|(\mathcal{A}-\lambda)^{-1}\big\| \le \frac{\gamma(\delta)}{\operatorname{dist}(\lambda, \mathcal{F})}.$$

Theorem 1.4.5 *If* $\overline{W^2(\mathcal{A})} = \mathcal{F}_1 \dot\cup \mathcal{F}_2$ *consists of two components, then*

$$\big\|(\mathcal{A}-\lambda)^{-1}\big\| \le \frac{\|\mathcal{A}\| + |\lambda|}{\operatorname{dist}(\lambda,\mathcal{F}_1)\operatorname{dist}(\lambda,\mathcal{F}_2)}, \quad \lambda \notin \overline{W^2(\mathcal{A})}.$$

Remark 1.4.6 The situation described in Theorem 1.4.4 occurs *e.g.* for $\mathcal{J}$-self-adjoint block operator matrices as in Proposition 1.3.9 provided that the numbers $a_\pm$ and $d_\pm$ defined therein satisfy $a_- < d_- < a_+ < d_+$. Then each matrix $\mathcal{A}_{f,g}$, $f \in S_{\mathcal{H}_1}$, $g \in S_{\mathcal{H}_2}$ has at most one eigenvalue in each of the intervals $\big[a_-, (a_- + d_-)/2\big]$ and $\big[(a_+ + d_+)/2, d_+\big]$.

The resolvent estimate in terms of the numerical range implies that the length of a Jordan chain at an eigenvalue lying on the boundary of the numerical range is at most one, *i.e.* there are no associated vectors. As a corollary of Theorem 1.4.1, we obtain an analogue for boundary points of the quadratic numerical range. Since the latter is no longer convex, we need the following definition.

Definition 1.4.7 Let $W \subset \mathbb{C}$. A boundary point $\lambda_0 \in \partial W$ is said to have the *exterior cone property* if there exists a closed cone K (having positive aperture) with vertex λ_0 such that, for some $r > 0$,

$$K \cap \big\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| \le r\big\} \cap \overline{W} = \{\lambda_0\}.$$

Corollary 1.4.8 *Let* $\lambda_0 \in \sigma_{\rm p}(\mathcal{A})$. *If* $\lambda_0 \in \partial W^2(\mathcal{A})$ *has the exterior cone property, then the length of a Jordan chain at* λ_0 *is at most two.*

If, in the situation of Theorem 1.4.4, $\lambda_0 \in \partial\mathcal{F}$ *has the exterior cone property and* $\operatorname{dist}\big(\lambda, W^2(\mathcal{A})\setminus\mathcal{F}\big) > 0$, *or, in the situation of Theorem* 1.4.5, $\lambda_0 \in \partial\mathcal{F}_1 \dot\cup \partial\mathcal{F}_2$ *has the exterior cone property, then the length of a Jordan chain at* λ_0 *is at most one, i.e. there are no associated vectors at* λ_0.

Proof. Assume that there exists a Jordan chain $\{x_0, x_1, x_2\}$ of length 3 at $\lambda_0 \in \partial W^2(\mathcal{A})$. Then, for $\lambda \notin \overline{W^2(\mathcal{A})}$,

$$\big\|(\mathcal{A}-\lambda)^{-1}x_2\big\| = \left\| \frac{1}{(\lambda_0-\lambda)^3}x_0 + \frac{1}{(\lambda_0-\lambda)^2}x_1 + \frac{1}{\lambda_0-\lambda}x_2 \right\| \ge C\frac{1}{|\lambda_0-\lambda|^3}$$

for $|\lambda_0 - \lambda|$ sufficiently small. If λ_0 has the exterior cone property and λ lies on the axis of the cone K, $|\lambda_0 - \lambda| \le r$, then $|\lambda_0 - \lambda| \le C' \operatorname{dist}\big(\lambda, W^2(\mathcal{A})\big)$ with some constant $C' > 0$ and hence

$$\big\|(\mathcal{A}-\lambda)^{-1}x_2\big\| \ge C'' \frac{1}{\operatorname{dist}\big(\lambda, W^2(\mathcal{A})\big)^3},$$

a contradiction. The proof of the other two assertions is similar. □

The following example shows that Jordan chains of length two may occur at boundary points of the quadratic numerical range.

Example 1.4.9 In Example 1.4.3, the point 0 lies on the boundary of $W^2(\mathcal{A}) = \{0\}$ and has the exterior cone property. If we choose $g \in \mathcal{H}_2$ such that $Bg \neq 0$, then $(Bg\ 0)^{\mathrm{t}}$, $(0\ g)^{\mathrm{t}}$ is a Jordan chain of $\mathcal{A}$ at 0 of length two.

1.5 Corners of the quadratic numerical range

For a bounded linear operator T in a Hilbert space it is well-known that a corner $\lambda_0 \in W(T)$ of the numerical range $W(T)$ is an eigenvalue of T. Moreover, every corner of $\lambda_0 \in \overline{W(T)}$ belongs to the spectrum of T (see [Kat95], [HJ91], [GR97, Theorem 1.5-5, Corollary 1.5-6]).

For the quadratic numerical range these statements do not generalize in a straightforward way; this can be seen *e.g.* from the quadratic numerical range of the 4×4 matrix $\mathcal{A}_2$ in Example 1.1.5 which has 8 corners (see Fig. 1.2). It turns out that here not only the spectrum of the block operator matrix itself but also the spectra of its diagonal elements come into play.

We begin by giving the precise definition of a corner of a subset of $\mathbb{C}$.

Definition 1.5.1 Let $W \subset \mathbb{C}$. A boundary point $\alpha \in \partial W$ is called *corner* of W if there exist $\psi \in [0,\pi)$, $\varphi \in [0, 2\pi)$, and $\varepsilon > 0$ so that

$$\varphi \leq \arg(\lambda - \alpha) \leq \varphi + \psi, \quad \lambda \in W,\ |\lambda - \alpha| < \varepsilon, \tag{1.5.1}$$

where $\arg(\,\cdot\,)$ is suitably defined. The infimum ψ_0 of all $\psi \in [0,\pi)$ such that there exist $\varphi \in [0, 2\pi)$ and $\varepsilon > 0$ with (1.5.1) is called *angle* of the corner α.

Theorem 1.5.2 *Let $\lambda_0 \in W^2(\mathcal{A})$ and let $x_0 \in S_{\mathcal{H}_1}$, $y_0 \in S_{\mathcal{H}_2}$ be such that λ_0 is a zero of*

$$\Delta(x_0, y_0; \lambda) = \det \begin{pmatrix} (Ax_0, x_0) - \lambda & (By_0, x_0) \\ (Cx_0, y_0) & (Dy_0, y_0) - \lambda \end{pmatrix}. \tag{1.5.2}$$

If λ_0 is a corner of $W^2(\mathcal{A})$, then at least one of the following holds:

i) *λ_0 is an eigenvalue of A with eigenvector x_0,*

ii) *λ_0 is an eigenvalue of D with eigenvector y_0,*

iii) *λ_0 is an eigenvalue of $\mathcal{A}$ with eigenvector $\big(x_0\ \gamma y_0\big)^{\mathrm{t}}$ where*

$$\gamma = -\frac{(Cx_0, y_0)}{\big((D-\lambda_0)y_0, y_0\big)} \quad \text{or} \quad \gamma = -\frac{\big((A-\lambda_0)x_0, x_0\big)}{(By_0, x_0)}.$$

Proof. Without loss of generality, we assume that $\lambda_0 = 0$. First we consider the case of a simple zero. For $y \in S_{\mathcal{H}_2}$ and $z \in \mathbb{C}$ we define

$$\begin{aligned} g_y(\lambda, z) := & \big((Ax_0, x_0) - \lambda\big)\big(\big(D(y_0 + zy), y_0 + \overline{z}y\big) - \lambda(y_0 + zy, y_0 + \overline{z}y)\big) \\ & - \big(B(y_0 + zy), x_0\big)\big(Cx_0, y_0 + \overline{z}y\big). \end{aligned}$$

Then $g_y(\lambda, 0) = \Delta(x_0, y_0; \lambda)$ and $g_y(\cdot, z)$ is a quadratic polynomial in λ. The latter has a zero $\lambda_y(z)$ such that λ_y is analytic in a neighbourhood U of 0 with $\lambda_y(0) = \lambda_0 = 0$ and which is given by

$$\begin{aligned} \lambda_y(z) = & \frac{(Ax_0, x_0)}{2} + \frac{\big(D(y_0+zy), y_0+\overline{z}y\big)}{2(y_0+zy, y_0+\overline{z}y)} \qquad (1.5.3) \\ & + \sqrt{\left(\frac{(Ax_0, x_0)}{2} - \frac{\big(D(y_0+zy), y_0+\overline{z}y\big)}{2(y_0 + zy, y_0 + \overline{z}y)}\right)^2 + \frac{\big(B(y_0+zy), x_0\big)\big(Cx_0, y_0+\overline{z}y\big)}{4(y_0 + zy, y_0 + \overline{z}y)}}; \end{aligned}$$

here the branch of the square root is chosen such that $\lambda_y(0) = 0$. Obviously, $\lambda_y(t) \in \sigma_{\mathrm{p}}(\mathcal{A}_{x_0, y_0+ty}) \subset W^2(\mathcal{A})$ for real $t \in U$ and, by assumption, $\lambda_y(0) = 0$ is a corner of $W^2(\mathcal{A})$. This implies that the curve $\lambda_y(t)$, $t \in U \cap \mathbb{R}$, does not have a tangent in the point 0 and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda_y(t)\Big|_{t=0} = 0. \qquad (1.5.4)$$

On the other hand, $g_y\big(\lambda_y(z), z\big) = 0$ for all $z \in \mathbb{C}$ and hence, for $t \in U \cap \mathbb{R}$,

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\, g_y\big(\lambda_y(t), t\big) \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\lambda_y(t)\Big(\big(D(y_0 + ty), y_0 + ty\big) - \lambda_y(t)\big(y_0 + ty, y_0 + ty\big)\Big) \\ &\quad + \big((Ax_0, x_0) - \lambda_y(t)\big)\Big((Dy, y_0 + ty) + (D(y_0 + ty), y) \\ &\quad - \frac{\mathrm{d}}{\mathrm{d}t}\lambda_y(t)\big(y_0 + ty, y_0 + ty\big) - \lambda_y(t)\big((y, y_0 + ty) + (y_0 + ty, y)\big)\Big) \\ &\quad - \big(By, x_0\big)\big(Cx_0, y_0 + ty\big) - \big(B(y_0 + ty), x_0\big)\big(Cx_0, y\big). \end{aligned}$$

For $t = 0$ we obtain, together with (1.5.4) and $\lambda_y(0) = 0$,

$$\begin{aligned} 0 &= (Ax_0, x_0)\big((Dy, y_0) + (Dy_0, y)\big) - (By, x_0)(Cx_0, y_0) - (By_0, x_0)(Cx_0, y) \\ &= \big(y, \overline{(Ax_0, x_0)}D^*y_0 - \overline{(Cx_0, y_0)}B^*x_0\big) + \big((Ax_0, x_0)Dy_0 - (By_0, x_0)Cx_0, y\big). \end{aligned}$$

Since $y \in S_{\mathcal{H}_2}$ was arbitrary, the above relation also holds with $\mathrm{i}y$ instead of y and so it follows that

$$(Ax_0, x_0)Dy_0 - (By_0, x_0)Cx_0 = 0, \tag{1.5.5}$$

$$\overline{(Ax_0, x_0)}D^*y_0 - \overline{(Cx_0, y_0)}B^*x_0 = 0. \tag{1.5.6}$$

In a similar way, for $x \in S_{\mathcal{H}_1}$ and $z \in \mathbb{C}$ we consider the polynomial

$$\begin{aligned} h_x(\lambda, z) :=& \big(\big(A(x_0 + zx), x_0 + \overline{z}x\big) - \lambda(x_0 + zx, x_0 + \overline{z}x)\big)\big((Dy_0, y_0) - \lambda\big) \\ &- \big(By_0, x_0 + \overline{z}x\big)\big(C(x_0 + zx), y_0\big) \end{aligned}$$

and arrive at

$$(Dy_0, y_0)Ax_0 - (Cx_0, y_0)By_0 = 0, \tag{1.5.7}$$

$$\overline{(Dy_0, y_0)}A^*x_0 - \overline{(By_0, x_0)}C^*y_0 = 0. \tag{1.5.8}$$

The numerical range $W(A)$ of A is contained in $W^2(\mathcal{A})$ if $\dim \mathcal{H}_2 \geq 2$ (and analogously for D, see Theorem 1.1.9). We distinguish the following cases:

a) $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = 1$: In this case $W^2(\mathcal{A})$ consists only of the two eigenvalues of $\mathcal{A}$, and the assertion is trivial.

b) $\dim \mathcal{H}_1 = 1$ or $\dim \mathcal{H}_2 = 1$: Let $\dim \mathcal{H}_2 = 1$; the case $\dim \mathcal{H}_1 = 1$ is analogous. Then D is the multiplication by a constant, say d. If $Ax_0 = 0$ or $d = 0$, the corner 0 is an eigenvalue of A or of D, respectively. If $Ax_0 \neq 0$ and $d \neq 0$, relation (1.5.7) yields that $(Cx_0, y_0) \neq 0$ and

$$Ax_0 + B\left(-\frac{(Cx_0, y_0)}{d}y_0\right) = 0.$$

Moreover, in fact $y_0 = 1$ and $Dy_0 = d$, so that we also have

$$Cx_0 + D\left(-\frac{(Cx_0, y_0)}{d}y_0\right) = 0.$$

Hence 0 is an eigenvalue of $\mathcal{A}$ with eigenvector

$$\begin{pmatrix} x_0 \\ -\dfrac{(Cx_0, y_0)}{(Dy_0, y_0)}y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ -\dfrac{1}{d}Cx_0 \end{pmatrix}.$$

Note that since $\dim \mathcal{H}_2 = 1$, we cannot conclude from $Ax_0 \neq 0$ that $(Ax_0, x_0) \neq 0$ and hence $(By_0, x_0) \neq 0$; the reason for this is that $(Ax_0, x_0) = 0$ shows that $0 \in W(A)$, but we cannot conclude that 0 is a corner of $W(A)$ because the numerical range of A need not be contained in $W^2(\mathcal{A})$ (see Theorem 1.1.9). Therefore, in this case we can only use the first form of the constant γ in the eigenvectors in iii).

c) $\dim \mathcal{H}_1 \geq 2$, $\dim \mathcal{H}_2 \geq 2$: First let $(Ax_0, x_0) = 0$. By Theorem 1.1.9, it follows that $W(A) \subset W^2(\mathcal{A})$ since $\dim\, \mathcal{H}_2 \geq 2$. Hence $0 \in W(A)$ is also a corner of $W(A)$. The well known theorem on corners of the numerical range (see [GR97, Theorem1.5-5]) now implies that $Ax_0 = 0$. If $(Dy_0, y_0) = 0$, a similar reasoning yields that $Dy_0 = 0$. If $(Ax_0, x_0) \neq 0$ and $(Dy_0, y_0) \neq 0$, then also $(By_0, x_0) \neq 0$ and (1.5.5), (1.5.7) imply that

$$Ax_0 + B\left(-\frac{(Cx_0, y_0)}{(Dy_0, y_0)}\, y_0\right) = 0,$$
$$Cx_0 + D\left(-\frac{(Ax_0, x_0)}{(By_0, x_0)}\, y_0\right) = 0.$$

Using the relation $(Ax_0, x_0)(Dy_0, y_0) - (By_0, x_0)(Cx_0, y_0) = 0$, we conclude that 0 is an eigenvalue of $\mathcal{A}$ with an eigenvector of the asserted form.

If λ_0 is a double zero, then, for every $y \in S_{\mathcal{H}_2}$, there are two root functions $\lambda_y^{(1)}(t)$, $\lambda_y^{(2)}(t)$, $t \in \mathbb{R}$, such that $g_y\big(\lambda_y^{(j)}(t), t\big) = 0$ near $t=0$ and $\lambda_y^{(j)}(0) = 0$, $j = 1, 2$, with Puiseux expansions (see *e.g.* [Kat95, Section II.1.2])

$$\lambda_y^{(j)}(t) = \alpha_1 \mathrm{e}^{\pi \mathrm{i} j} t^{1/2} + \alpha_2 \mathrm{e}^{2\pi \mathrm{i} j} t + \ldots, \quad j = 1, 2.$$

If $\alpha_1 \neq 0$, the four one-sided tangents of the functions $\lambda_y^{(1)}(t)$, $\lambda_y^{(2)}(t)$, $\lambda_y^{(1)}(-t)$, and $\lambda_y^{(2)}(-t)$, $t \geq 0$, divide the plane into four sectors of angle $\pi/2$. This contradicts the fact that 0 is a corner of $W^2(\mathcal{A})$. If $\alpha_1 = 0$, then $\lambda_y^{(j)}(t)$ are differentiable at 0 and the claim follows in the same way as in the case of a simple zero. □

Remark 1.5.3 From equations (1.5.6) and (1.5.8), it follows that in case iii) of Theorem 1.5.2 the point $\overline{\lambda_0}$ is an eigenvalue of $\mathcal{A}^*$ with eigenvector $(x_0\ \widetilde{\gamma} y_0)^{\mathrm{t}}$ where

$$\widetilde{\gamma} = -\frac{(B^* x_0, y_0)}{\big((D^* - \overline{\lambda_0}) y_0, y_0\big)} \quad \text{or} \quad \widetilde{\gamma} = -\frac{\big((A^* - \overline{\lambda_0}) x_0, x_0\big)}{(C^* y_0, x_0)}.$$

Remark 1.5.4 In order to prove relation (1.5.5), it is sufficient to consider roots $\lambda_{y_1}(t)$, $\lambda_{y_2}(t)$ (for real t) with

$$y_1 = (Ax_0, x_0) Dy_0 - (By_0, x_0) Cx_0 \quad \text{and} \quad y_2 = \mathrm{i} y_1,$$

and, for the proof of relation (1.5.7),

$$y_1 = (Dy_0, y_0) Ax_0 - (Cx_0, y_0) By_0 \ \text{ and } \ y_2 = \mathrm{i} y_1.$$

Example 1.5.5 Consider the 4×4 matrices

$$\mathcal{A}_5 := \left(\begin{array}{cc|cc} 1 & 3+\mathrm{i} & 2 & \mathrm{i} \\ 3+\mathrm{i} & 1 & \mathrm{i} & 2 \\ \hline -2 & \mathrm{i} & 1 & 3+\mathrm{i} \\ \mathrm{i} & -2 & 3+\mathrm{i} & 1 \end{array}\right), \quad \mathcal{A}_6 := \left(\begin{array}{cc|cc} 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \\ \hline -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & -1 \end{array}\right).$$

Their quadratic numerical ranges, displayed in Fig. 1.7, both have 6 corners: For $\mathcal{A}_5$ the four corners $-2-\mathrm{i}(1-\sqrt{5})$, $-2-\mathrm{i}(1+\sqrt{5})$, $4+\mathrm{i}(1+\sqrt{5})$, $4+\mathrm{i}(1-\sqrt{5})$ are the eigenvalues of $\mathcal{A}_5$ (marked by black dots), the corners $4+\mathrm{i}$, $-2-\mathrm{i}$ are the eigenvalues of the left upper corner A (and, at the same time, of the right lower corner D). For $\mathcal{A}_6$ the four corners $-1+2\mathrm{i}$, $-1-2\mathrm{i}, 1+2\mathrm{i}, 1-2\mathrm{i}$ are the eigenvalues of $\mathcal{A}_6$ (marked by black dots), the corners $-1, 1$ are the eigenvalues of A (and, at the same time, of D).

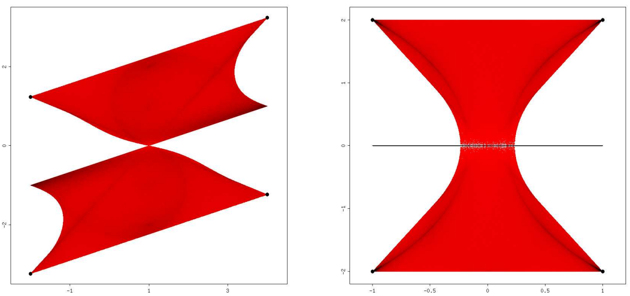

Figure 1.7 Quadratic numerical ranges of $\mathcal{A}_5$ and $\mathcal{A}_6$.

Next we consider corners of the quadratic numerical range $W^2(\mathcal{A})$ which do not belong to $W^2(\mathcal{A})$, but only to its closure. For this purpose, we use the well-known method of Banach limits (see [Ber62]). By passing to a Hilbert space formed by bounded sequences of $\mathcal{H}$, we convert points of the spectrum into eigenvalues of a corresponding linear operator and apply the previous Theorem 1.5.2 to the latter.

Definition 1.5.6 Let $\mathcal{H}$ be an arbitrary Hilbert space, fix a Banach limit LIM on the space of bounded sequences in $\mathbb{C}$ with values in $\mathbb{C}$ (that is, a linear mapping which coincides with the usual limit for convergent sequences and is non-negative for non-negative sequences), let $\mathcal{R}$ be the linear space of all bounded sequences $x = (x_n)_1^\infty \subset \mathcal{H}$ with the (non-negative, but degenerate) inner product

$$[x, y] := \operatorname*{LIM}_{n\to\infty} (x_n, y_n), \quad x = (x_n)_1^\infty,\ y = (y_n)_1^\infty \in \mathcal{R},$$

and let $\mathcal{R}_0$ be the subspace of all $x = (x_n)_1^\infty \in \mathcal{R}$ with

$$\operatorname*{LIM}_{n\to\infty} (x_n, x_n) = 0.$$

Then we define the Hilbert space $\widetilde{\mathcal{H}}$ as the completion of the quotient space $\mathcal{R}/\mathcal{R}_0$ with respect to the norm generated by the inner product $[\,\cdot\,,\,\cdot\,]$. For

Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, we associate with $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ the operator

$$\widetilde{T} \in L(\widetilde{\mathcal{H}}_1, \widetilde{\mathcal{H}}_2), \quad \widetilde{T}\,\widetilde{x} := \big[(Tx_n)_1^\infty\big], \quad \widetilde{x} = (x_n)_1^\infty \in \widetilde{\mathcal{H}}_1,$$

where $[\,\cdot\,]$ denotes the equivalence class in $\mathcal{R}/\mathcal{R}_0$.

The following observations are easy to check (see [Ber62]).

Remark 1.5.7 Let $\mathcal{H}$, $\mathcal{H}_1$, and $\mathcal{H}_2$ be Hilbert spaces.

i) The mapping $T \mapsto \widetilde{T}$ is an isometry from $L(\mathcal{H}_1, \mathcal{H}_2)$ into $L(\widetilde{\mathcal{H}}_1, \widetilde{\mathcal{H}}_2)$.
ii) For $T \in L(\mathcal{H})$, we have $\sigma\big(\widetilde{T}\big) = \sigma_{\mathrm{p}}\big(\widetilde{T}\big) = \sigma_{\mathrm{app}}(T)$.

Theorem 1.5.8 *If* $\lambda_0 \in \overline{W^2(\mathcal{A})}$ *is a corner of* $W^2(\mathcal{A})$, *then*

$$\lambda_0 \in \sigma(A) \cup \sigma(D) \cup \sigma(\mathcal{A}).$$

Proof. Since $\lambda_0 \in \overline{W^2(\mathcal{A})}$, there exist a sequence $(\lambda_n)_1^\infty \subset W^2(\mathcal{A})$ with $\lambda_n \to \lambda_0$, $n \to \infty$, and sequences $(x_n^0)_1^\infty \subset S_{\mathcal{H}_1}$, $(y_n^0)_1^\infty \subset S_{\mathcal{H}_2}$ such that

$$\Delta(x_n^0, y_n^0; \lambda_n) = \det \begin{pmatrix} (Ax_n^0, x_n^0) - \lambda & (By_n^0, x_n^0) \\ (Cx_n^0, y_n^0) & (Dy_n^0, y_n^0) - \lambda \end{pmatrix} = 0.$$

We may assume that, for some neighbourhood V of λ_0, all quadratic polynomials $\Delta(x_n^0, y_n^0; \,\cdot\,)$, $n \in \mathbb{N}$, have either one zero or two zeroes in V.

By means of a Banach limit, we construct the space $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}_1 \oplus \widetilde{\mathcal{H}}_2$ and the operator

$$\widetilde{\mathcal{A}} = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix} : \widetilde{\mathcal{H}}_1 \oplus \widetilde{\mathcal{H}}_2 \to \widetilde{\mathcal{H}}_1 \oplus \widetilde{\mathcal{H}}_2$$

according to Definition 1.5.6. First we consider the case of a simple zero. Since the sequences $(x_n^0)_1^\infty$ and $(y_n^0)_1^\infty$ are bounded, we may assume without loss of generality (by passing to suitable subsequences) that all sequences of the form

$$\big((Fu_n, v_n)\big)_1^\infty \tag{1.5.9}$$

converge where F is one of the operators A, B, C, D, A^*, B^*, C^*, D^* or a product of two or three of them, and u_n, v_n are the elements x_n^0 or y_n^0, whenever the inner products in (1.5.9) are defined. Now let $\widetilde{x}^{\,0} = (x_n^0)_1^\infty \in \widetilde{\mathcal{H}}_1$, $\widetilde{y}^{\,0} = (y_n^0)_1^\infty \in \widetilde{\mathcal{H}}_2$. From Hurwitz's Theorem (see *e.g.* [Tit68, Chapter III, 3.45], it follows that λ_0 is a simple root of

$$\det \begin{pmatrix} (\widetilde{A}\widetilde{x}^{\,0}, \widetilde{x}^{\,0}) - \lambda & (\widetilde{B}\widetilde{y}^{\,0}, \widetilde{x}^{\,0}) \\ (\widetilde{C}\widetilde{x}^{\,0}, \widetilde{y}^{\,0}) & (\widetilde{D}\widetilde{y}^{\,0}, \widetilde{y}^{\,0}) - \lambda \end{pmatrix} = 0.$$

Hence $\lambda_0 \in W^2(\widetilde{\mathcal{A}})$.

Following the lines of the proof of Theorem 1.5.2, we derive the analogues of equalities (1.5.5), (1.5.7) for the operators $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, and $\widetilde{D}$. We introduce the quadratic polynomial $g_{\widetilde{y}}(\lambda, z)$ with its root $\lambda_{\widetilde{y}}(z)$ given by a formula analogous to (1.5.3). By Remark 1.5.4, for the proof of the analogues of (1.5.5), (1.5.7), it is sufficient to consider *e.g.* elements $\widetilde{y} = (y_n)_1^\infty$ which are certain linear combinations of the four vectors $\widetilde{A}\widetilde{x}^0$, $\widetilde{B}\widetilde{y}^0$, $\widetilde{C}\widetilde{x}^0$, and $\widetilde{D}\widetilde{y}^0$. Since all sequences of the form (1.5.9) converge, we can use the ordinary limit instead of the Banach limit in the construction of the quadratic forms occurring in the formula for the root $\lambda_{\widetilde{y}}(z)$. This means that the root $\lambda_{\widetilde{y}}(z)$ is a limit of the corresponding roots $\lambda_{y_n}(z)$. By assumption, all roots $\lambda_y(t)$ for real $t \in U$ lie in a (closed) sector with vertex λ_0 and angle $< \pi$. Hence the roots $\lambda_{\widetilde{y}}(t)$ lie in the same sector, and we obtain the analogues of (1.5.5) and (1.5.7) for $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, $\widetilde{D}$, $\widetilde{x}^0$, and $\widetilde{y}^0$. Now the proof for the case of a simple zero can be completed in a similar way as in the proof of Theorem 1.5.2; note that here we only have to consider case c) since $\dim \widetilde{\mathcal{H}}_1 = \dim \widetilde{\mathcal{H}}_2 = \infty$.

As a result of this and of Remark 1.5.7, we have $\lambda_0 \in \sigma_{\rm p}(\widetilde{A}) = \sigma_{\rm app}(A) \subset \sigma(A)$ or $\lambda_0 \in \sigma_{\rm p}(\widetilde{D}) = \sigma_{\rm app}(D) \subset \sigma(D)$ or $\lambda_0 \in \sigma_{\rm p}(\widetilde{\mathcal{A}}) = \sigma_{\rm app}(\mathcal{A}) \subset \sigma(\mathcal{A})$.

If λ_0 is a double zero, the proof is analogous to the corresponding part of the proof of Theorem 1.5.2. □

1.6 Schur complements and their factorization

In the spectral theory of block operator matrices an important role is played by the so-called Schur complements. For a 2×2 block operator matrix (1.1.3), there exist two Schur complements, one associated with each of the diagonal elements A and D. The Schur complements are analytic operator functions defined outside of the spectrum of D and of A, respectively.

First we prove that the numerical ranges of these analytic operator functions are contained in the quadratic numerical range. Further, we show that if the closure of the quadratic numerical range consists of two components, then a linear operator factor can be split off the Schur complements.

Definition 1.6.1 For a block operator matrix $\mathcal{A}$ given by (1.1.3) the analytic operator functions $S_1 : \mathbb{C}\backslash\sigma(D) \to L(\mathcal{H}_1)$ and $S_2 : \mathbb{C}\backslash\sigma(A) \to L(\mathcal{H}_2)$,

$$S_1(\lambda) := A - \lambda - B(D-\lambda)^{-1}C, \quad \lambda \notin \sigma(D),$$
$$S_2(\lambda) := D - \lambda - C(A-\lambda)^{-1}B, \quad \lambda \notin \sigma(A),$$

are called *Schur complements* of $\mathcal{A}$.

If $\mathcal{H}$ is a Hilbert space, $\Omega \subset \mathbb{C}$ is open, and $S : \Omega \to L(\mathcal{H})$ is an analytic operator function, the resolvent set $\rho(S)$, the spectrum $\sigma(S)$, and the point spectrum $\sigma_{\rm p}(S)$ are defined as (see *e.g.* [Mar88, § 11.2])

$$\begin{aligned}\rho(S) &:= \{\lambda \in \Omega : S(\lambda) \text{ bijective in } \mathcal{H}\},\\ \sigma(S) &:= \Omega \setminus \rho(S),\\ \sigma_{\rm p}(S) &:= \{\lambda \in \Omega : S(\lambda) \text{ not injective in } \mathcal{H}\}.\end{aligned}$$

The numerical range $W(S)$ is defined as the set (see *e.g.* [Mar88, § 26.2])

$$W(S) = \bigl\{\lambda \in \Omega : \exists\, f \in \mathcal{H},\, f \neq 0,\ \bigl(S(\lambda)f, f\bigr) = 0\bigr\}. \tag{1.6.1}$$

Obviously, in the special case $S(\lambda) = T - \lambda$, $\lambda \in \mathbb{C}$, with a linear operator $T \in L(\mathcal{H})$, all these notions coincide with the usual definitions of the resolvent set, spectrum, point spectrum, and numerical range of the linear operator T.

It is well-known (see *e.g.* [Mar88, Theorem 26.6]) that $\sigma(S) \subset \overline{W(S)}$ if there exists a $\lambda_0 \in \Omega$ so that $0 \notin \overline{W(S(\lambda_0))}$. For the Schur complements, this condition is always satisfied for λ_0 large enough since

$$\|A\| + \|B(D-\lambda_0)^{-1}C\| < |\lambda_0| \quad \Longrightarrow \quad 0 \in \rho\bigl(S_1(\lambda_0)\bigr).$$

Hence $\sigma(S_1) \subset \overline{W(S_1)}$ and $\sigma(S_2) \subset \overline{W(S_2)}$.

The following *Frobenius-Schur factorization* of the block operator matrix $\mathcal{A}$ ties its spectral properties closely to those of its Schur complements.

Proposition 1.6.2 *For $\lambda \notin \sigma(D)$ and $\lambda \notin \sigma(A)$, respectively, we have*

$$\mathcal{A} - \lambda = \begin{pmatrix} I & B(D-\lambda)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S_1(\lambda) & 0 \\ 0 & D-\lambda \end{pmatrix} \begin{pmatrix} I & 0 \\ (D-\lambda)^{-1}C & I \end{pmatrix}, \tag{1.6.2}$$

$$\mathcal{A} - \lambda = \begin{pmatrix} I & 0 \\ C(A-\lambda)^{-1} & I \end{pmatrix} \begin{pmatrix} A-\lambda & 0 \\ 0 & S_2(\lambda) \end{pmatrix} \begin{pmatrix} I & (A-\lambda)^{-1}B \\ 0 & I \end{pmatrix}, \tag{1.6.3}$$

and hence

$$\sigma(\mathcal{A}) \setminus \sigma(D) = \sigma(S_1), \quad \sigma(\mathcal{A}) \setminus \sigma(A) = \sigma(S_2).$$

Theorem 1.6.3 *$W(S_1) \cup W(S_2) \subset W^2(\mathcal{A})$.*

Proof. Let $\lambda \in W(S_1)$. Then there exists an $f \in \mathcal{H}_1$, $f \neq 0$, such that $\bigl(S_1(\lambda)f, f\bigr) = 0$. If $Cf = 0$, then $\bigl(S_1(\lambda)f, f\bigr) = \bigl((A-\lambda)f, f\bigr)$ and $\Delta(f, g; \lambda) = \bigl((A-\lambda)f, f\bigr)\bigl((D-\lambda)g, g\bigr)$. Thus $\Delta(f, g; \lambda) = 0$ for every

$g \in \mathcal{H}_2$, $g \neq 0$, and so $\lambda \in W^2(\mathcal{A})$. If $Cf \neq 0$, then $(D-\lambda)^{-1}Cf \neq 0$ and

$$\begin{aligned}\Delta\big(f,(D-\lambda)^{-1}Cf;\lambda\big) =& \big((A-\lambda)f,f\big)\big(Cf,(D-\lambda)^{-1}Cf\big)\\ &- \big(B(D-\lambda)^{-1}Cf,f\big)\big(Cf,(D-\lambda)^{-1}Cf\big)\\ =& \big(S_1(\lambda)f,f\big)\,\big(Cf,(D-\lambda)^{-1}Cf\big). \qquad (1.6.4)\end{aligned}$$

Hence $\big(S_1(\lambda)f,f\big) = 0$ implies that $\Delta\big(f,(D-\lambda)^{-1}Cf;\lambda\big) = 0$ and thus $\lambda \in W^2(\mathcal{A})$. The proof for $W(S_2)$ is similar. □

Theorem 1.6.4 *Suppose that* $\dim\mathcal{H}_1 \geq 2$, $\dim\mathcal{H}_2 \geq 2$, *and assume that* $\overline{W^2(\mathcal{A})} = \mathcal{F}_1 \,\dot\cup\, \mathcal{F}_2$ *consists of two components. Then* $\mathcal{F}_1$, $\mathcal{F}_2$ *can be enumerated such that*

$$W(S_1) \cap \mathcal{F}_1 \neq \emptyset, \quad W(S_2) \cap \mathcal{F}_2 \neq \emptyset.$$

Proof. Due to the dimension conditions, Corollary 1.1.10 i) shows that we can enumerate the components $\mathcal{F}_1$, $\mathcal{F}_2$ such that

$$\overline{W(A)} \subset \mathcal{F}_1, \quad \overline{W(D)} \subset \mathcal{F}_2. \qquad (1.6.5)$$

The claim is trivial if either $B = 0$ or $C = 0$; in this case $W(S_1) = W(A)$ and $W(S_2) = W(D)$. So we may assume that $B \neq 0$ and $C \neq 0$.

Both components $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\overline{W^2(\mathcal{A})}$ are connected disjoint compact subsets of $\mathbb{C}$. Hence there exists a piecewise smooth simply closed Jordan curve Γ_1 such that one of the components is located inside of Γ_1 while the other one is located outside of Γ_1 (see [LMMT01, Lemma 4.2]).

First we consider the case that $\mathcal{F}_1$ lies in the bounded component $\mathcal{U}_1$ of $\mathbb{C} \setminus \Gamma_1$. Then $\mathcal{U}_1$ is a bounded simply connected domain with

$$\mathcal{F}_1 \subset \mathcal{U}_1, \quad \mathcal{F}_2 \cap \overline{\mathcal{U}_1} = \emptyset$$

and the boundary Γ_1 of $\mathcal{U}_1$ is a piecewise smooth simply closed Jordan curve. We choose an element $f \in \mathcal{H}_1$ such that $Cf \neq 0$ and set

$$g(\lambda) := (D-\lambda)^{-1}Cf, \quad \lambda \in \overline{\mathcal{U}_1}.$$

For fixed $\lambda \in \overline{\mathcal{U}_1}$, we consider the function

$$\varphi_\lambda(\mu) := \Delta\big(f, g(\lambda); \mu\big), \quad \mu \in \mathbb{C}.$$

By Proposition 1.1.3, every zero of the quadratic polynomial φ_λ lies in $W^2(\mathcal{A})$. Since $\overline{W^2(\mathcal{A})} = \mathcal{F}_1 \,\dot\cup\, \mathcal{F}_2$, it follows that φ_λ has exactly one zero $\mu(\lambda) \in \mathcal{F}_1$. The function $\mu : \overline{\mathcal{U}_1} \to \mathcal{F}_1$ is continuous and maps $\overline{\mathcal{U}_1}$ into itself. Since $\mathcal{U}_1$ is simply connected and its boundary is a piecewise smooth simply closed Jordan curve, $\overline{\mathcal{U}_1}$ is homeomorphic to the closed unit disc (see *e.g.* [Hur64], Chapter III.6.4). Hence the Brouwer fixed point theorem (see *e.g.*

[DS88a, Sections V.10, V.12]) applies and yields that there exists at least one point $\lambda_1 \in \mathcal{F}_1$ such that $\mu(\lambda_1) = \lambda_1$. Then $\Delta\big(f, g(\lambda_1); \lambda_1\big) = 0$. On the other hand, (see (1.6.4)),

$$\begin{aligned}\Delta\big(f, g(\lambda_1); \lambda_1\big) &= \big(S_1(\lambda_1)f, f\big)\big(Cf, (D-\lambda_1)^{-1}Cf\big)\\ &= \big(S_1(\lambda_1)f, f\big)\big((D-\lambda_1)g(\lambda_1), g(\lambda_1)\big). \qquad (1.6.6)\end{aligned}$$

The second factor is non-zero since $g(\lambda_1) \neq 0$ and $\lambda_1 \notin W(D)$ (note that $W(D) \cap \mathcal{F}_1 = \emptyset$). Thus (1.6.6) implies $\big(S_1(\lambda_1)f, f\big) = 0$, that is, $\lambda_1 \in W(S_1)$.

Now consider the component $\mathcal{F}_2$ and the second Schur complement S_2. If there also exists a piecewise smooth simply closed Jordan curve Γ_2 such that $\mathcal{F}_2$ lies in the interior of Γ_2 and $\mathcal{F}_1$ in its exterior, then the proof of $W(S_2) \cap \mathcal{F}_2 \neq \emptyset$ is analogous to the above reasoning. Otherwise, we let $\mathcal{V}_1 := \overline{\mathbb{C}} \setminus \overline{\mathcal{U}_1}$, where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the extended complex plane, and we suppose for simplicity that $0 \in \mathcal{U}_1$. Since $B \neq 0$, there exists an element $g \in \mathcal{H}_2$ such that $Bg \neq 0$. Set

$$f(\lambda) := (A-\lambda)^{-1}Bg, \quad \lambda \in \mathbb{C} \setminus \mathcal{U}_1.$$

If $\lambda \in \mathbb{C} \setminus \mathcal{U}_1$ $\big(= \overline{\mathcal{V}_1} \setminus \{\infty\}\big)$ is fixed, the function

$$\psi_\lambda(\eta) := \Delta\big(f(\lambda), g; \eta\big), \quad \eta \in \mathbb{C},$$

has exactly one zero $\eta(\lambda) \in \mathcal{F}_2$. Evidently, the same holds for the function $\widetilde{\psi}_\lambda(\eta) := |\lambda|^2\psi_\lambda(\eta)$, $\eta \in \mathbb{C}$, which we can consider also for $\lambda = \infty$. More exactly, let $\widetilde{\psi}_\infty(\eta) := \lim_{\lambda\to\infty} \widetilde{\psi}_\lambda(\eta)$. Since $\lim_{\lambda\to\infty} \lambda(A-\lambda)^{-1} = -I$, it is easy to see that $\widetilde{\psi}_\infty(\eta) = \Delta(Bg, g; \eta)$. Hence the function $\widetilde{\psi}_\infty$ has exactly one zero $\eta(\infty) \in \mathcal{F}_2$. The function $\eta : \overline{\mathcal{V}_1} \to \mathcal{F}_2$ is continuous on $\overline{\mathcal{V}_1}$ and maps $\overline{\mathcal{V}_1}$ into itself. Since the boundary of $\mathcal{V}_1$ is a piecewise smooth simply closed Jordan curve, it follows that there exists at least one point $\lambda_2 \in \mathcal{F}_2$ such that $\eta(\lambda_2) = \lambda_2$, that is, $\Delta\big(f(\lambda_2), g; \lambda_2\big) = 0$. Using the equality

$$\Delta\big(f(\lambda_2), g; \lambda_2\big) = \big(S_2(\lambda_2)g, g\big)\big((A-\lambda_2)f(\lambda_2), f(\lambda_2)\big)$$

instead of (1.6.6), we obtain that $\big(S_2(\lambda_2)g, g\big) = 0$, that is, $\lambda_2 \in W(S_2)$.

In the second case that $\mathcal{F}_2$ lies in the bounded component of $\mathbb{C} \setminus \Gamma_1$, we consider the functions f on $\overline{\mathcal{U}_1}$ and g on $\mathbb{C} \setminus \mathcal{U}_1$ and proceed in the same way as above. □

Remark 1.6.5 It is an open question whether the components of $\overline{W^2(\mathcal{A})}$ are simply connected, and, if no, one component may lie in a "hole" of the other one.

In the latter case, the second part of the above proof involving the construction with ∞ necessary. In fact, then there do not exist two piecewise smooth simply closed Jordan curves Γ_i, $i = 1, 2$, such that $\mathcal{F}_i$ lies in the interior of Γ_i and in the exterior of the other curve. One of these curves, say Γ_2, can only be chosen to be a Cauchy contour; in fact, it may be chosen as the union of two piecewise smooth simply closed Jordan curves, one of them being the curve Γ_1 with opposite orientation, the other one being the positively oriented circle $\{z \in \mathbb{C} : |z| = R\}$ of radius $R > 0$ so that $\|\mathcal{A}\| < R$.

Theorem 1.6.6 *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be separable Hilbert spaces with $\dim \mathcal{H}_1 \geq 2$, $\dim \mathcal{H}_2 \geq 2$. Suppose that $\overline{W^2(\mathcal{A})} = \mathcal{F}_1 \dot\cup \mathcal{F}_2$ consists of two components, enumerated so that $W(S_1) \cap \mathcal{F}_1 \neq \emptyset$, $W(S_2) \cap \mathcal{F}_2 \neq \emptyset$. Then:*

i) *for $f \in \mathcal{H}_1$, $f \neq 0$, the function $\big(S_1(\cdot)f, f\big)$ has exactly one zero in $\mathcal{F}_1$, for $g \in \mathcal{H}_2$, $g \neq 0$, the function $\big(S_2(\cdot)g, g\big)$ has exactly one zero in $\mathcal{F}_2$;*

ii) *the Schur complements S_1 and S_2 admit factorizations*

$$S_j(\lambda) = M_j(\lambda)(Z_j - \lambda), \quad \lambda \in \mathcal{F}_j, \tag{1.6.7}$$

for $j = 1, 2$ where $M_j : \mathcal{F}_j \to L(\mathcal{H}_j)$ is an analytic operator function such that $M_j(\lambda)$ is boundedly invertible for all $\lambda \in \mathcal{F}_j$ and $Z_j \in L(\mathcal{H}_j)$ is such that $\sigma(Z_j) \subset \mathcal{F}_j$.

Proof. Let the bounded set $\mathcal{U}_1$ and the curve Γ_1 parametrizing its boundary be chosen as in the proof of Theorem 1.6.4 with $\mathcal{F}_1 \subset \mathcal{U}_1$, $\mathcal{F}_2 \cap \overline{\mathcal{U}_1} = \emptyset$ (the case $\mathcal{F}_2 \subset \mathcal{U}_1$, $\mathcal{F}_1 \cap \overline{\mathcal{U}_1} = \emptyset$ is treated analogously).

i) We prove the claim for S_1; the proof for S_2 is completely analogous.

First we assume that $\dim \mathcal{H}_1 =: n_1 < \infty$, $\dim \mathcal{H}_2 =: n_2 < \infty$. From [KMM93, Lemma 6] it follows that for arbitrary $f \in \mathcal{H}_1$, $f \neq 0$,

$$\operatorname{ind}_{\Gamma_1} \det\, S_1(\cdot) = n_1 \operatorname{ind}_{\Gamma_1} \big(S_1(\cdot)f, f\big) =: n_1 l_1, \tag{1.6.8}$$

where $\operatorname{ind}_{\Gamma_1} \big(S_1(\cdot)f, f\big)$ denotes the number l_1 of zeroes of $\big(S_1(\cdot)f, f\big)$ in $\mathcal{F}_1$. The factorization (1.6.2) implies that

$$\det(\mathcal{A} - \lambda) = \det S_1(\lambda)\, \det(D - \lambda), \quad \lambda \notin \sigma(D).$$

Since $\mathcal{F}_2 \cap \overline{\mathcal{U}_1} = \emptyset$ and $\sigma(D) \subset W(D) \subset \mathcal{F}_2$, the function $\lambda \mapsto \det(D - \lambda)$ does not have zeroes in the interior $\mathcal{U}_1$ of Γ_1, and hence

$$\operatorname{ind}_{\Gamma_1} \det(\mathcal{A} - \cdot) = \operatorname{ind}_{\Gamma_1} \det S_1(\cdot).$$

Therefore the operator $\mathcal{A}$ has exactly $n_1 l_1$ eigenvalues in $\mathcal{F}_1$, counted according to their algebraic multiplicities. Analogously, if for $g \in \mathcal{H}_2$, $g \neq 0$, we denote by l_2 the number of zeroes of the function $(S_2(\cdot)g, g)$ in the set $\mathcal{F}_2$, then the operator $\mathcal{A}$ has exactly $n_2 l_2$ eigenvalues in $\mathcal{F}_2$, again counted according to their algebraic multiplicities. Since the total number of eigenvalues of $\mathcal{A}$ is equal to $\dim \mathcal{H} = n_1 + n_2$, we obtain the equality

$$n_1 l_1 + n_2 l_2 = n_1 + n_2.$$

Due to Theorem 1.6.4, we have $l_1 \geq 1$, $l_2 \geq 1$ and hence $l_1 = l_2 = 1$. This proves i) in the finite-dimensional case.

In the general case, we choose sequences of orthogonal projections $P_n \in L(\mathcal{H}_1)$ and $Q_n \in L(\mathcal{H}_2)$, $n \in \mathbb{N}$, that converge strongly to the respective identity operators and have finite-dimensional ranges, $\dim R(P_n) \geq 1$, $\dim R(Q_n) \geq 1$. For $n \in \mathbb{N}$, we set

$$A_n := P_n A P_n, \quad B_n := P_n B Q_n, \quad C_n := Q_n C P_n, \quad D_n := Q_n D Q_n$$

and consider the matrix

$$\mathcal{A}_n := \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}$$

in the space $R(P_n) \oplus R(Q_n)$. It is easy to see that, for all $n \in \mathbb{N}$,

$$W^2(\mathcal{A}_n) \subset W^2(\mathcal{A})$$

and that also $W^2(\mathcal{A}_n) = \overline{W^2(\mathcal{A}_n)}$ consists of two components, $W^2(\mathcal{A}_n) = \mathcal{F}_1^n \,\dot\cup\, \mathcal{F}_2^n$ with $W(A_n) \subset \mathcal{F}_1^n \subset \mathcal{F}_1$ and $W(D_n) \subset \mathcal{F}_2^n \subset \mathcal{F}_2$. Set

$$S_1^{(n)}(\lambda) := A_n - \lambda - B_n(D_n - \lambda)^{-1} C_n, \quad \lambda \notin \mathcal{F}_2.$$

If we prove that, for every $f \in \mathcal{H}_1$,

$$\sup_{\lambda \in \Gamma_1} \left| \big(S_1(\lambda) f, f\big) - \big(S_1^{(n)}(\lambda) f, f\big) \right| \longrightarrow 0, \quad n \to \infty, \tag{1.6.9}$$

then, according to what was shown in the first part of the proof for the finite-dimensional case,

$$\operatorname{ind}_{\Gamma_1} \big(S_1(\cdot) f, f\big) = \operatorname{ind}_{\Gamma_1} \big(S_1^{(n)}(\cdot) f, f\big) = 1$$

for every $f \in \mathcal{H}_1$, $f \neq 0$; this completes the proof of claim i) for S_1 in the general case.

In order to prove (1.6.9), let $f \in \mathcal{H}_1$ be arbitrary and write

$$\begin{aligned}
&\Big(\big(S_1(\lambda) - S_1^{(n)}(\lambda)\big) f, f\Big) \\
&= \big((A - A_n) f, f\big) - \big(B_n (D_n - \lambda)^{-1} (C - C_n) f, f\big) \\
&\quad - \Big(B_n \big((D - \lambda)^{-1} - (D_n - \lambda)^{-1}\big) C f, f\Big) - \big((B - B_n)(D - \lambda)^{-1} C f, f\big) \\
&= \big((A - A_n) f, f\big) - \big(B_n (D_n - \lambda)^{-1} (C - C_n) f, f\big) \\
&\quad - \big(B_n (D_n - \lambda)^{-1} (D_n - D)(D - \lambda)^{-1} C f, f\big) - \big((B - B_n)(D - \lambda)^{-1} C f, f\big).
\end{aligned}$$

Now $A_n \to A$ (strongly) for $n \to \infty$ implies that

$$\big((A - A_n)f, f\big) \longrightarrow 0, \quad n \to \infty.$$

Since $W(D_n) \subset W(D) \subset \mathcal{F}_2$ and $\mathcal{F}_2 \cap \Gamma_1 \subset \mathcal{F}_2 \cap \overline{\mathcal{U}_1} = \emptyset$, we have

$$\big\|(D_n - \lambda)^{-1}\big\|, \big\|(D - \lambda)^{-1}\big\| \leq \frac{1}{\operatorname{dist}(\Gamma_1, \mathcal{F}_2)}, \quad \lambda \in \Gamma_1, \ n \in \mathbb{N}.$$

Together with $\|B_n\| \leq \|B\|$ and $C_n \to C$ (strongly) for $n \to \infty$, we obtain

$$\sup_{\lambda\in\Gamma_1}\big|\big(B_n(D_n - \lambda)^{-1}(C - C_n)f, f\big)\big| \leq \|B\| \frac{1}{\operatorname{dist}(\Gamma_1, \mathcal{F}_2)}\big\|(C - C_n)f\big\| \, \|f\| \longrightarrow 0, \quad n \to \infty.$$

Further, since also $B_n^* \to B^*$ (strongly) for $n \to \infty$, it follows that

$$\sup_{\lambda\in\Gamma_1}\big|\big((B - B_n)(D - \lambda)^{-1}Cf, f\big)\big| \leq \frac{1}{\operatorname{dist}(\Gamma_1, \mathcal{F}_2)} \|Cf\| \, \big\|(B^* - B_n^*)f\big\| \longrightarrow 0, \quad n \to \infty.$$

Finally, the facts that $D_n \to D$ (strongly) for $n \to \infty$ and that the set $\big\{(D - \lambda)^{-1}Cf : \lambda \in \Gamma_1\big\} \subset \mathcal{H}_2$ is compact imply that

$$\sup_{\lambda\in\Gamma_1}\big\|(D_n - D)(D - \lambda)^{-1}Cf\big\| \longrightarrow 0, \quad n \to \infty.$$

Hence

$$\begin{aligned}&\big\|B_n(D_n - \lambda)^{-1}(D_n - D)(D - \lambda)^{-1}Cf\big\| \\ &\leq \|B\|\frac{1}{\operatorname{dist}(\Gamma_1, \mathcal{F}_2)}\big\|(D_n - D)(D - \lambda)^{-1}Cf\big\| \longrightarrow 0, \quad n \to \infty,\end{aligned}$$

uniformly for $\lambda \in \Gamma_1$. This proves (1.6.9).

ii) Since $\mathcal{U}_1$ is simply connected, we obtain the factorization (1.6.7) for S_1 from the factorization theorem [MM75, Theorem 2, Remark 1)]. If also for $\mathcal{F}_2$ there exists a bounded simply connected domain $\mathcal{U}_2$ such that $\mathcal{F}_2 \subset \mathcal{U}_2$, $\mathcal{F}_1 \cap \overline{\mathcal{U}_2} = \emptyset$, the second relation in (1.6.7) follows from the same factorization theorem. If this is not the case, we consider the domain $\mathcal{U}_2 := \overline{\mathbb{C}} \setminus \overline{\mathcal{U}_1}$ which is simply connected in the extended plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let $\mathcal{V}_2$ be the image of $\mathcal{U}_2$ under the inversion $\mu = \lambda^{-1}$ and set

$$W(\mu) := \mu \, S_2(\mu^{-1}), \quad \mu \in \mathcal{V}_2, \ \mu \neq 0. \tag{1.6.10}$$

If we define $W(0) := \lim_{\mu\to 0} W(\mu) = -I$, then W is an analytic operator function in $\mathcal{V}_2$. It is easy to check that for $f \in \mathcal{H}_1$, $f \neq 0$, the function $\big(W(\cdot)f, f\big)$ has exactly one zero in $\mathcal{V}_2$. Therefore, by [MM75, Theorem 2, Remark 1)], the operator function W admits a factorization

$$W(\mu) = Q(\mu)\,(Y - \mu), \quad \mu \in \mathcal{V}_2, \tag{1.6.11}$$

where $Q : \mathcal{V}_2 \to L(\mathcal{H}_2)$ is an analytic operator function on $\mathcal{V}_2$ with boundedly invertible values and $Y \in L(\mathcal{H}_2)$ with $\sigma(Y) \subset \mathcal{V}_2$. The operator Y is invertible since $Q(0)\,Y = W(0) = -I$. Moreover, formula (1.6.11) implies that

$$S_2(\lambda) = \lambda\, W(\lambda^{-1}) = \lambda\, Q(\lambda^{-1})\left(Y - \lambda^{-1}\right) = -Q(\lambda^{-1})\, Y \left(Y^{-1} - \lambda\right).$$

If we set $M_2(\lambda) := -Q(\lambda^{-1})\, Y$, $\lambda \in \mathcal{U}_2$, and $Z_2 := Y^{-1}$, the factorization (1.6.7) for S_2 follows. □

Remark 1.6.7 In Theorem 1.6.6 ii), the operator function M_1 is even analytic on $\rho(D)$ and M_2 is analytic on $\rho(A)$.
This follows by analytic continuation from the identity (1.6.7) since S_1 is analytic on $\rho(D)$ and S_2 is analytic on $\rho(A)$.

1.7 Block diagonalization

In this section we show that the quadratic numerical range yields a criterion for the block diagonalizability of a block operator matrix: If the closure of the quadratic numerical range consists of two components, then the block operator matrix can be transformed into diagonal form.

In order to prove this, we show that the two spectral subspaces corresponding to the two disjoint parts of the spectrum admit so-called angular operator representations. The diagonalizing matrix is constructed by means of the angular operators which, in addition, turn out to be solutions of Riccati equations associated with the block operator matrix.

In the following, a subset $\sigma \subset \sigma(\mathcal{A})$ is called an isolated part of $\sigma(\mathcal{A})$ if both σ and $\sigma(\mathcal{A}) \setminus \sigma$ are closed. Associated with an isolated part σ of $\sigma(\mathcal{A})$ is a *spectral subspace* $\mathcal{L}_\sigma$ which is defined as the range of the Riesz projection

$$P_\sigma := -\frac{1}{2\pi \mathrm{i}} \int_\Gamma (\mathcal{A} - z)^{-1} \mathrm{d}z$$

of $\mathcal{A}$ corresponding to σ; here Γ is a Cauchy contour (that is, the finite union of simply closed rectifiable Jordan curves) such that σ lies in its interior and $\sigma(\mathcal{A}) \setminus \sigma$ in its exterior. The spectral subspace $\mathcal{L}_\sigma$ is an invariant subspace of $\mathcal{A}$, *i.e.* $\mathcal{A}\mathcal{L}_\sigma \subset \mathcal{L}_\sigma$, and $\sigma\big(\mathcal{A}|_{\mathcal{L}_\sigma}\big) = \sigma$ (see *e.g.* [GGK90, Chapter I.2]).

Theorem 1.7.1 *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be separable Hilbert spaces with* $\dim \mathcal{H}_1 \geq 2$, $\dim \mathcal{H}_2 \geq 2$. *Suppose that $\overline{W^2(\mathcal{A})} = \mathcal{F}_1 \dot\cup \mathcal{F}_2$ consists of two components, enumerated such that $W(S_1) \cap \mathcal{F}_1 \neq \emptyset$, $W(S_2) \cap \mathcal{F}_2 \neq \emptyset$.*

Then the spectrum $\sigma(\mathcal{A})$ separates into two parts, $\sigma(\mathcal{A}) = \sigma_1 \dot\cup \sigma_2$ with

$$\sigma_1 := \sigma(\mathcal{A}) \cap \mathcal{F}_1 \neq \emptyset, \quad \sigma_2 := \sigma(\mathcal{A}) \cap \mathcal{F}_2 \neq \emptyset,$$

and there exist bounded linear operators $K_1 \in L(\mathcal{H}_1, \mathcal{H}_2)$, $K_2 \in L(\mathcal{H}_2, \mathcal{H}_1)$ such that the following hold:

i) *the spectral subspaces $\mathcal{L}_1$ and $\mathcal{L}_2$ corresponding to σ_1 and σ_2, respectively, have angular operator representations*

$$\mathcal{L}_1 = \left\{ \begin{pmatrix} x \\ K_1 x \end{pmatrix} : x \in \mathcal{H}_1 \right\}, \quad \mathcal{L}_2 = \left\{ \begin{pmatrix} K_2 y \\ y \end{pmatrix} : y \in \mathcal{H}_2 \right\}; \qquad (1.7.1)$$

ii) *the angular operators K_1 and K_2 satisfy the Riccati equations*

$$K_1 B K_1 + K_1 A - D K_1 - C = 0, \quad K_2 C K_2 + K_2 D - A K_2 - B = 0;$$

iii) *if Γ_j, $j = 1, 2$, is a Cauchy contour such that $\mathcal{F}_j$ lies in its interior and the other component $\overline{W^2(\mathcal{A})} \setminus \mathcal{F}_j$ in its exterior and if Z_j, M_j are the operators and operator functions, respectively, in the factorization* (1.6.7) *of the Schur complement S_j, $j = 1, 2$, then*

$$K_1 = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} (D - \lambda)^{-1} C (Z_1 - \lambda)^{-1} \mathrm{d}\lambda, \qquad (1.7.2)$$

$$K_2 = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_2} (A - \lambda)^{-1} B (Z_2 - \lambda)^{-1} \mathrm{d}\lambda, \qquad (1.7.3)$$

and

$$Z_1 = A + B K_1, \quad M_1(\lambda) = I - B(D - \lambda)^{-1} K_1, \quad \lambda \in \rho(D),$$
$$Z_2 = D + C K_2, \quad M_2(\lambda) = I - C(A - \lambda)^{-1} K_2, \quad \lambda \in \rho(A).$$

Proof. Let P and Q be the Riesz projections of $\mathcal{A}$ corresponding to σ_1 and σ_2, respectively. With respect to $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, we write them as

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, \quad Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}.$$

By [LMMT01, Lemma 4.2], at least one of the contours Γ_1 and Γ_2, say Γ_1, can be chosen to be a piecewise smooth simply closed Jordan curve. First we prove the statements for K_1.

i) The Frobenius-Schur factorization (1.6.2) implies that, for $\lambda \in \Gamma_1$,

$$(\mathcal{A} - \lambda)^{-1} \qquad (1.7.4)$$
$$= \begin{pmatrix} S_1(\lambda)^{-1} & -S_1(\lambda)^{-1} B (D-\lambda)^{-1} \\ -(D-\lambda)^{-1} C S_1(\lambda)^{-1} & (D-\lambda)^{-1} + (D-\lambda)^{-1} C S_1(\lambda)^{-1} B (D-\lambda)^{-1} \end{pmatrix}.$$

As the resolvent of D is holomorphic inside Γ_1, the matrix entries of the Riesz projection P are given by

$$\begin{aligned}
P_{11} &= -\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_1} S_1(\lambda)^{-1}\mathrm{d}\lambda,\\
P_{12} &= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_1} S_1(\lambda)^{-1}B(D-\lambda)^{-1}\mathrm{d}\lambda,\\
P_{21} &= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_1} (D-\lambda)^{-1}CS_1(\lambda)^{-1}\mathrm{d}\lambda,\\
P_{22} &= -\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_1} (D-\lambda)^{-1}CS_1(\lambda)^{-1}B(D-\lambda)^{-1}\mathrm{d}\lambda.
\end{aligned}$$

We define the operator K_1 by (1.7.2). By [DK74a, Theorem I.3.2] (see also [GGK90, Theorem I.4.1]), it is the unique solution of the operator equation

$$K_1Z_1 - DK_1 = C$$

with $Z_1 = A + BK_1$. By (1.6.7) and Remark 1.6.7, we have $S_1(\lambda)^{-1} = (Z_1-\lambda)^{-1}M_1(\lambda)^{-1}$, $\lambda \in \mathbb{C}\setminus\rho(D)$. This and [DK74a, Lemma I.2.1] lead to

$$\begin{aligned}
P_{22} &= -\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_1} (D-\lambda)^{-1}C(Z_1-\lambda)^{-1}M_1(\lambda)^{-1}B(D-\lambda)^{-1}\mathrm{d}\lambda\\
&= \left(\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_1}(D-\lambda)^{-1}C(Z_1-\lambda)^{-1}\mathrm{d}\lambda\right)\\
&\qquad\cdot\left(\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_1}(Z_1-\lambda)^{-1}M_1(\lambda)^{-1}B(D-\lambda)^{-1}\mathrm{d}\lambda\right) = K_1P_{12},\\
P_{21} &= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_1} (D-\lambda)^{-1}C(Z_1-\lambda)^{-1}M_1(\lambda)^{-1}\mathrm{d}\lambda\\
&= \left(\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_1}(D-\lambda)^{-1}C(Z_1-\lambda)^{-1}\mathrm{d}\lambda\right)\\
&\qquad\cdot\left(-\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_1}(Z_1-\lambda)^{-1}M_1(\lambda)^{-1}\mathrm{d}\lambda\right) = K_1P_{11}.
\end{aligned}$$

These relations yield

$$P = \begin{pmatrix} P_{11} & P_{12}\\ K_1P_{11} & K_1P_{12}\end{pmatrix} = \begin{pmatrix} I\\ K_1\end{pmatrix}\begin{pmatrix} P_{11} & P_{12}\end{pmatrix}.$$

Since Γ_1 is a piecewise smooth simply closed Jordan curve, P_{11} is bijective (see [MM75, Theorem 3]). Therefore the range $R\big((P_{11}\ P_{12})\big)$ is $\mathcal{H}_1$ and so

$$\mathcal{L}_1 = R(P) = R\left(\begin{pmatrix} I\\ K_1\end{pmatrix}\right),$$

which proves the representation of $\mathcal{L}_1$ in (1.7.1).

ii) Let $x \in \mathcal{H}_1$ be arbitrary. Since the spectral subspace $\mathcal{L}_1$ is invariant for $\mathcal{A}$, there exists a $z \in \mathcal{H}_1$ such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x \\ K_1 x \end{pmatrix} = \begin{pmatrix} z \\ K_1 z \end{pmatrix} \quad \Longleftrightarrow \quad \begin{aligned} Ax + BK_1x &= z, \\ Cx + DK_1x &= K_1 z. \end{aligned}$$

Inserting the first equation into the second, we find that

$$\bigl(C + DK_1 - K_1A - K_1BK_1\bigr)\, x = 0.$$

iii) By the definition of K_1 in (1.7.2), we have

$$\begin{aligned} BK_1 =& \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} B(D-\lambda)^{-1}C(Z_1-\lambda)^{-1}\mathrm{d}\lambda \\ =& \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} \bigl(A - \lambda - S_1(\lambda)\bigr)(Z_1-\lambda)^{-1}\mathrm{d}\lambda \\ =& \frac{1}{2\pi \mathrm{i}} A\int_{\Gamma_1} (Z_1-\lambda)^{-1}\mathrm{d}\lambda - \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} \lambda(Z_1-\lambda)^{-1}\mathrm{d}\lambda - \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} M_1(\lambda)\mathrm{d}\lambda \\ =& -A + Z_1; \end{aligned}$$

here we have used that, according to Theorem 1.6.6 ii), the spectrum of Z_1 lies in the interior of Γ_1 and that M_1 is analytic on $\rho(D)$ and hence in the interior of Γ_1 by Remark 1.6.7.

To prove the representation of $M_1(\lambda)$, we use the relation $Z_1 = A + BK_1$ and the Riccati equation for K_1 to obtain that, for $\lambda \in \rho(D)$,

$$\begin{aligned} &\bigl(I - B(D-\lambda)^{-1}K_1\bigr)\,(Z_1 - \lambda) \\ &= \bigl(I - B(D-\lambda)^{-1}K_1\bigr)\,(A + BK_1 - \lambda) \\ &= A - \lambda - B(D-\lambda)^{-1}\bigl(K_1(A + BK_1) - (D-\lambda)K_1 - \lambda K_1\bigr) \\ &= A - \lambda - B(D-\lambda)^{-1}C \\ &= S_1(\lambda). \end{aligned}$$

This completes the proof of all statements involving K_1.

The proofs of the statements for K_2 in i) to iii) are analogous if the contour Γ_2 can also be chosen to be a piecewise smooth simply closed Jordan curve.

Otherwise, for the Riesz projection Q of $\mathcal{A}$ corresponding to σ_2, the bijectivity of Q_{22} remains to be proved. In this case the contour Γ_2 can be chosen to consist of the two simply closed Jordan curves $-\Gamma_1$ (*i.e.* Γ_1 with opposite orientation) and the positively oriented circle $\{\lambda \in \mathbb{C} : |\lambda| = M\}$ where $M > 0$ is such that $\mathcal{F}_2$ is in its interior. The (piecewise smooth) contour Γ_2 is the boundary of a domain $\mathcal{U}_2$ such that $\mathcal{F}_2 \subset \mathcal{U}_2$. Without loss of generality, we assume that 0 lies inside Γ_1. Denote by Λ_2 the image of Γ_2 and by $\mathcal{V}_2$ the image of $\mathcal{U}_2$ under the inversion $\mu = \lambda^{-1}$. Then Λ_2 consists of the Jordan curves $\Lambda_2^+ = \{\mu \in \mathbb{C} : |\mu| = 1/M\}$ (inner boundary) and Λ_2^- (outer boundary, the image of $-\Gamma_1$), and their orientation is again positive with respect to the image of $\mathcal{F}_2$. Then

$$Q_{22} = -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_2} S_2(\lambda)^{-1} \mathrm{d}\lambda = \frac{1}{2\pi \mathrm{i}} \int_{\Lambda_2} S_2(\mu^{-1})^{-1} \mu^{-2} \mathrm{d}\mu = \frac{1}{2\pi \mathrm{i}} \int_{\Lambda_2} W(\mu)^{-1} \mu^{-1} \mathrm{d}\mu,$$

where $W(\mu) = \mu \, S_2(\mu^{-1})$ for $\mu \in \mathcal{V}_2$ as in (1.6.10). By (1.6.11),

$$\mu^{-1} W(\mu)^{-1} - Y^{-1} W(\mu)^{-1} = \mu^{-1} Y^{-1} Q(\mu)^{-1}.$$

The operator function on the right hand side is analytic on $\mathcal{V}_2$ since $0 \notin \mathcal{V}_2$. It follows that

$$Q_{22} = \frac{1}{2\pi \mathrm{i}} Y^{-1} \int_{\Lambda_2} W(\mu)^{-1} \mathrm{d}\mu = \frac{1}{2\pi \mathrm{i}} Y^{-1} \left(\int_{\Lambda_2^+} W(\mu)^{-1} \mathrm{d}\mu + \int_{\Lambda_2^-} W(\mu)^{-1} \mathrm{d}\mu \right).$$

The operator function $W(\cdot)^{-1}$ is analytic in the circle $\{\mu \in \mathbb{C} : |\mu| \le 1/M\}$, therefore the first integral equals 0. In the proof of Theorem 1.6.6 it was already used that the operator function W fulfils the assumptions of [MM75, Theorem 2] with respect to Λ_2^-. By [MM75, Theorem 3], the operator

$$\frac{1}{2\pi \mathrm{i}} \int_{\Lambda_2^-} W(\mu)^{-1} \mathrm{d}\mu$$

is bijective and hence so is the operator Q_{22}. □

Corollary 1.7.2 *Under the assumptions of Theorem* 1.7.1*, the block operator matrix* $\mathcal{A}$ *is similar to the block diagonal operator matrix*

$$\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} = \begin{pmatrix} A + BK_1 & 0 \\ 0 & D + CK_2 \end{pmatrix};$$

in fact,

$$\begin{pmatrix} I & K_2 \\ K_1 & I \end{pmatrix}^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & K_2 \\ K_1 & I \end{pmatrix} = \begin{pmatrix} A + BK_1 & 0 \\ 0 & D + CK_2 \end{pmatrix}.$$

Proof. The operator $\begin{pmatrix} I \\ K_1 \end{pmatrix}$ maps $\mathcal{H}_1$ isomorphically on $\mathcal{L}_1$ and, by Theorem 1.7.1 ii) and iii),

$$\mathcal{A} \begin{pmatrix} I \\ K_1 \end{pmatrix} = \begin{pmatrix} I \\ K_1 \end{pmatrix} Z_1, \quad \mathcal{A} \begin{pmatrix} K_2 \\ I \end{pmatrix} = \begin{pmatrix} K_2 \\ I \end{pmatrix} Z_2. \tag{1.7.5}$$

Therefore the restriction of the operator $\mathcal{A}$ to $\mathcal{L}_1$ is an operator which is similar to Z_1. Similarly, the restriction of the operator $\mathcal{A}$ to $\mathcal{L}_2$ is similar to Z_2. The second statement is immediate from (1.7.5). □

Corollary 1.7.3 *If dim $\mathcal{H}_1 = n_1 < \infty$, then $\mathcal{F}_1$ contains exactly n_1 eigenvalues of $\mathcal{A}$ (counting multiplicities), and the first components of the corresponding eigenvectors and associated vectors form a basis in $\mathcal{H}_1$.*

Corollary 1.7.4 *If $\mathcal{A} = \mathcal{A}^*$ in Theorem 1.7.1, then*

$$\mathcal{L}_1 = \left\{ \begin{pmatrix} x \\ Kx \end{pmatrix} : x \in \mathcal{H}_1 \right\}, \quad \mathcal{L}_2 = \left\{ \begin{pmatrix} -K^*y \\ y \end{pmatrix} : y \in \mathcal{H}_2 \right\} \tag{1.7.6}$$

with a uniform contraction $K \in L(\mathcal{H}_1, \mathcal{H}_2)$ (i.e. $\|K\| < 1$).

Proof. Since $\mathcal{A}$ is self-adjoint, we have $\mathcal{L}_1 \perp \mathcal{L}_2$ and hence $K_2 = -K_1^*$. The strict inequality $\|K_1\| < 1$ cannot be proved with the methods used in this section; later, in Theorem 2.7.7, this is shown even for unbounded diagonal elements A, D (see also [AL95, Lemma 2.2]). □

1.8 Spectral supporting subspaces

If a self-adjoint block operator matrix $\mathcal{A}$ has separated diagonal elements, $\sup W(D) < \alpha < \inf W(A)$ for some $\alpha \in \mathbb{R}$, then $\overline{W^2(\mathcal{A})}$ consists of two components. Then, by (1.7.1) (see also (1.7.6) above), the spectral subspace $\mathcal{L}_{(\alpha,\infty)}(\mathcal{A})$ is the graph of a bounded linear operator $K_1 \in L(\mathcal{H}_1, \mathcal{H}_2)$.

In this section we drop the separation condition for the diagonal elements. We show that for intervals $\Delta \subset \rho(D)$, the corresponding spectral subspace $\mathcal{L}_\Delta(\mathcal{A})$ is the graph of a closed linear operator $K_1^\Delta : \mathcal{H}_1 \to \mathcal{H}_2$ which may only be defined on a subspace $\mathcal{H}_1^\Delta := \mathcal{D}(K_1^\Delta)$ of $\mathcal{H}_1$; the operator K_1^Δ is bounded if $\overline{\Delta} \subset \rho(D)$.

The main results of this section concern a description of the so-called spectral supporting subspace $\mathcal{H}_1^\Delta$ in terms of the Schur complement S_1. Analogous results hold for intervals $\Delta \subset \rho(A)$ and the corresponding spectral supporting subspaces $\mathcal{H}_2^\Delta$ (see [LMMT03]).

Theorem 1.8.1 *Suppose that $\mathcal{A} = \mathcal{A}^*$. Let $\Delta \subset \mathbb{R}$ be an interval such that $\overline{\Delta} \subset \rho(D)$. Then there exists a subspace $\mathcal{H}_1^\Delta \subset \mathcal{H}_1$ and a bounded linear operator $K_1^\Delta : \mathcal{H}_1^\Delta \to \mathcal{H}_2$ such that*

$$\mathcal{L}_\Delta(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix} : x \in \mathcal{H}_1^\Delta \right\}. \tag{1.8.1}$$

Proof. A subspace $\mathcal{L} \subset \mathcal{H}_1 \oplus \mathcal{H}_2$ is the graph of a linear operator if it contains no elements of the form $(0\ y)^{\mathrm{t}}$ with $y \neq 0$; it is the graph of a bounded linear operator if there is a $\gamma \geq 0$ with $\|y\| \leq \gamma \|x\|$ for all $(x\ y)^{\mathrm{t}} \in \mathcal{L}$.

Let $\alpha := \inf \Delta$, $\beta := \sup \Delta$ and set $\lambda_0 := (\alpha + \beta)/2$, $\delta := (\beta - \alpha)/2$. Then, for every $h = (x\ y)^{\mathrm{t}} \in \mathcal{L}_\Delta(\mathcal{A})$, we have $\|(\mathcal{A} - \lambda_0)h\| \leq \delta\,\|h\|$. Thus

$$\|(D - \lambda_0)y + B^* x\| \leq \|(\mathcal{A} - \lambda_0)h\| \leq \delta\left(\|x\|^2 + \|y\|^2\right)^{1/2} \leq \delta\,(\|x\| + \|y\|).$$

Denote $\varrho := \operatorname{dist}\big(\lambda_0, \sigma(D)\big)\ (> \delta)$. Then

$$\|(D - \lambda_0)y\| \geq \frac{1}{\big\|(D - \lambda_0)^{-1}\big\|}\|y\| \geq \varrho\,\|y\|,$$

and hence

$$\varrho\,\|y\| - \|B\|\,\|x\| \leq \big\|(D - \lambda_0)y + B^* x\big\| \leq \delta\,(\|x\| + \|y\|),$$

or $\|y\| \leq \dfrac{\delta + \|B\|}{\varrho - \delta}\|x\|$. □

Definition 1.8.2 If Δ is an interval as in Theorem 1.8.1, then $\mathcal{H}_1^\Delta$ is called the Δ-*spectral supporting subspace* of $\mathcal{A}$ in $\mathcal{H}_1$.

Together with the analogue of Theorem 1.8.1 for intervals $\Delta \subset \mathbb{R}$ with $\overline{\Delta} \subset \rho(A)$, we obtain the following corollary.

Corollary 1.8.3 *Suppose that $\mathcal{A} = \mathcal{A}^*$. Let $\Delta \subset \mathbb{R}$ be an interval such that $\overline{\Delta} \subset \rho(A) \cap \rho(D)$. Then there exist subspaces $\mathcal{H}_1^\Delta \subset \mathcal{H}_1$, $\mathcal{H}_2^\Delta \subset \mathcal{H}_2$ and a bijective bounded linear operator $K_1^\Delta : \mathcal{H}_1^\Delta \to \mathcal{H}_2^\Delta$ such that*

$$\mathcal{L}_\Delta(\mathcal{A}) = \left\{\begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix} : x \in \mathcal{H}_1^\Delta\right\} = \left\{\begin{pmatrix} \left(K_1^\Delta\right)^{-1} y \\ y \end{pmatrix} : y \in \mathcal{H}_2^\Delta\right\}. \qquad (1.8.2)$$

If we only assume that $\Delta \subset \rho(D)$, then the operator K_1^Δ is no longer bounded, but still closed:

Theorem 1.8.4 *Suppose that $\mathcal{A} = \mathcal{A}^*$. Let $\Delta \subset \mathbb{R}$ be an open or half-open interval such that $\Delta \subset \rho(D)$. Then there exists a closed linear operator $K_1^\Delta : \mathcal{H}_1 \to \mathcal{H}_2$ with domain $\mathcal{H}_1^\Delta := \mathcal{D}(K_1^\Delta)$ such that*

$$\mathcal{L}_\Delta(\mathcal{A}) = \left\{\begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix} : x \in \mathcal{H}_1^\Delta\right\}. \qquad (1.8.3)$$

Proof. Let *e.g.* $\Delta = (\alpha, \beta) \subset \rho(D)$ with $\alpha \in \sigma(D)$. We use the same reasoning and notations as in the proof of Theorem 1.8.1. Assume that an element $h := (0\ \ y)^{\mathrm{t}}$ with $y \neq 0$ belongs to $\mathcal{L}_\Delta(\mathcal{A})$. Then we have $\|(D - \lambda_0)y\| = \|(\mathcal{A} - \lambda_0)h\| < \delta\,\|h\| = \delta\,\|y\|$ and, at the same time,

$$\|(D - \lambda_0)y\| \geq \frac{1}{\big\|(D - \lambda_0)^{-1}\big\|}\|y\| \geq \delta\,\|y\| > \|(D - \lambda_0)y\|,$$

a contradiction. This shows that $\mathcal{L}_\Delta(\mathcal{A})$ is the graph of a linear operator K_1. Since the subspace $\mathcal{L}_\Delta(\mathcal{A})$ is closed, K_1 is a closed operator. □

Note that $\mathcal{H}_1^\Delta$ is the first component of $\mathcal{L}_\Delta(\mathcal{A})$. The subspace $\mathcal{H}_1^\Delta$ is closed if $\overline{\Delta} \subset \rho(D)$; it is not necessarily closed if only $\Delta \subset \rho(D)$.

Proposition 1.8.5 *Suppose that $\mathcal{A} = \mathcal{A}^*$. Let $\Delta = [\alpha, \beta] \subset \rho(D)$, $\gamma \in (\alpha, \beta)$, and $\Delta_1 := [\alpha, \gamma]$, $\Delta_2 := (\gamma, \beta]$. Then*

$$\mathcal{H}_1^\Delta = \mathcal{H}_1^{\Delta_1} \dot{+} \mathcal{H}_1^{\Delta_2};$$

the subspaces $\mathcal{H}_1^{\Delta_1}$ and $\mathcal{H}_1^{\Delta_2}$ are orthogonal with respect to the inner product

$$\langle x, y \rangle := \left(\begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix}, \begin{pmatrix} y \\ K_1^\Delta y \end{pmatrix} \right), \quad x, y \in \mathcal{H}_1^\Delta,$$

where K_1^Δ is the angular operator in the representation (1.8.1) *of $\mathcal{L}_\Delta(\mathcal{A})$.*

Proof. Since $\mathcal{A}$ is self-adjoint, we have

$$\mathcal{L}_\Delta(\mathcal{A}) = \mathcal{L}_{\Delta_1}(\mathcal{A}) \oplus \mathcal{L}_{\Delta_2}(\mathcal{A}).$$

On the other hand, by Theorem 1.8.1,

$$\mathcal{L}_\Delta(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix} : x \in \mathcal{H}_1^\Delta \right\}, \quad \mathcal{L}_{\Delta_j}(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ K_1^{\Delta_j} x \end{pmatrix} : x \in \mathcal{H}_1^{\Delta_j} \right\}, \ j = 1, 2,$$

with bounded linear operators K_1^Δ, $K_1^{\Delta_1}$, $K_1^{\Delta_2} : \mathcal{H}_1 \to \mathcal{H}_2$. By projection onto the first component, $\mathcal{L}_\Delta(\mathcal{A})$ is mapped isomorphically onto $\mathcal{H}_1^\Delta$ and $\mathcal{L}_{\Delta_j}(\mathcal{A})$ isomorphically onto $\mathcal{H}_1^{\Delta_j}$, $j = 1, 2$. This implies the first statement. The second statement follows if we observe that $K_1^{\Delta_1}$, $K_1^{\Delta_2}$ are the restrictions of K_1^Δ to $\mathcal{H}_1^{\Delta_1}$ and $\mathcal{H}_1^{\Delta_2}$, respectively. □

Proposition 1.8.6 *Suppose that $\mathcal{A} = \mathcal{A}^*$. Let $\Delta = [\alpha, \beta] \subset \rho(D)$ and denote by Ω the component of $\rho(D)$ containing Δ. Let $\mathcal{H}_1^\Delta$ be the spectral supporting subspace corresponding to Δ and let $P_1^\Delta : \mathcal{H}_1 \to \mathcal{H}_1^\Delta$ be the projection of $\mathcal{H}_1$ onto $\mathcal{H}_1^\Delta$. Then the block operator matrix*

$$\mathcal{A}_\Delta := \begin{pmatrix} P_1^\Delta A P_1^\Delta & P_1^\Delta B \\ B^* P_1^\Delta & D \end{pmatrix} \tag{1.8.4}$$

in $\mathcal{H}^\Delta := \mathcal{H}_1^\Delta \oplus \mathcal{H}_2$ satisfies

i) $\mathcal{L}_\Delta(\mathcal{A}_\Delta) = \mathcal{L}_\Delta(\mathcal{A})$;

ii) $\sigma(\mathcal{A}_\Delta) \cap \Omega \subset \Delta$.

Proof. i) The subspace $\mathcal{L}_\Delta(\mathcal{A}) \subset \mathcal{H}^\Delta$ is invariant under $\mathcal{A}$ and hence also under $\mathcal{A}_\Delta$. Moreover, $\mathcal{A}_\Delta|_{\mathcal{L}_\Delta(\mathcal{A})} = \mathcal{A}|_{\mathcal{L}_\Delta(\mathcal{A})}$ so that $\sigma\big(\mathcal{A}_\Delta|_{\mathcal{L}_\Delta(\mathcal{A})}\big) = \sigma\big(\mathcal{A}|_{\mathcal{L}_\Delta(\mathcal{A})}\big) \subset \Delta$. This shows that $\mathcal{L}_\Delta(\mathcal{A}) \subset \mathcal{L}_\Delta(\mathcal{A}_\Delta)$.

Using this inclusion and applying Theorem 1.8.1 to the operator $\mathcal{A}_\Delta$ in $\mathcal{H}^\Delta$, we find that there exists a subspace $\widetilde{\mathcal{H}}_1^\Delta \subset \mathcal{H}_1^\Delta$ and a bounded linear operator $\widetilde{K}_1^\Delta : \widetilde{\mathcal{H}}_1^\Delta \to \mathcal{H}_2$ so that

$$\mathcal{L}_\Delta(\mathcal{A}_\Delta) = \left\{ \begin{pmatrix} x \\ \widetilde{K}_1^\Delta x \end{pmatrix} : x \in \widetilde{\mathcal{H}}_1^\Delta \right\} \supset \left\{ \begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix} : x \in \mathcal{H}_1^\Delta \right\} = \mathcal{L}_\Delta(\mathcal{A}).$$

As a consequence, we obtain $\widetilde{\mathcal{H}}_1^\Delta = \mathcal{H}_1^\Delta$, $\widetilde{K}_1^\Delta = K_1^\Delta$, and hence the claim.

ii) It is sufficient to show that for every closed interval $\widetilde{\Delta}$ such that $\Delta \subset \widetilde{\Delta} \subset \Omega$, we have $\sigma(\mathcal{A}_\Delta) \cap \widetilde{\Delta} = \sigma(\mathcal{A}_\Delta) \cap \Delta$.

By i), $\mathcal{L}_\Delta(\mathcal{A}) = \mathcal{L}_\Delta(\mathcal{A}_\Delta) \subset \mathcal{L}_{\widetilde{\Delta}}(\mathcal{A}_\Delta)$. By Theorem 1.8.1 applied to $\mathcal{A}_\Delta$ in $\mathcal{H}^\Delta$ and the interval $\widetilde{\Delta}$, there exists a subspace $\widetilde{\mathcal{H}}_1^{\widetilde{\Delta}} \subset \mathcal{H}_1^\Delta$ and a bounded linear operator $\widetilde{K}_1^{\widetilde{\Delta}} : \widetilde{\mathcal{H}}_1^{\widetilde{\Delta}} \to \mathcal{H}_2$ such that

$$\mathcal{L}_{\widetilde{\Delta}}(\mathcal{A}_\Delta) = \left\{ \begin{pmatrix} x \\ \widetilde{K}_1^{\widetilde{\Delta}} x \end{pmatrix} : x \in \widetilde{\mathcal{H}}_1^{\widetilde{\Delta}} \right\} \supset \left\{ \begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix} : x \in \mathcal{H}_1^\Delta \right\} = \mathcal{L}_\Delta(\mathcal{A}).$$

As in part i), we find $\widetilde{\mathcal{H}}_1^{\widetilde{\Delta}} = \mathcal{H}_1^\Delta$, $\widetilde{K}_1^{\widetilde{\Delta}} = K_1^\Delta$, and hence $\mathcal{L}_{\widetilde{\Delta}}(\mathcal{A}_\Delta) = \mathcal{L}_\Delta(\mathcal{A}) = \mathcal{L}_\Delta(\mathcal{A}_\Delta)$. This proves $\sigma(\mathcal{A}_\Delta) \cap \widetilde{\Delta} = \sigma(\mathcal{A}_\Delta) \cap \Delta$. □

In the following we relate the spectral supporting subspace $\mathcal{H}_1^\Delta$ of the block operator matrix $\mathcal{A}$ to its Schur complement S_1. It turns out that $\mathcal{H}_1^\Delta$ is the maximal spectral subspace of S_1 corresponding to the interval Δ.

Theorem 1.8.7 *Suppose that $\mathcal{A} = \mathcal{A}^*$. Let $\Delta = [\alpha, \beta] \subset \rho(D)$ and let Γ_Δ be a simply closed Jordan curve surrounding Δ and intersecting $\mathbb{R}$ orthogonally in α and β. Define*

$$Q_\Delta := -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\Delta}' S_1(z)^{-1} \,\mathrm{d}z, \tag{1.8.5}$$

where $\int'$ denotes the Cauchy principal value at $\mathbb{R}$. Then the range of Q_Δ is given by $R(Q_\Delta) = \mathcal{H}_1^\Delta$.

Proof. Let $P_\Delta(\mathcal{A}) \in L(\mathcal{H})$ denote the orthogonal projection onto $\mathcal{L}_\Delta(\mathcal{A})$ and set $\Delta^\circ := (\alpha, \beta)$. We introduce the operator

$$\widehat{P}_\Delta(\mathcal{A}) := \frac{1}{2}\big(P_\Delta(\mathcal{A}) + P_{\Delta^\circ}(\mathcal{A})\big) = -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\Delta}' (\mathcal{A} - z)^{-1} \,\mathrm{d}z.$$

Evidently, $R\big(\widehat{P}_\Delta(\mathcal{A})\big) = \mathcal{L}_\Delta(\mathcal{A})$. The inclusion $R(Q_\Delta) \subset \mathcal{H}_1^\Delta$ follows from

$$P_1 \widehat{P}_\Delta(\mathcal{A}) \begin{pmatrix} x \\ 0 \end{pmatrix} = Q_\Delta x, \quad x \in \mathcal{H}_1,$$

where $P_1 : \mathcal{H} \to \mathcal{H}_1$ is the projection of $\mathcal{H}$ onto its first component $\mathcal{H}_1$. In order to show that the range $R(Q_\Delta)$ is dense in $\mathcal{H}_1^\Delta$, consider $x_0 \in \mathcal{H}_1^\Delta$

with $x_0 \perp P_1 \widehat{P}_\Delta(\mathcal{A})\,(x\ 0)^{\mathrm{t}} = Q_\Delta x$, $x \in \mathcal{H}_1$. This implies

$$\left(\widehat{P}_\Delta(\mathcal{A})\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x_0 \\ 0 \end{pmatrix}\right) = 0, \quad x \in \mathcal{H}_1.$$

Since the projection $\widehat{P}_\Delta(\mathcal{A})$ is non-negative, we conclude $\widehat{P}_\Delta(\mathcal{A})(x_0\ 0)^{\mathrm{t}} = 0$. On the other hand, $\big(x_0\ K_1^\Delta x_0\big)^{\mathrm{t}} \in R\big(\widehat{P}_\Delta(\mathcal{A})\big)$ and thus

$$\|x_0\|^2 = \left(\begin{pmatrix} x_0 \\ K_1^\Delta x_0 \end{pmatrix}, \begin{pmatrix} x_0 \\ 0 \end{pmatrix}\right) = 0.$$

This shows that $R\big(Q_\Delta\big)$ is dense in $\mathcal{H}_1^\Delta$. The proof that $R\big(Q_\Delta\big) = \mathcal{H}_1^\Delta$ uses a similar reasoning: Suppose that $R\big(Q_\Delta\big)$ is not closed. Then there exists a sequence $(x_n)_1^\infty \subset \mathcal{H}_1^\Delta$, $\|x_n\| = 1$, such that $\|Q_\Delta x_n\| \to 0$ if $n \to \infty$. Hence

$$\left\|\widehat{P}_\Delta(\mathcal{A})^{1/2}\begin{pmatrix} x_n \\ 0 \end{pmatrix}\right\|^2 = \left(\widehat{P}_\Delta(\mathcal{A})\begin{pmatrix} x_n \\ 0 \end{pmatrix}, \begin{pmatrix} x_n \\ 0 \end{pmatrix}\right) = \left(P_1 \widehat{P}_\Delta(\mathcal{A})\begin{pmatrix} x_n \\ 0 \end{pmatrix}, x_n\right)$$

$$= (Q_\Delta x_n, x_n) \longrightarrow 0, \quad n \to \infty.$$

Obviously, $\widehat{P}_\Delta(\mathcal{A})(x_n\ 0)^{\mathrm{t}} \to 0$ implies that $P_\Delta(\mathcal{A})(x_n\ 0)^{\mathrm{t}} \to 0$ for $n \to \infty$. Since $\big(x_n\ K_1^\Delta x_n\big)^{\mathrm{t}} \in \mathcal{L}_\Delta(\mathcal{A})$, we obtain the contradiction

$$1 = \|x_n\|^2 = \left(\begin{pmatrix} x_n \\ 0 \end{pmatrix}, \begin{pmatrix} x_n \\ K_1^\Delta x_n \end{pmatrix}\right) = \left(\begin{pmatrix} x_n \\ 0 \end{pmatrix}, P_\Delta(\mathcal{A})\begin{pmatrix} x_n \\ K_1^\Delta x_n \end{pmatrix}\right)$$

$$= \left(P_\Delta(\mathcal{A})\begin{pmatrix} x_n \\ 0 \end{pmatrix}, \begin{pmatrix} x_n \\ K_1^\Delta x_n \end{pmatrix}\right) \longrightarrow 0, \quad n \to \infty. \qquad \square$$

Next we describe a complementary subspace of the Δ-spectral supporting subspace $\mathcal{H}_1^\Delta$ in $\mathcal{H}_1$ in terms of certain spectral subspaces of the Schur complement S_1.

Theorem 1.8.8 *Let $\mathcal{A} = \mathcal{A}^*$ and $\Delta = [\alpha, \beta] \subset \rho(D)$.*

i) *If there exists a $\gamma > 0$ such that one of the conditions*

$$(\alpha, \alpha + \gamma) \cup (\beta - \gamma, \beta) \subset \rho(\mathcal{A}), \tag{1.8.6}$$

$$(\alpha - \gamma, \alpha) \cup (\beta, \beta + \gamma) \subset \rho(\mathcal{A}), \tag{1.8.7}$$

is satisfied, then

$$\mathcal{H}_1 = \overline{\mathcal{L}_{(-\infty,0)}\big(S_1(\alpha)\big) \dot{+} \mathcal{H}_1^\Delta \dot{+} \mathcal{L}_{(0,\infty)}\big(S_1(\beta)\big)}. \tag{1.8.8}$$

ii) *If, in addition, α and β are isolated points of $\sigma(\mathcal{A})$ or in $\rho(\mathcal{A})$, then*

$$\mathcal{H}_1 = \mathcal{L}_{(-\infty,0)}\big(S_1(\alpha)\big) \dot{+} \mathcal{H}_1^\Delta \dot{+} \mathcal{L}_{(0,\infty)}\big(S_1(\beta)\big) \tag{1.8.9}$$

and hence

$$\dim\big(\mathcal{H}_1 \ominus \mathcal{H}_1^\Delta\big) = \dim \mathcal{L}_{(-\infty,0)}\big(S_1(\alpha)\big) + \dim \mathcal{L}_{(0,\infty)}\big(S_1(\beta)\big). \tag{1.8.10}$$

Remark 1.8.9 The dimension formula (1.8.10) can also be written as

$$\dim\big(\mathcal{H}_1 \ominus \mathcal{H}_1^{\Delta}\big) = \dim\big(S_1(\alpha)\big)_- + \dim\big(S_1(\beta)\big)_+,$$

where, for a bounded self-adjoint operator T, we denote by $T_\pm := \frac{1}{2}\big(T \pm |T|\big)$ the positive and negative part of T, respectively.
Moreover, since we have $\mathcal{H}_1^{\{\alpha\}} = \mathcal{L}_{\{0\}}\big(S_1(\alpha)\big) = \ker S_1(\alpha)$, we can replace $\mathcal{L}_{(-\infty,0)}\big(S_1(\alpha)\big)$ by $\mathcal{L}_{(-\infty,0]}\big(S_1(\alpha)\big)$ in the relations (1.8.8) and (1.8.9) and, at the same time, $\mathcal{H}_1^{\Delta}$ by $\mathcal{H}_1^{(\alpha,\beta]}$. The same holds for the point β.

For the proof of Theorem 1.8.8, we provide a number of auxiliary lemmata concerning the monotonicity of the Schur complement S_1. In fact, S_1 is a monotonically decreasing operator function on $\rho(D) \cap \mathbb{R}$ with

$$S_1'(\lambda) = -I - B(D-\lambda)^{-2}B^* \le -I, \quad \lambda \in \rho(D) \cap \mathbb{R}, \tag{1.8.11}$$

and $\lim_{\lambda\to\infty} S_1(\lambda) = -\infty$. These observations are crucial in the following.

Lemma 1.8.10 *Suppose that $\mathcal{A} = \mathcal{A}^*$ and let $\Delta = [\alpha, \beta] \subset \rho(D)$.*

i) *If $x = u + v$ where $u \in \mathcal{H}_1^{\Delta}$ and $\big(S_1(\beta)v, v\big) \ge 0$, then $\big(S_1(\alpha)x, x\big) \ge 0$.*
ii) *If $x = u + v$ where $u \in \mathcal{H}_1^{\Delta}$ and $\big(S_1(\alpha)v, v\big) \le 0$, then $\big(S_1(\beta)x, x\big) \le 0$.*

Proof. We prove claim i); the proof of ii) is similar. Let $P_j : \mathcal{H} \to \mathcal{H}_j$ be the projection of $\mathcal{H}$ onto $\mathcal{H}_j$, $j = 1, 2$. For every $\mathbf{x} = (x\ y)^{\mathrm{t}} \in \mathcal{H}$, we have

$$\begin{aligned} S_1(\lambda)x &= (A-\lambda)x + By - B(D-\lambda)^{-1}B^*x - By \\ &= P_1(\mathcal{A}-\lambda)\mathbf{x} - B(D-\lambda)^{-1}\big(B^*x + (D-\lambda)y\big) \\ &= P_1(\mathcal{A}-\lambda)\mathbf{x} - B(D-\lambda)^{-1}P_2(\mathcal{A}-\lambda)\mathbf{x}. \end{aligned}$$

Let $\gamma := \max_{\lambda\in\Delta}\big(1 + \|B(D-\lambda)^{-1}\|\big) \le 1 + \|B\|\big(\operatorname{dist}\big(\Delta, \sigma(D)\big)\big)^{-1}$. Then

$$\|S_1(\lambda)x\| \le \big(1 + \|B(D-\lambda)^{-1}\|\big)\,\|(\mathcal{A}-\lambda)\mathbf{x}\| \ \le \gamma\,\|(\mathcal{A}-\lambda)\mathbf{x}\|. \tag{1.8.12}$$

Now let $x = u + v$ with $u \in \mathcal{H}_1^{\Delta}$ and $v \in \mathcal{H}_1$ such that $\big(S_1(\beta)v, v\big) \ge 0$. For arbitrary $n \in \mathbb{N}$, we decompose the interval Δ into n subintervals

$$\begin{aligned} \Delta_k &:= \left[\alpha + \frac{(k-1)(\beta-\alpha)}{n},\ \alpha + \frac{k(\beta-\alpha)}{n}\right), \quad k = 1, 2, \dots, n-1, \\ \Delta_n &:= \left[\alpha + \frac{(n-1)(\beta-\alpha)}{n},\ \beta\right]. \end{aligned}$$

Let $E_{\mathcal{A}}(\Delta)$, $E_{\mathcal{A}}(\Delta_k)$ be the corresponding spectral projections of the operator $\mathcal{A}$. Then, for $\mathbf{u} = \big(u\ K_1^{\Delta}u\big)^{\mathrm{t}} \in \mathcal{L}_\Delta(\mathcal{A})$, we have $E_{\mathcal{A}}(\Delta)\mathbf{u} = \mathbf{u}$. If we set $\mathbf{u_k} := E_{\mathcal{A}}(\Delta_k)\mathbf{u}$ and $u_k := P_1\mathbf{u_k}$, $k = 1, \dots, n$, then $u = P_1\mathbf{u} = \sum_{k=1}^n u_k$. For $k = 1, 2, \dots, n$, we choose arbitrary points $\lambda_k \in \Delta_k$. By (1.8.12), we have

$$\big\|S_1(\lambda_k)u_k\big\| \le \gamma \,\big\|(\mathcal{A}-\lambda_k)\mathbf{u_k}\big\| \le \gamma\,\frac{\beta-\alpha}{n}\,\|\mathbf{u_k}\|.$$

Since S_1 is decreasing on $[\alpha,\beta]$ and all operators $S_1(\lambda)$ are self-adjoint for $\lambda \in \mathbb{R}$, it follows that

$$\begin{aligned}
\big(S_1(\alpha)x,x\big) &= \big(S_1(\alpha)(u+v),u+v\big) \ge \big(S_1(\lambda_1)(u+v),u+v\big)\\
&= \Big(S_1(\lambda_1)\Big(\sum_{k=2}^n u_k+v\Big),\sum_{k=2}^n u_k+v\Big) + \big(S_1(\lambda_1)u_1,u+v\big)\\
&\quad + \Big(\sum_{k=2}^n u_k+v, S_1(\lambda_1)u_1\Big)\\
&\ge \Big(S_1(\lambda_2)\Big(\sum_{k=2}^n u_k+v\Big),\sum_{k=2}^n u_k+v\Big) - \|S_1(\lambda_1)u_1\|\,\|u+v\|\\
&\quad - \Big\|\sum_{k=2}^n u_k+v\Big\|\,\big\|S_1(\lambda_1)u_1\big\|.
\end{aligned}$$

Since $\mathbf{u_1},\dots,\mathbf{u_n}$ are pairwise orthogonal, we have

$$\Big\|\sum_{j=k}^n u_j\Big\| \le \Big\|\sum_{j=k}^n \mathbf{u_j}\Big\| \le \|\mathbf{u}\|, \quad k=1,2,\dots,n.$$

Hence, if we let $\gamma' := \gamma\,(\|\mathbf{u}\|+\|v\|)$, then

$$\big(S_1(\alpha)x,x\big) \ge \Big(S_1(\lambda_2)\Big(\sum_{k=2}^n u_k+v\Big),\sum_{k=2}^n u_k+v\Big) - 2\gamma',\frac{\beta-\alpha}{n}\,\|\mathbf{u_1}\|.$$

Repeating these considerations, we finally obtain

$$\begin{aligned}
\big(S_1(\alpha)x,x\big) &\ge \Big(S_1(\lambda_3)\Big(\sum_{k=3}^n u_k+v\Big),\sum_{k=3}^n u_k+v\Big) - 2\gamma'\frac{\beta-\alpha}{n}\big(\|\mathbf{u_1}\|+\|\mathbf{u_2}\|\big)\\
&\ge \dots\\
&\ge \big(S_1(\lambda_n)v,v\big) - 2\gamma'\,\frac{\beta-\alpha}{n}\sum_{k=1}^n\|\mathbf{u_k}\|\\
&\ge \big(S_1(\beta)v,v\big) - 2\gamma'\,\frac{\beta-\alpha}{n}\sum_{k=1}^n\|\mathbf{u_k}\|\\
&\ge -\,2\gamma'\,\frac{\beta-\alpha}{n}\,\sqrt{n}\,\Big(\sum_{k=1}^n\|\mathbf{u_k}\|^2\Big)^{1/2}\\
&= -2\gamma'\,\frac{\beta-\alpha}{\sqrt{n}}\,\|\mathbf{u}\|.
\end{aligned}$$

Since n can be chosen arbitrarily large, it follows that $\big(S_1(\alpha)x,x\big) \ge 0$. □

Corollary 1.8.11 *Let $\mathcal{A} = \mathcal{A}^*$ and $\Delta = [\alpha, \beta] \subset \rho(D)$. Then*

$$\big(S_1(\alpha)x, x\big) \geq 0, \quad \big(S_1(\beta)x, x\big) \leq 0, \qquad x \in \mathcal{H}_1^\Delta;$$

if Ω denotes the component of $\rho(D)$ containing Δ, then

$$\big(S_1(\lambda)x, x\big) \geq \ \ (\alpha - \lambda)\,\|x\|^2, \quad x \in \mathcal{H}_1^\Delta, \quad \text{if } \lambda \in \Omega,\ \lambda < \alpha,$$
$$\big(S_1(\lambda)x, x\big) \leq -(\lambda - \beta)\,\|x\|^2, \quad x \in \mathcal{H}_1^\Delta, \quad \text{if } \lambda \in \Omega,\ \lambda > \beta.$$

Proof. The first two claims are immediate from Lemma 1.8.10 ii) if we set $v = 0$. The last claim follows from relation (1.8.11). □

Corollary 1.8.12 *Let $\mathcal{A} = \mathcal{A}^*$ and $\Delta = [\alpha, \beta] \subset \rho(D)$. Then*

$$x \in \mathcal{H}_1^\Delta + \mathcal{L}_{[0,\infty)}\big(S_1(\beta)\big) \quad \Longrightarrow \quad \big(S_1(\alpha)x, x\big) \geq 0.$$

Proof. Since $\big(S_1(\beta)v, v\big) \geq 0$ for $v \in \mathcal{L}_{[0,\infty)}\big(S_1(\beta)\big)$, the claim follows immediately from Lemma 1.8.10. □

The next lemma is a well-known fact about the spectra of positive perturbations of self-adjoint operators (see *e.g.* [BS87, (9.4.4)]).

Lemma 1.8.13 *Let T_1, T_2 be bounded self-adjoint operators in a Hilbert space so that $(\mu, \nu) \subset \rho(T_1)$ and $\|T_2 - T_1\| < \nu - \mu$. Then*

$$T_1 \leq T_2 \quad \Longrightarrow \quad \big(\mu + \|T_2 - T_1\|, \nu\big) \subset \rho(T_2),$$
$$T_2 \leq T_1 \quad \Longrightarrow \quad \big(\mu, \nu - \|T_2 - T_1\|\big) \subset \rho(T_2).$$

Lemma 1.8.14 *Suppose that $\mathcal{A} = \mathcal{A}^*$, $[\alpha, \alpha+\gamma) \subset \rho(D)$ for some $\gamma > 0$, and $0 \in \rho\big(S_1(\lambda)\big)$ for all $\lambda \in (\alpha, \alpha + \gamma)$. If, for such λ, we set*

$$a(\lambda) := \max\big(\sigma\big(S_1(\lambda)\big) \cap (-\infty, 0)\big), \quad b(\lambda) := \min\big(\sigma\big(S_1(\lambda)\big) \cap (0, \infty)\big),$$

then a and b are continuous non-increasing functions on $(\alpha, \alpha + \gamma)$, and

$$\big(0, b(\lambda_0)\big) \subset \rho\big(S_1(\alpha)\big), \quad \big(a(\lambda_0), 0\big) \subset \rho\big(S_1(\alpha+\gamma)\big), \quad \lambda_0 \in (\alpha, \alpha + \gamma).$$

Proof. First suppose that $a(\lambda)$ and $b(\lambda)$ are finite for all $\lambda \in (\alpha, \alpha + \gamma)$. The continuity of a and b on $(\alpha, \alpha+\gamma)$ follows from [Kat95, Remark V.4.9] since S_1 is a continuous function of λ with respect to the operator norm. In order to show that b is non-increasing, it is sufficient to prove that for arbitrary $\lambda_0 \in (\alpha, \alpha + \gamma)$ there exists an $\varepsilon > 0$ such that

$$b(\lambda) \geq b(\lambda_0), \quad \lambda_0 - \varepsilon < \lambda < \lambda_0. \tag{1.8.13}$$

Choose $\varepsilon > 0$ such that $\|S_1(\lambda) - S_1(\lambda_0)\| < |a(\lambda_0)|$ for all $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$. Then, since $\big(a(\lambda_0), b(\lambda_0)\big) \subset \rho\big(S_1(\lambda_0)\big)$ and $S_1(\lambda) > S_1(\lambda_0)$, Lemma 1.8.13 implies that for $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$ we have

$$\big(0, b(\lambda_0)\big) \subset \big(a(\lambda_0) + \|S_1(\lambda) - S_1(\lambda_0)\|\,, b(\lambda_0)\big) \subset \rho\big(S_1(\lambda)\big),$$

and (1.8.13) follows.

In order to prove the last statement for b, let $\lambda_0 \in (\alpha, \alpha+\gamma)$ be arbitrary and suppose that there exists a $\beta \in \big(0, b(\lambda_0)\big)$ such that $\beta \in \sigma\big(S_1(\alpha)\big)$. Then, again since S_1 is continuous in the operator norm, by [Kat95, Remark V.4.9] there exists a $\lambda' < \lambda_0$ in the neighbourhood of α such that

$$\big(0, b(\lambda_0)\big) \cap \sigma\big(S_1(\lambda')\big) \neq \emptyset.$$

Hence $b(\lambda') < b(\lambda_0)$, a contradiction to the fact that b is non-increasing. The proofs for the function a are analogous; it is also easy to check that all assertions remain true if a or b are no longer finite everywhere. □

Lemma 1.8.15 *Let $\mathcal{A} = \mathcal{A}^*$ and $\alpha \in \rho(D) \cap \mathbb{R}$. Then*

$$(\alpha, \alpha+\gamma) \subset \rho(\mathcal{A}) \text{ for some } \gamma > 0 \iff (0, \delta) \subset \rho\big(S_1(\alpha)\big) \text{ for some } \delta > 0,$$

$$(\alpha-\gamma, \alpha) \subset \rho(\mathcal{A}) \text{ for some } \gamma > 0 \iff (-\delta, 0) \subset \rho\big(S_1(\alpha)\big) \text{ for some } \delta > 0.$$

Proof. We prove the first relation; the proof of the second one is analogous. If $(\alpha, \alpha+\gamma) \subset \rho(\mathcal{A})$ for some $\gamma > 0$, then $0 \in \rho\big(S_1(\lambda)\big)$ for all $\lambda \in (\alpha, \alpha+\gamma)$, and the assertion is immediate from Lemma 1.8.14. Conversely, if $(0,\delta) \subset \rho\big(S_1(\alpha)\big)$ for some $\delta > 0$, then $(-\varepsilon, \delta-\varepsilon) \subset \rho\big(S_1(\alpha)-\varepsilon\big)$ for arbitrary $\varepsilon > 0$. It is not difficult to see (*e.g.* using the resolvent identity for D) that $S_1(\alpha+\varepsilon) < S_1(\alpha) - \varepsilon$. Now choose $\varepsilon_0 > 0$ such that

$$\big\|S_1(\alpha) - \varepsilon - S_1(\alpha+\varepsilon)\big\| < \delta - \varepsilon, \quad 0 < \varepsilon < \varepsilon_0.$$

Applying Lemma 1.8.13 (for the case $T_2 < T_1$), we conclude that

$$\big(-\varepsilon, \delta - \varepsilon - \|S_1(\alpha) - \varepsilon - S_1(\alpha+\varepsilon)\|\big) \subset \rho\big(S_1(\alpha+\varepsilon)\big).$$

Thus $0 \in \rho\big(S_1(\alpha+\varepsilon)\big)$, $0 < \varepsilon < \varepsilon_0$, and hence $(\alpha, \alpha+\varepsilon_0) \subset \rho(\mathcal{A})$. □

Lemma 1.8.16 *Let T_n, $n \in \mathbb{N}$, and T be bounded self-adjoint operators in a Hilbert space $\mathcal{H}$ such that T_n is invertible, $\|T_n^{-1}\|\,\|T - T_n\| \le \omega$ for some $\omega > 0$, $n \in \mathbb{N}$, and $\|T - T_n\| \to 0$, $n \to \infty$. If $x, \widehat{x} \in \mathcal{H}$ are such that $x = T\,\widehat{x}$, then*

$$\lim_{n\to\infty} \big(T_n^{-1}x, x\big) = (T\,\widehat{x}, \widehat{x}). \tag{1.8.14}$$

Proof. We have $T_n^{-1}x = T_n^{-1}T\,\widehat{x} = \widehat{x} + T_n^{-1}(T - T_n)\widehat{x}$ and hence

$$\|T_n^{-1}x\| \le (1+\omega)\,\|\widehat{x}\|, \quad n \in \mathbb{N}. \tag{1.8.15}$$

Further,

$$\big(T_n^{-1}x, x\big) = (\widehat{x}, x) + \big(T_n^{-1}(T - T_n)\widehat{x}, x\big) = (\widehat{x}, T\,\widehat{x}) + \big((T - T_n)\widehat{x}, T_n^{-1}x\big),$$

and therefore, by (1.8.15),

$$\big|\big(T_n^{-1}x,x\big)-(\widehat{x},T\,\widehat{x})\big|\le\|T-T_n\|\,\|\widehat{x}\|\,\|T_n^{-1}x\|\le\|T-T_n\|\,(1+\omega)\,\|\widehat{x}\|.$$

Now (1.8.14) follows from the assumption $\|T-T_n\|\to 0$, $n\to\infty$. □

Lemma 1.8.17 *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be subspaces of a Hilbert space $\mathcal{H}$ and let T be a bounded self-adjoint operator in $\mathcal{H}$. If*

$$(Tx,x)>0,\quad x\in\mathcal{L}_1,\ x\neq 0,\qquad (Ty,y)\le 0\quad y\in\mathcal{L}_2,\tag{1.8.16}$$

then $\mathcal{L}_1\cap\mathcal{L}_2=\{0\}$. If the first inequality in (1.8.16) *is sharpened to*

$$(Tx,x)\ge\delta\,\|x\|^2,\quad x\in\mathcal{L}_1,\tag{1.8.17}$$

with some $\delta>0$, then $\mathcal{L}_1\dot{+}\mathcal{L}_2$ is closed.

Proof. The first assertion is obvious. In order to prove the second assertion, it is sufficient to show that there do not exist sequences $(x_n)_1^\infty\subset\mathcal{L}_1$ and $(y_n)_1^\infty\subset\mathcal{L}_2$ such that

$$\|x_n\|=1,\qquad \|x_n-y_n\|\longrightarrow 0,\quad n\to\infty,\tag{1.8.18}$$

(see *e.g.* [GK92, Theorem 2.1.1]). Assume to the contrary that such sequences do exist. Then the sequence $(y_n)_1^\infty$ is bounded and hence

$$\begin{aligned}\big|(Tx_n,x_n)-(Ty_n,y_n)\big|&\le\big|(Tx_n,x_n)-(Tx_n,y_n)\big|+\big|(Tx_n,y_n)-(Ty_n,y_n)\big|\\&\le\|T\|\,\|x_n\|\,\|x_n-y_n\|+\|T\|\,\|x_n-y_n\|\,\|y_n\|\\&\longrightarrow 0,\quad n\to\infty.\end{aligned}$$

Therefore $(Ty_n,y_n)\le 0$ implies $\limsup_{n\to\infty}(Tx_n,x_n)\le 0$, a contradiction to (1.8.17). □

Now we are ready for the proof of Theorem 1.8.8 on the description of complementary subspaces of spectral supporting subspaces.

***Proof of Theorem* 1.8.8.** i) Due to Lemma 1.8.15, the assumptions in (1.8.6) are equivalent to $(0,\delta)\subset\rho\big(S_1(\alpha)\big)$ and $(-\delta,0)\subset\rho\big(S_1(\beta)\big)$ for some $\delta>0$.

First we show that the sum in (1.8.8) is direct. By Corollary 1.8.11 and Lemma 1.8.17 (with $T=S_1(\beta)$), the sum $\mathcal{H}_1^\Delta+\mathcal{L}_{(0,\infty)}\big(S_1(\beta)\big)$ is direct. From Corollary 1.8.12 and Lemma 1.8.17 it follows that also the sum $\mathcal{L}_{(-\infty,0)}\big(S_1(\alpha)\big)+\big(\mathcal{H}_1^\Delta\dot{+}\mathcal{L}_{(0,\infty)}\big(S_1(\beta)\big)\big)$ is direct.

In order to prove (1.8.8), assume to the contrary that an element $x_0\neq 0$ is orthogonal to the right hand side of (1.8.8). The relation $x_0\perp\mathcal{H}_1^{(\alpha,\beta)}$ implies that $(x_0\ 0)^{\rm t}\perp\mathcal{L}_{(\alpha,\beta)}(\mathcal{A})$, therefore the vector function

$(\mathcal{A}-z)^{-1}(x_0 \;\; 0)^{\mathrm{t}}$, defined *e.g.* for all non-real z, has an analytic continuation onto the whole interval (α,β), and the same holds for the scalar function

$$\varphi(z) := \left((\mathcal{A}-z)^{-1}\begin{pmatrix} x_0 \\ 0 \end{pmatrix}, \begin{pmatrix} x_0 \\ 0 \end{pmatrix}\right) = \big(S_1(z)^{-1}x_0, x_0\big), \quad z \in \mathbb{C}\setminus\mathbb{R}. \tag{1.8.19}$$

Obviously,

$$\varphi'(z) = \left((\mathcal{A}-z)^{-2}\begin{pmatrix} x_0 \\ 0 \end{pmatrix}, \begin{pmatrix} x_0 \\ 0 \end{pmatrix}\right), \quad z \in \mathbb{C}\setminus\mathbb{R}.$$

Hence $\varphi'(\lambda) > 0$ for $\lambda \in (\alpha,\beta)$ so that φ is increasing on (α,β).

First we prove (1.8.8) under the assumption (1.8.6). For every sequence $(\lambda_n)_1^\infty \subset (\alpha, \alpha+\gamma)$ such that $\lambda_n \searrow \alpha$, $n \to \infty$, the operators $T_n := S_1(\lambda_n)$ and $T := S_1(\alpha)$ satisfy the assumptions of Lemma 1.8.16. Indeed, since S_1 is continuous we have $\big\|S_1(\alpha) - S_1(\lambda_n)\big\| \to 0$, $n \to \infty$. Moreover, for $n \in \mathbb{N}$,

$$\begin{aligned}
\big\|S_1(\lambda_n)^{-1}\big\| &\le \big\|(\mathcal{A}-\lambda_n)^{-1}\big\| \le (\lambda_n - \alpha)^{-1}, \\
\big\|S_1(\lambda_n) - S_1(\alpha)\big\| &= \big\|\alpha - \lambda_n - (\alpha-\lambda_n)B(D-\lambda_n)^{-1}(D-\alpha)^{-1}B^*\big\| \\
&\le \big\|I + B(D-\lambda_n)^{-1}(D-\alpha)^{-1}B^*\big\|\,(\lambda_n - \alpha) \\
&\le \omega\,(\lambda_n - \alpha)
\end{aligned}$$

with some constant $\omega > 0$. Thus $\big\|S_1(\lambda_n)^{-1}\big\|\,\big\|S_1(\lambda_n) - S_1(\alpha)\big\| \le \omega$.

Now let $\widetilde{S} := S_1(\alpha)\big|_{\mathcal{L}_{(0,\infty)}(S_1(\alpha))}$. By definition, we have $\sigma\big(\widetilde{S}\big) \subset [0,\infty)$ and $\ker \widetilde{S} = \{0\}$. Since $(0,\delta) \subset \rho\big(S_1(\alpha)\big)$, we also have $(0,\delta) \subset \rho\big(\widetilde{S}\big)$; therefore, 0 is either a point of $\rho\big(\widetilde{S}\big)$ or an isolated point of $\sigma\big(\widetilde{S}\big)$. Because every isolated spectral point of a self-adjoint operator is an eigenvalue, the second case is excluded and so the operator $\widetilde{S}$ is boundedly invertible. From the assumption that $x_0 \perp \mathcal{L}_{(-\infty,0]}\big(S_1(\alpha)\big)$ it follows that $x_0 \in \mathcal{L}_{(0,\infty)}\big(S_1(\alpha)\big)$. If we set $\widehat{x}_0 := \widetilde{S}^{-1}x_0 \in \mathcal{D}\big(\widetilde{S}\big) = \mathcal{L}_{(0,\infty)}\big(S_1(\alpha)\big)$, then $x_0 = S_1(\alpha)\widehat{x}_0$ and, by Lemma 1.8.16,

$$\lim_{n\to\infty}\varphi(\lambda_n) = \lim_{n\to\infty}\big(S_1(\lambda_n)^{-1}x_0, x_0\big) = \big(S_1(\alpha)\,\widehat{x}_0, \widehat{x}_0\big) > 0.$$

Analogously we prove that for every sequence $(\mu_n)_1^\infty \subset (\beta-\gamma,\beta)$ such that $\mu_n \nearrow \beta$, $n \to \infty$, there exists an $\widehat{x}_1 \in \mathcal{L}_{(-\infty,0)}\big(S_1(\beta)\big)$ such that

$$\lim_{n\to\infty}\varphi(\mu_n) = \lim_{n\to\infty}\big(S_1(\mu_n)^{-1}x_0, x_0\big) = \big(S_1(\beta)\,\widehat{x}_1, \widehat{x}_1\big) < 0.$$

Altogether, for sufficiently large $n \in \mathbb{N}$, we have $\lambda_n < \mu_n$, $\varphi(\lambda_n) > 0$, and $\varphi(\mu_n) < 0$, which contradicts the fact that φ is increasing on (α,β).

The proof that (1.8.7) implies (1.8.8) is similar. Here we consider the subspace $\mathcal{R} := \mathcal{L}_{\mathbb{R}\setminus[\alpha,\beta]}(\mathcal{A}) = \mathcal{L}_{[\alpha,\beta]}(\mathcal{A})^\perp$ and the operator $\widetilde{\mathcal{A}} := \mathcal{A}|_{\mathcal{R}}$, for which $(\alpha-\gamma,\alpha)\cup(\beta,\beta+\gamma)\subset\rho\big(\widetilde{\mathcal{A}}\big)$. Then sequences $(\lambda_n)_1^\infty\subset(\alpha-\gamma,\alpha)$ and $(\mu_n)_1^\infty\subset(\beta,\beta+\gamma)$ with $\lambda_n\nearrow\alpha$ and $\mu_n\searrow\beta$, $n\to\infty$, are constructed leading to an analogous contradiction as above.

ii) It remains to be proved that the direct sum in (1.8.9) is closed. By Corollary 1.8.11,

$$\big(S_1(\beta)x,x\big)\leq 0,\quad x\in\mathcal{H}_1^\Delta.$$

The assumptions in ii) are equivalent to the fact that $R(\mathcal{A}-\alpha)$, $R(\mathcal{A}-\beta)$ are closed. From the Schur factorization (1.6.2) it is clear that this is equivalent to $R\big(S_1(\alpha)\big)$, $R\big(S_1(\beta)\big)$ being closed. Hence 0 is an isolated point of $\sigma\big(S_1(\alpha)\big)$ $\big(\sigma\big(S_1(\beta)\big)$, respectively$\big)$ or in $\rho\big(S_1(\alpha)\big)$ $\big(\rho\big(S_1(\beta)\big)$, respectively$\big)$. This implies that there exists a $\delta>0$ with

$$\big(S_1(\beta)x,x\big)\geq\delta\,\|x\|^2,\quad x\in\mathcal{L}_{(0,\infty)}\big(S_1(\beta)\big).$$

Then it follows from Lemma 1.8.17 ii) that the sum $\mathcal{H}_1^\Delta+\mathcal{L}_{(0,\infty)}\big(S_1(\beta)\big)$ is closed. By Corollary 1.8.12,

$$\big(S_1(\alpha)x,x\big)\geq 0,\quad x\in\mathcal{H}_1^\Delta\dot{+}\mathcal{L}_{(0,\infty)}\big(S_1(\beta)\big).$$

As 0 is an isolated point of $\sigma\big(S_1(\alpha)\big)$ or in $\rho\big(S_1(\alpha)\big)$, there is a $\delta>0$ with

$$\big(S_1(\alpha)x,x\big)\leq-\delta\,\|x\|^2,\quad x\in\mathcal{L}_{(-\infty,0)}\big(S_1(\alpha)\big).$$

Again by Lemma 1.8.17, it follows that the sum in (1.8.9) is closed. □

Corollary 1.8.18 *Let* $\alpha>\max\sigma(D)$. *Then*

$$\mathcal{H}_1=\overline{\mathcal{L}_{(-\infty,0)}\big(S_1(\alpha)\big)\dot{+}\mathcal{H}_1^{[\alpha,\infty)}},\quad\mathcal{H}_1=\mathcal{L}_{(-\infty,0)}\big(S_1(\alpha)\big)\dot{+}\mathcal{H}_1^{[\alpha,\infty)}.$$

Proof. If we let $\beta>\max\sigma(\mathcal{A})$ arbitrarily large in Theorem 1.8.8, then $\mathcal{L}_{(0,\infty)}\big(S_1(\beta)\big)=\{0\}$ in (1.8.8) and (1.8.9) since $\lim_{\lambda\to\infty}S_1(\lambda)=-\infty$. □

Corollary 1.8.19 *If* $\Delta=[\alpha,\beta]\subset\rho(D)$, *then the following are equivalent:*

i) $\mathcal{H}_1^\Delta=\mathcal{H}_1$;

ii) $S_1(\alpha)\geq 0$, $S_1(\beta)\leq 0$.

Proof. That i) implies ii) follows immediately from Corollary 1.8.11. Conversely, if ii) holds, then we choose two sequences $(\alpha_n)_1^\infty$, $(\beta_n)_1^\infty$ such that $\alpha_n\nearrow\alpha$, $\beta_n\searrow\beta$, $n\to\infty$, and $[\alpha_1,\beta_1]\subset\rho(D)$. According to (1.5.7), we have $S_1(\alpha_n)\gg 0$, $S_1(\beta_n)\ll 0$, $n\in\mathbb{N}$. Hence, by Theorem 1.8.8 or by [MS96] applied for every interval $\Delta_n:=[\alpha_n,\beta_n]$, we obtain $\mathcal{H}_1^{\Delta_n}=\mathcal{H}_1$. Obviously, $\mathcal{L}_\Delta(\mathcal{A})=\bigcap_{n=1}^\infty\mathcal{L}_{\Delta_n}(\mathcal{A})$, and hence

$$\mathcal{H}_1^\Delta=\bigcap_{n=1}^\infty\mathcal{H}_1^{\Delta_n}=\mathcal{H}_1.$$ □

Remark 1.8.20 Certain analogues of the dimension formula (1.8.10), but not of the decomposition (1.8.9), were proved in the papers [MM87], [Kru93], [ADS00] for a rather general class of operator functions S, including even the non-analytic case. In the last two papers, the condition $S'(\lambda) \gg 0$ (or rather $S'(\lambda) \gg 0$ in [MM87]) was replaced by a weaker condition, called Virozub–Matsaev condition in [Kru93] and condition (**S**) in [ADS00]. In all three papers a strong additional assumption is imposed which implies the discreteness of the spectrum of S; as a consequence, in (1.8.10) instead of $\mathcal{H}_1^\Delta$ the closed span of all eigenvectors of S to eigenvalues in Δ appears. It is easy to prove that, in the situation of [MM87] (or [Mar88, § 33]), also an analogue of (1.8.9) holds.

1.9 Variational principles for eigenvalues in gaps

The classical variational principles (see *e.g.* [WS72], [RS78]) provide a characterization of the eigenvalues of a semi-bounded self-adjoint operator $\mathcal{A}$ below or above its essential spectrum in terms of the *Rayleigh functional*

$$p(x) = \frac{(\mathcal{A}x, x)}{\|x\|^2}, \quad x \in \mathcal{D}(\mathcal{A}),\ x \neq 0.$$

For example, if $\lambda_1 \le \lambda_2 \le \cdots$ are the eigenvalues of $\mathcal{A}$ below $\sigma_{\rm ess}(\mathcal{A})$, then

$$\lambda_n = \min_{\substack{\mathcal{L}\subset\mathcal{D}(\mathcal{A})\\ \dim\mathcal{L}=n}} \max_{\substack{x\in\mathcal{L}\\ x\neq 0}} p(x), \quad n = 1, 2, \ldots .$$

Note that the range of the Rayleigh functional is the numerical range of $\mathcal{A}$,

$$W(\mathcal{A}) = \big\{p(x) : x \in \mathcal{D}(\mathcal{A}),\ x \neq 0\big\}.$$

Even for bounded self-adjoint operators, eigenvalues in gaps of the essential spectrum cannot be characterized by such simple min-max principles. However, after suitable decomposition of the underlying Hilbert space, we may use that, for a self-adjoint block operator matrix $\mathcal{A}$, the quadratic numerical range is the union of the ranges $\Lambda_\pm(\mathcal{A})$ of the functionals $\lambda_\pm$:

$$W^2(\mathcal{A}) = \Lambda_-(\mathcal{A}) \cup \Lambda_+(\mathcal{A})$$

where (see Corollary 1.1.4)

$$\lambda_\pm\begin{pmatrix} x\\ y\end{pmatrix} = \frac{1}{2}\left(\frac{(Ax,x)}{\|x\|^2} + \frac{(Dy,y)}{\|y\|^2} \pm \sqrt{\left(\frac{(Ax,x)}{\|x\|^2} - \frac{(Dy,y)}{\|y\|^2}\right)^2 + 4\frac{|(By,x)|^2}{\|x\|^2\,\|y\|^2}}\right),$$

$$\Lambda_\pm(\mathcal{A}) = \left\{\lambda_\pm\begin{pmatrix} x\\ y\end{pmatrix} : x \in \mathcal{H}_1,\ y \in \mathcal{H}_2,\ x, y \neq 0\right\}.$$

Theorem 1.9.1 *Let $\mathcal{A} = \mathcal{A}^*$. Assume that there exists an $\alpha > \sup W(D)$ so that $\big(\sup W(D), \alpha\big) \subset \rho(\mathcal{A})$ and define*

$$\lambda_{\mathrm{e}} := \min\big(\sigma_{\mathrm{ess}}(\mathcal{A}) \cap (\sup W(D), \infty)\big). \tag{1.9.1}$$

Further, let

$$\kappa := \kappa_-(\lambda) := \dim \mathcal{L}_{(-\infty,0)}\big(S_1(\lambda)\big) < \infty, \quad \lambda \in \big(\sup W(D), \alpha\big].$$

If $\lambda_1 \le \lambda_2 \le \cdots$ is the finite or infinite sequence of the eigenvalues of $\mathcal{A}$ in the interval $\big(\sup W(D), \lambda_{\mathrm{e}}\big)$ counted with multiplicities, then

$$\lambda_n = \min_{\substack{\mathcal{L} \subset \mathcal{H}_1 \\ \dim \mathcal{L} = \kappa + n}} \max_{\substack{x \in \mathcal{L} \\ x \neq 0}} \max_{\substack{y \in \mathcal{H}_2 \\ y \neq 0}} \lambda_+\begin{pmatrix} x \\ y \end{pmatrix}, \quad n = 1, 2, \dots. \tag{1.9.2}$$

Proof. First we observe that the index shift κ in (1.9.2) is independent of λ. In fact, by means of continuity arguments, it can be shown that $\kappa_-(\cdot)$ is constant on each subinterval of $\rho(S_1)$ (see [EL04]). Define

$$\mu_{n+\kappa} := \inf_{\substack{\mathcal{L} \subset \mathcal{H}_1 \\ \dim \mathcal{L} = \kappa + n}} \sup_{\substack{x \in \mathcal{L} \\ x \neq 0}} \sup_{\substack{y \in \mathcal{H}_2 \\ y \neq 0}} \lambda_+\begin{pmatrix} x \\ y \end{pmatrix}, \quad n = 1, 2, \dots.$$

Let $\alpha' \in \big(\sup W(D), \alpha\big) \subset \rho(\mathcal{A})$. Then $\alpha' \in \rho(S_1)$ by Proposition 1.6.2. First we prove that $\lambda_n \le \mu_{\kappa+n}$. To this end, set $\Delta := [\lambda_n, \infty)$. By Theorem 1.8.1, the spectral subspace $\mathcal{L}_\Delta(\mathcal{A})$ admits the representation

$$\mathcal{L}_\Delta(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix} : x \in \mathcal{H}_1^\Delta \right\} \tag{1.9.3}$$

where $\mathcal{H}_1^\Delta \subset \mathcal{H}_1$ is a subspace and $K_1^\Delta : \mathcal{H}_1^\Delta \to \mathcal{H}_2$ is a bounded linear operator. According to Corollary 1.8.18, we have the decompositions

$$\mathcal{H}_1 = \mathcal{H}_1^{[\alpha',\infty)} \dot{+} \mathcal{L}_{(-\infty,0)}\big(S_1(\alpha')\big) = \mathcal{H}_1^\Delta \dot{+} \mathcal{L}_{(-\infty,0)}\big(S_1(\lambda_n)\big). \tag{1.9.4}$$

Since $\big(\sup W(D), \alpha\big) \subset \rho(\mathcal{A})$ and $\alpha' < \alpha$, we have $\Delta = [\lambda_n, \infty) \subset [\alpha', \infty)$ and hence $\mathcal{H}_1^\Delta \subset \mathcal{H}_1^{[\alpha',\infty)}$. By (1.9.4) and by definition of κ, we obtain

$$\operatorname{codim}_{\mathcal{H}_1} \mathcal{H}_1^\Delta = \dim \mathcal{L}_{(-\infty,0)}\big(S_1(\alpha')\big) + \operatorname{codim}_{\mathcal{H}_1^{[\alpha',\infty)}} \mathcal{H}_1^\Delta = \kappa + n - 1.$$

Now let $\mathcal{L} \subset \mathcal{H}_1$ be an arbitrary subspace with $\dim \mathcal{L} = \kappa + n$. Then $\dim \mathcal{L} > \operatorname{codim}_{\mathcal{H}_1} \mathcal{H}_1^\Delta$ and hence there exists an $x \in \mathcal{L} \cap \mathcal{H}_1^\Delta$, $x \neq 0$. If $K_1^\Delta x \neq 0$, then (1.9.3) implies that

$$\begin{aligned} \lambda_n &\le \frac{1}{\|x\|^2 + \|K_1^\Delta x\|^2} \left(\mathcal{A}\begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix}, \begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix} \right) \\ &= \frac{1}{\|x\|^2 + \|K_1^\Delta x\|^2} \left(\mathcal{A}_{x, K_1^\Delta x}\begin{pmatrix} \|x\| \\ \|K_1^\Delta x\| \end{pmatrix}, \begin{pmatrix} \|x\| \\ \|K_1^\Delta x\| \end{pmatrix} \right)_{\mathbb{C}^2} \le \lambda_+\begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix}; \end{aligned}$$

for the last inequality we have used that $\lambda_+(x, K_1^\Delta x)$ is the larger of the two eigenvalues of $\mathcal{A}_{x,K_1^\Delta x}$. If $K_1^\Delta x = 0$, then (1.2.6) shows that for every $y \in \mathcal{H}_2$, $y \neq 0$,

$$\lambda_n \leq \frac{1}{\|x\|^2 + \|K_1^\Delta x\|^2}\left(\mathcal{A}\begin{pmatrix} x \\ K_1^\Delta x\end{pmatrix}, \begin{pmatrix} x \\ K_1^\Delta x\end{pmatrix}\right) = \frac{(Ax,x)}{\|x\|^2} \leq \lambda_+\begin{pmatrix} x \\ y\end{pmatrix}.$$

So in each case we have found elements $x \in \mathcal{L}$, $y \in \mathcal{H}_2$ with $\lambda_n \leq \lambda_+(x,y)$ and hence $\lambda_n \leq \mu_{\kappa+n}$.

Next we prove that $\lambda_n \geq \mu_{\kappa+n}$. In the same way as above, we see $\lambda_n \geq \mu_{\kappa+n}$ that

$$\mathcal{H}_1 = \mathcal{H}_1^{[\alpha',\infty)} \dot{+} \mathcal{L}_{(-\infty,0)}\big(S_1(\alpha')\big) = \mathcal{H}_1^{(\lambda_n,\infty)} \dot{+} \mathcal{L}_{(-\infty,0]}\big(S_1(\lambda_n)\big)$$

and

$$\dim \mathcal{L}_{(-\infty,0]}\big(S_1(\lambda_n)\big) = \kappa + n.$$

Now choose $\mathcal{L} = \mathcal{L}_{(-\infty,0]}\big(S_1(\lambda_n)\big)$. If $x \in \mathcal{L}$, $y \in \mathcal{H}_2$, $x, y \neq 0$, and ξ, $\eta \in \mathbb{C}$ are arbitrary and we set $\mathbf{u} := (u\ v)^{\mathrm{t}} := \big(\xi\, x/\|x\|\ \ \eta\, y/\|y\|\big)^{\mathrm{t}}$, then we have $u \in \mathcal{L}_{(-\infty,0]}\big(S_1(\lambda_n)\big)$ and

$$\frac{1}{|\xi|^2 + |\eta|^2}\left(\mathcal{A}_{x,y}\begin{pmatrix} \xi \\ \eta\end{pmatrix}, \begin{pmatrix} \xi \\ \eta\end{pmatrix}\right)_{\mathbb{C}^2} = \frac{(\mathcal{A}\mathbf{u}, \mathbf{u})}{\|\mathbf{u}\|^2}.$$

Using the Frobenius-Schur factorization (1.6.2) of $\mathcal{A} - \lambda_n$, we obtain

$$\begin{aligned}&\left(\big(\mathcal{A} - \lambda_n\big)\begin{pmatrix} u \\ v\end{pmatrix}, \begin{pmatrix} u \\ v\end{pmatrix}\right)\\ &= \left(\begin{pmatrix} S_1(\lambda_n) & 0 \\ 0 & D-\lambda_n\end{pmatrix}\begin{pmatrix} I & 0 \\ (D-\lambda_n)^{-1}B^* & I\end{pmatrix}\begin{pmatrix} u \\ v\end{pmatrix}, \begin{pmatrix} I & 0 \\ (D-\lambda_n)^{-1}B^* & I\end{pmatrix}\begin{pmatrix} u \\ v\end{pmatrix}\right)\\ &= \big(S_1(\lambda_n)u, u\big) + \big((D-\lambda_n)w, w\big) \leq 0\end{aligned}$$

where $w := (D-\lambda_n)^{-1}B^*u + v$. This implies $W(\mathcal{A}_{x,y}) \subset (-\infty, \lambda_n]$ and hence $\lambda_+(x,y) \leq \lambda_n$. Thus the proof of $\lambda_n = \mu_{\kappa+n}$ is complete.

It remains to be shown that the infimum and the suprema in $\mu_{\kappa+n}$ are all attained. Since $\lambda_n \in \sigma_{\mathrm{p}}(\mathcal{A}) \subset W^2(\mathcal{A})$, there exists an eigenvector $(x_n\ y_n)^{\mathrm{t}} = (x_n\ K_1^\Delta x_n)^{\mathrm{t}} \in \mathcal{L}_\Delta(\mathcal{A})$ such that $\lambda_n = \lambda_+(\widehat{x}_n, \widehat{y}_n)$ where $\widehat{x}_n$, $\widehat{y}_n$ are such that $x_n = \|x_n\|\widehat{x}_n$, $y_n = \|y_n\|\widehat{y}_n$. □

We will further extend on variational principles for eigenvalues in gaps in Sections 2.10 and 2.11. There we generalize the above result to unbounded block operator matrices with real quadratic numerical range, including self-adjoint and certain $\mathcal{J}$-self-adjoint block operator matrices. The proof in the unbounded case uses variational principles for the Schur complements. We also present a method to calculate the index shift and establish two-sided eigenvalue estimates in terms of the entries of the block operator matrix. The results in Sections 2.10 and 2.11 may easily be specialized to bounded block operator matrices by observing that in this case $\mathcal{D}(A) = \mathcal{D}(B^*) = \mathcal{H}_1$ and $\mathcal{D}(B) = \mathcal{D}(D) = \mathcal{H}_2$.

1.10 $\mathcal{J}$-self-adjoint block operator matrices

$\mathcal{J}$-self-adjoint block operator matrices arise naturally when studying self-adjoint operators in Krein spaces. In fact, a Krein space is an inner product space $(\mathcal{K}, [\cdot,\cdot])$ that admits a decomposition $\mathcal{K} = \mathcal{K}_+[+]\mathcal{K}_-$ so that $\mathcal{H}_1 = (\mathcal{K}_+, [\cdot,\cdot])$ and $\mathcal{H}_2 = (\mathcal{K}_-, -[\cdot,\cdot])$ are Hilbert spaces (see [Lan82, Section I.1]). With respect to the decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$, we have $[\cdot,\cdot] = (\mathcal{J}\cdot,\cdot)$ with $\mathcal{J} = \operatorname{diag}(I, -I)$ as in Definition 1.1.14 and every bounded self-adjoint operator in $\mathcal{K}$ has a block operator matrix representation (1.1.3) with $A = A^*$, $D = D^*$, and $C = -B^*$, *i.e.* $\mathcal{A}$ is $\mathcal{J}$-self-adjoint.

In the following, we use the block operator techniques developed in the previous sections to study the spectrum of $\mathcal{J}$-self-adjoint block operator matrices; in particular, we identify intervals on which the spectrum is of definite type; on such intervals, the operator possesses a local spectral function. We also study the corresponding spectral supporting subspaces, thus establishing results analogous to those derived in Section 1.8 for the self-adjoint case.

We start with some elementary properties of the quadratic numerical range of $\mathcal{J}$-self-adjoint block operator matrices. Here an important role is played by the sets $\Lambda_\pm(\mathcal{A})$ introduced in Corollary 1.1.4; in the particular case $C = -B^*$, we have

$$\operatorname{dis}_{\mathcal{A}}(x,y) = \left(\frac{(Ax,x)}{\|x\|^2} - \frac{(Dy,y)}{\|g\|^2}\right)^2 - 4\frac{|(By,x)|^2}{\|x\|^2\,\|y\|^2}$$

for $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$, $x, y \neq 0$, and, if $\operatorname{dis}_{\mathcal{A}}(x,y) \geq 0$,

$$\lambda_\pm\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\left(\frac{(Ax,x)}{\|x\|^2} + \frac{(Dy,y)}{\|y\|^2} \pm \sqrt{\left(\frac{(Ax,x)}{\|x\|^2} - \frac{(Dy,y)}{\|y\|^2}\right)^2 - 4\frac{|(By,x)|^2}{\|x\|^2\,\|y\|^2}}\right),$$

$$\Lambda_\pm := \Lambda_\pm(\mathcal{A}) = \left\{\lambda_\pm\begin{pmatrix} x \\ y \end{pmatrix} : x \in \mathcal{H}_1,\, y \in \mathcal{H}_2,\, x, y \neq 0,\, \operatorname{dis}_{\mathcal{A}}(x,y) \geq 0\right\}.$$

If $C = -B^*$ and $W^2(\mathcal{A})$ is real, then we have $\operatorname{dis}_{\mathcal{A}}(x,y) \geq 0$ for all $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$, $x, y \neq 0$, and hence $W^2(\mathcal{A}) = \Lambda_- \cup \Lambda_+$. In this case, continuity arguments show that the sets $\Lambda_\pm$ are intervals. In general, we can only prove a weaker statement (see Proposition 1.10.3 ii) below).

Lemma 1.10.1 *For all* $(x_1\ y_1)^{\mathrm{t}}, (x_2\ y_2)^{\mathrm{t}} \in \mathcal{H}_1 \oplus \mathcal{H}_2$, $x_1, x_2, y_1, y_2 \neq 0$, *there exists a curve* $\big(x(t)\ y(t)\big)^{\mathrm{t}}$, $t \in [0,1]$, *such that*

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad \begin{pmatrix} x(1) \\ y(1) \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \qquad x(t),\, y(t) \neq 0, \quad t \in [0,1].$$

Proof. There exist at most two different points $\zeta \in \mathbb{C}$, say ζ_1, ζ_2, such that $\zeta x_1 + (1-\zeta)x_2 = 0$ or $\zeta y_1 + (1-\zeta)y_2 = 0$. Let $\zeta(t)$, $t \in [0,1]$, be a curve in the complex plane with $\zeta(0) = 0$, $\zeta(1) = 1$ not passing through ζ_1 and ζ_2 . Then a curve with the required properties is given by

$$\zeta(t) \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \big(1 - \zeta(t)\big) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \quad t \in [0,1]. \qquad \square$$

In the following we always assume that $\dim \mathcal{H} > 2$; otherwise, $\mathcal{A}$ is a 2×2 matrix and the questions considered here are trivial.

Proposition 1.10.2 *Suppose* $\dim \mathcal{H} > 2$ *and let* $\mathcal{A}$ *be* $\mathcal{J}$*-self-adjoint.*

i) *If* $W^2(\mathcal{A}) \setminus \mathbb{R} \neq \emptyset$, *then* $W^2(\mathcal{A})$ *is connected.*
ii) *If* $\overline{W^2(\mathcal{A})}$ *consists of two components, then* $\sigma(\mathcal{A}) \subset \mathbb{R}$.

Proof. i) Let $z_1 \in W^2(\mathcal{A}) \setminus \mathbb{R}$. Then $z_1 \in \sigma_{\mathrm{p}}(\mathcal{A}_{x_1,y_1})$ for some $x_1 \in \mathcal{H}_1$, $y_1 \in \mathcal{H}_2$, $x_1, y_1 \neq 0$, with $\mathrm{dis}_{\mathcal{A}}(x_1, y_1) < 0$. Since $\dim \mathcal{H} > 2$, we either have $\dim \mathcal{H}_1 \geq 2$ or $\dim \mathcal{H}_2 \geq 2$ and thus, by Theorem 1.1.9, either $W(D) \subset W^2(\mathcal{A})$ or $W(A) \subset W^2(\mathcal{A})$. Because A and D are self-adjoint, this implies that there exists a $z_2 \in W^2(\mathcal{A}) \cap \mathbb{R}$. Hence $z_2 \in \sigma_{\mathrm{p}}(\mathcal{A}_{x_2,y_2})$ for some $x_2 \in \mathcal{H}_1$, $y_2 \in \mathcal{H}_2$, $x_2, y_2 \neq 0$ with $\mathrm{dis}_{\mathcal{A}}(x_2, y_2) \geq 0$.

Let $\big(x(t)\ y(t)\big)^{\mathrm{t}}$, $t \in [0,1]$, be a curve in $\mathcal{H}$ as in Lemma 1.10.1 connecting $(x_1\ y_1)^{\mathrm{t}}$ and $(x_2\ y_2)^{\mathrm{t}}$. Since $t \mapsto \mathrm{dis}_{\mathcal{A}}\big(x(t), y(t)\big)$ is continuous, there exists a $t_0 \in [0,1]$ such that $\mathrm{dis}_{\mathcal{A}}\big(x(t_0), y(t_0)\big) = 0$. This means that the matrix $\mathcal{A}_{x(t_0),y(t_0)}$ has a double eigenvalue, which we denote by z_0.

Now let $z, z' \in W^2(\mathcal{A})$ be arbitrary, $z \in \sigma_{\mathrm{p}}(\mathcal{A}_{x,y})$, $z' \in \sigma_{\mathrm{p}}(\mathcal{A}_{x',y'})$ for some $x, x' \in \mathcal{H}_1$, $y, y' \in \mathcal{H}_2$, $x, x', y, y' \neq 0$. By Lemma 1.10.1, there exists a curve $\big(x(t)\, y(t)\big)^{\mathrm{t}}$, $t \in [0,1]$, in $\mathcal{H}$ connecting $(x\, y)^{\mathrm{t}}$ with $\big(x(t_0)\ y(t_0)\big)^{\mathrm{t}}$. If $z = \lambda_\pm(x,y)$, then $\lambda_\pm\big(x(t), y(t)\big)$, $t \in [0,1]$, is a curve in $W^2(\mathcal{A})$ from z to z_0. A curve from z_0 to z' in $W^2(\mathcal{A})$ is constructed analogously.

ii) Since $W^2(\mathcal{A})$ consists of at most two components and $\sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A})}$, the claim is immediate from i). $\square$

In the following, we investigate the sets $\Lambda_\pm$ in greater detail. As in Proposition 1.3.9, we use the notations

$$a_- := \inf W(A), \quad a_+ := \sup W(A),$$
$$d_- := \inf W(D), \quad d_+ := \sup W(D).$$

Proposition 1.10.3 *Let $\mathcal{A}$ be $\mathcal{J}$-self-adjoint.*

i) *Then* $\overline{\operatorname{conv}(\Lambda_- \cup \Lambda_+)} \subset \overline{\operatorname{conv}\big(W(A) \cup W(D)\big)}$ *or, equivalently,*

$$\inf \Lambda_- \geq \min\{a_-, d_-\}, \quad \sup \Lambda_+ \leq \max\{a_+, d_+\},$$

and equality holds if $\dim \mathcal{H}_1 \geq 2$ *and* $\dim \mathcal{H}_2 \geq 2$.

ii) *If* $\inf \Lambda_+ < \sup \Lambda_-$, *then*

$$(\inf \Lambda_-, \inf \Lambda_+] \subset \Lambda_-, \quad [\sup \Lambda_-, \sup \Lambda_+) \subset \Lambda_+.$$

Proof. We prove the second statements in i) and ii); the proof of the first claims is similar.

i) For all $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$, $x, y \neq 0$, with $\operatorname{dis}_{\mathcal{A}}(x, y) \geq 0$, we have

$$\begin{aligned}\lambda_+ \begin{pmatrix} x \\ y \end{pmatrix} &\leq \frac{1}{2}\left(\frac{(Ax,x)}{\|x\|^2} + \frac{(Dy,y)}{\|y\|^2} + \left|\frac{(Ax,x)}{\|x\|^2} - \frac{(Dy,y)}{\|y\|^2}\right|\right) \\ &= \max\left\{\frac{(Ax,x)}{\|x\|^2}, \frac{(Dy,y)}{\|y\|^2}\right\} \leq \max\{a_+, d_+\}\end{aligned}$$

and thus $\sup \Lambda_+ \leq \max\{a_+, d_+\}$. Since $\dim \mathcal{H}_1 \geq 2$, $\dim \mathcal{H}_2 \geq 2$, we have $W(A) \subset W^2(\mathcal{A})$, $W(D) \subset W^2(\mathcal{A})$ by Theorem 1.1.9, and so equality follows.

ii) Assume to the contrary that there exists a $\lambda_0 \in [\sup \Lambda_-, \sup \Lambda_+)$ with $\lambda_0 \notin \Lambda_+$. Since $\lambda_0 < \sup \Lambda_+$, there is a $\lambda_1 \in \Lambda_+$ with $\lambda_0 < \lambda_1$. By assumption, there exists $\lambda_2 \in \Lambda_- \cap \Lambda_+$. Then $\lambda_2 < \lambda_0 < \lambda_1$ and there exist $x_1, x_2 \in \mathcal{H}_1$ and $y_1, y_2 \in \mathcal{H}_2$, $x_1, y_1, x_2, y_2 \neq 0$, with $\lambda_1 := \lambda_+(x_1, y_1)$ and $\lambda_2 := \lambda_+(x_2, y_2)$. Let $\big(x(t)\ y(t)\big)^{\mathrm{t}}$, $t \in [0,1]$, be a curve in $\mathcal{H}$ connecting $(x_1\ y_1)^{\mathrm{t}}$ and $(x_2\ y_2)^{\mathrm{t}}$ as in Lemma 1.10.1. Then $\lambda_+\big(x(t), y(t)\big)$, $t \in [0,1]$, is a curve in $W^2(\mathcal{A})$ connecting λ_1 and λ_2. This curve cannot stay in $\mathbb{R}$ since $\lambda_2 < \lambda_0 < \lambda_1$ and $\lambda_0 \notin \Lambda_+$; hence $\operatorname{dis}_{\mathcal{A}}\big(x(t), y(t)\big) < 0$ for some $t \in [0,1]$. Let $t_0 \in [0,1]$ be the minimal zero of $\operatorname{dis}_{\mathcal{A}}\big(x(\cdot), y(\cdot)\big)$. Then $\lambda_+\big(x(t_0), y(t_0)\big) = \lambda_-\big(x(t_0), y(t_0)\big) \in \Lambda_+ \cap \Lambda_-$. On the other hand, $\lambda_+\big(x(t), y(t)\big) \in \Lambda_+$, $t \in [0, t_0]$, and $\lambda_+\big(x(0), y(0)\big) = \lambda_1 > \lambda_0$; hence $\lambda_+\big(x(t_0), y(t_0)\big) > \lambda_0 \geq \sup \Lambda_-$, a contradiction to $\lambda_+\big(x(t_0), y(t_0)\big) \in \Lambda_-$. □

In the following we identify a subinterval $[\nu, \mu] \subset W^2(\mathcal{A}) \cap \mathbb{R}$ such that outside of this interval $\mathcal{A}$ possesses a local spectral function. The interval is defined such that its complement lies outside of the intersection of Λ_- and Λ_+ and does not contain accumulation points of the non-real spectrum.

Proposition 1.10.4 *Let $\mathcal{A}$ be $\mathcal{J}$-self-adjoint. Define*

$$\Lambda_0 := \big\{\lambda \in \mathbb{R} : \exists\, (\lambda_n)_1^\infty \subset W^2(\mathcal{A}) \setminus \mathbb{R},\ \lim_{n\to\infty} \lambda_n = \lambda\big\} \tag{1.10.1}$$

and

$$\begin{aligned} \nu &:= \min\big\{\inf \Lambda_+,\ \inf \Lambda_0,\ \max\{a_-, d_-\}\big\},\\ \mu &:= \max\big\{\sup \Lambda_-,\ \sup \Lambda_0,\ \min\{a_+, d_+\}\big\}. \end{aligned} \tag{1.10.2}$$

Then the interval $[\nu, \mu]$ satisfies the inclusions

i) $\big[\max\{a_-, d_-\}, \min\{a_+, d_+\}\big] \subset [\nu, \mu] \subset \left[\dfrac{a_- + d_-}{2}, \dfrac{a_+ + d_+}{2}\right]$,

ii) $\Lambda_- \cap \Lambda_+ \ \subset\ [\inf \Lambda_+, \sup \Lambda_-] \ \subset\ [\nu, \mu]$.

Proof. The left inclusion in i) and the right inclusion in ii) are immediate from the definition of ν and μ. The left inclusion in ii) is obvious from the inequalities $\inf \Lambda_+ \geq \inf \Lambda_-$, $\sup \Lambda_+ \geq \sup \Lambda_-$. For the right inclusion in i), we observe that *e.g.* $\min\{a_+, d_+\} \leq (a_+ + d_+)/2$, $\lambda_-(x, y) \leq (a_+ + d_+)/2$ for all $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$, $x, y \neq 0$, by definition of λ_-, and $\operatorname{Re}\big(W^2(\mathcal{A}) \setminus \mathbb{R}\big) \subset \big[(a_- + d_-)/2, (a_+ + d_+)/2\big]$ by Proposition 1.3.9 ii). □

Remark 1.10.5 The interval $[\nu, \mu]$ was defined differently in [LLMT05, (2.4)], not taking into account the accumulation of the non-real points of $W^2(\mathcal{A})$. It was assumed, but not proved there, that non-real points of $W^2(\mathcal{A})$ can accumulate at the real axis only within $\Lambda_- \cap \Lambda_+$. This question is still open and hence the definition of ν, μ had to be modified here to ensure the completeness of the proof of [LLMT05, Theorem 3.1] (presented as Theorem 1.10.9 below).

In the following we introduce the notions of spectral points of definite type of $\mathcal{J}$-self-adjoint operators and of $\mathcal{J}$-nonnegative and $\mathcal{J}$-nonpositive subspaces (see [Lan82, Sections II.4, I.1]).

Definition 1.10.6 Let $\mathcal{A} \in L(\mathcal{H})$ be $\mathcal{J}$-self-adjoint and let $[\cdot,\cdot] := (\mathcal{J}\cdot,\cdot)$.

i) An eigenvector $\mathbf{x_0}$ at an eigenvalue λ_0 of $\mathcal{A}$ is called of *$\mathcal{J}$-positive type* if

$$[\mathbf{x_0}, \mathbf{x_0}] > 0;$$

an eigenvalue λ_0 of $\mathcal{A}$ is called of *$\mathcal{J}$-positive type* if all corresponding eigenvectors are of $\mathcal{J}$-positive type.

iii) A spectral point $\lambda_0 \in \sigma_{\rm app}(\mathcal{A}) \cap \mathbb{R}$ is called of *$\mathcal{J}$-positive type* if

$$\|\mathbf{x_n}\| = 1,\ \lim_{n\to\infty} \|(\mathcal{A} - \lambda_0)\mathbf{x_n}\| = 0 \implies \liminf_{n\to\infty}\, [\mathbf{x_n}, \mathbf{x_n}] > 0.$$

The definition of eigenvectors, eigenvalues, and spectral points of $\mathcal{J}$-*negative type* is analogous.

Definition 1.10.7 A subspace $\mathcal{L} \subset \mathcal{H}$ is called $\mathcal{J}$*-nonnegative* ($\mathcal{J}$*-positive, uniformly* $\mathcal{J}$*-positive*, respectively) if $[x, x] \geq 0$ for all $x \in \mathcal{L}$, $x \neq 0$ (> 0 or $\geq \gamma \|x\|^2$ with some $\gamma > 0$, respectively). A $\mathcal{J}$-nonnegative subspace $\mathcal{L}$ is called *maximal* $\mathcal{J}$*-nonnegative* if it is not properly contained in another $\mathcal{J}$-nonnegative subspace.

If for an interval $(\alpha, \beta) \subset \mathbb{R}$ the points of $(\alpha, \beta) \cap \sigma(\mathcal{A})$ are all of $\mathcal{J}$-positive or all of $\mathcal{J}$-negative type for $\mathcal{A}$, then $\mathcal{A}$ has a local spectral function $E_{\mathcal{A}}(\Delta)$ on (α, β) (see [LMM97, Theorem 3.1]); the spectral function is defined for all Δ belonging to the semi-ring $\mathcal{M}(\alpha, \beta)$ generated by all closed, open or semi-closed intervals whose closure is contained in (α, β).

The subspace $\mathcal{L}_\Delta(\mathcal{A}) := E_{\mathcal{A}}(\Delta)\mathcal{H}$ is called the *spectral subspace* of $\mathcal{A}$ corresponding to Δ. If $\Delta \in \mathcal{M}(\alpha, \beta)$ is so that $\Delta \cap \sigma(\mathcal{A}) \neq \emptyset$ and consists of points of $\mathcal{J}$-positive type, then the corresponding spectral subspace $\mathcal{L}_\Delta(\mathcal{A})$ is a uniformly $\mathcal{J}$-positive subspace of $\mathcal{H}$ (see [Lan82, Remark 1]).

A useful property of $\mathcal{J}$-definite subspaces is that they admit angular operator representations (see *e.g.* [Lan82], [Bog74, Theorem II.11.7]).

Remark 1.10.8 A closed subspace $\mathcal{L} \subset \mathcal{H}$ is $\mathcal{J}$-nonnegative if and only if there is a closed subspace $\mathcal{H}_1^{\mathcal{L}} \subset \mathcal{H}_1$ and a contraction $K : \mathcal{H}_1^{\mathcal{L}} \to \mathcal{H}_2$ with

$$\mathcal{L} = \left\{ \begin{pmatrix} x \\ Kx \end{pmatrix} : x \in \mathcal{H}_1^{\mathcal{L}} \right\}.$$

The subspace $\mathcal{L}$ is $\mathcal{J}$-positive if and only if K is a strict contraction (*i.e.* $\|Kx\| < \|x\|$ for all $x \in \mathcal{H}_1^{\mathcal{L}}$, $x \neq 0$), and $\mathcal{L}$ is uniformly $\mathcal{J}$-positive if and only if K is a uniform contraction on $\mathcal{H}_1^{\mathcal{L}}$ (*i.e.* $\|K\| < 1$); the subspace $\mathcal{L}$ is maximal $\mathcal{J}$-nonnegative if and only if $\mathcal{H}_1^{\mathcal{L}} = \mathcal{H}_1$.

Now we are ready to classify spectral points of $\mathcal{J}$-self-adjoint block operator matrices according to their types.

Theorem 1.10.9 *Let* $\mathcal{A}$ *be* $\mathcal{J}$*-self-adjoint and let* ν, μ *be defined as in* (1.10.2)*. Then*

i) *the spectral points of* $\mathcal{A}$ *in* (μ, ∞) *are of*

$$\begin{aligned} &\mathcal{J}\text{-positive type} \quad \text{if } \; d_+ < a_+, \\ &\mathcal{J}\text{-negative type} \quad \text{if } \; a_+ < d_+; \end{aligned}$$

ii) *the spectral points of $\mathcal{A}$ in $(-\infty,\nu)$ are of*

$$\begin{aligned} &\mathcal{J}\text{-negative type} \quad \text{if } d_- < a_-,\\ &\mathcal{J}\text{-positive type} \quad \text{if } a_- < d_-; \end{aligned}$$

iii) *$\mathcal{A}$ has a local spectral function $E_{\mathcal{A}}$ on $(-\infty,\nu)$ and (μ,∞).*

Remark 1.10.10 From Proposition 1.10.3 i) it follows that

$$\begin{aligned} a_+ = d_+ &\implies \sigma(\mathcal{A}) \cap (\mu,\infty) = \emptyset,\\ a_- = d_- &\implies \sigma(\mathcal{A}) \cap (-\infty,\nu) = \emptyset. \end{aligned}$$

Proof. i), ii) We restrict ourselves to the interval (μ,∞) and to the case $d_+ < a_+$; all other cases are analogous.

Let $\lambda \in \sigma(\mathcal{A}) \cap (\mu,\infty)$. Then there exists a sequence $(\mathbf{x_n})_1^\infty \subset \mathcal{H}$, $\mathbf{x_n} = (x_n\ y_n)^{\mathrm{t}}$, such that $\|\mathbf{x_n}\|^2 = \|x_n\|^2 + \|y_n\|^2 = 1$ and

$$\mathbf{u_n} := (\mathcal{A}-\lambda)\mathbf{x_n} \longrightarrow 0, \quad n \to \infty. \tag{1.10.3}$$

Without loss of generality, we assume that $\lim_{n\to\infty} \|x_n\|$ exists. In order to prove that λ is of $\mathcal{J}$-positive type, it suffices to show that $\lim_{n\to\infty} \|x_n\|^2 > 1/2$ since $[\mathbf{x_n},\mathbf{x_n}] = \|x_n\|^2 - \|y_n\|^2 = 2\|x_n\|^2 - 1$. Evidently, if $\lim_{n\to\infty} \|y_n\| = 0$, the claim is trivial. Otherwise, we let $\mathbf{u_n} = (u_n\ v_n)^{\mathrm{t}}$, take inner products of the rows in (1.10.3) with x_n and y_n, respectively, and arrive at

$$(Ax_n,x_n) - \lambda\|x_n\|^2 + (By_n,x_n) \;=\; (u_n,x_n), \tag{1.10.4}$$

$$-(B^*x_n,y_n) + (Dy_n,y_n) - \lambda\|y_n\|^2 \;=\; (v_n,y_n). \tag{1.10.5}$$

If $\lim_{n\to\infty} \|x_n\| = 0$, then (1.10.5) would imply $(Dy_n,y_n)/\|y_n\|^2 \to \lambda$, $n\to\infty$, a contradiction to $\lambda > d_+$. So, in the following, we may assume without loss of generality that $x_n, y_n \neq 0$ and $0 < \lim_{n\to\infty} \|x_n\| < 1$. If we set

$$a_n := \frac{(Ax_n,x_n)}{\|x_n\|^2}, \quad b_n := \frac{(By_n,x_n)}{\|x_n\|\,\|y_n\|}, \quad d_n := \frac{(Dy_n,y_n)}{\|y_n\|^2},$$

then (1.10.4), (1.10.5), and (1.10.3) imply that

$$\big(\mathcal{A}_{x_n,y_n} - \lambda\big)\begin{pmatrix}\|x_n\|\\ \|y_n\|\end{pmatrix} = \begin{pmatrix} a_n-\lambda & b_n\\ -\overline{b_n} & d_n-\lambda\end{pmatrix}\begin{pmatrix}\|x_n\|\\ \|y_n\|\end{pmatrix} = \begin{pmatrix}\frac{(u_n,x_n)}{\|x_n\|}\\ \frac{(v_n,y_n)}{\|y_n\|}\end{pmatrix} \longrightarrow 0, \; n\to\infty.$$

Therefore $\operatorname{dist}\big(\lambda, \sigma(\mathcal{A}_{x_n,y_n})\big) \to 0$, $n \to \infty$, by Lemma 1.3.2. Because $\lambda > \mu \geq \max\{\sup\Lambda_-, \sup\Lambda_0\}$, neither points of Λ_- nor non-real points of $W^2(\mathcal{A})$ can accumulate at λ; hence $\lim_{n\to\infty} \lambda_+(x_n,y_n) = \lambda$.

Adding (1.10.4) and the complex conjugate of (1.10.5), we see that

$$a_n\|x_n\|^2 + d_n(1-\|x_n\|^2) - \lambda = (u_n, x_n) + (y_n, v_n) =: \varepsilon_n \longrightarrow 0, \quad n \to \infty.$$

Since $\lambda > \mu \geq d_+ \geq d_n$, there exists an $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$,

$$0 < \frac{\lambda - d_n + \varepsilon_n}{\|x_n\|^2} = a_n - d_n \leq a_+ - d_-, \quad \left|\lambda - \lambda_+\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right| < \frac{\lambda - \mu}{4}. \quad (1.10.6)$$

Then we have, for all $n \geq n_0$,

$$\begin{aligned}\Big(\|x_n\|^2 - \frac{1}{2}\Big)\Big(a_+ - d_-\Big) \geq \Big(\|x_n\|^2 - \frac{1}{2}\Big)\Big(a_n - d_n\Big) &= \lambda - \frac{a_n + d_n}{2} + \varepsilon_n \\ &= \lambda - \frac{1}{2}\left(\lambda_+\begin{pmatrix} x_n \\ y_n \end{pmatrix} + \lambda_-\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right) + \varepsilon_n \\ &\geq \frac{\lambda - \mu}{2} - \frac{1}{2}\left(\lambda - \lambda_+\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right) \\ &\geq \frac{\lambda - \mu}{4} > 0.\end{aligned}$$

Since $a_+ > d_+ \geq d_-$, we obtain the desired inequality $\lim_{n\to\infty} \|x_n\|^2 > 1/2$.

iii) The existence of the local spectral function on $(-\infty, \nu)$ and on (μ, ∞) follows from ii) (see [LMM97, Theorem 3.1 and Lemma 1.4]). □

Corollary 1.10.11 *Let Δ be an interval with $\overline{\Delta} > \mu$ and let $\mathcal{L}_\Delta(\mathcal{A}) = E_\mathcal{A}(\Delta)\mathcal{H}$. If $d_+ < a_+$, then there exist a subspace $\mathcal{H}_1^\Delta \subset \mathcal{H}_1$ and a strict contraction $K_1^\Delta \in L\big(\mathcal{H}_1^\Delta, \mathcal{H}_2\big)$ such that*

$$\mathcal{L}_\Delta(\mathcal{A}) = \left\{\begin{pmatrix} x \\ K_1^\Delta x \end{pmatrix} : x \in \mathcal{H}_1^\Delta\right\};$$

if $a_+ < d_+$, then there exist a subspace $\mathcal{H}_2^\Delta \subset \mathcal{H}_2$ and a strict contraction $K_2^\Delta \in L\big(\mathcal{H}_2^\Delta, \mathcal{H}_1\big)$ such that

$$\mathcal{L}_\Delta(\mathcal{A}) = \left\{\begin{pmatrix} K_2^\Delta y \\ y \end{pmatrix} : y \in \mathcal{H}_2^\Delta\right\}.$$

Proof. Theorem 1.10.9 shows that $\mathcal{L}_\Delta(\mathcal{A}) = E_\mathcal{A}(\Delta)\mathcal{H}$ is uniformly positive if $d_+ < a_+$ and $\Delta \cap \sigma(\mathcal{A}) \neq \emptyset$, and uniformly negative if $a_+ < d_+$ and $\Delta \cap \sigma(\mathcal{A}) \neq \emptyset$. Now both claims follow from Remark 1.10.8. □

As in the self-adjoint case (see Theorem 1.8.7), we call the subspaces $\mathcal{H}_1^\Delta$ and $\mathcal{H}_2^\Delta$ Δ-spectral supporting subspaces of $\mathcal{A}$ in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively; for the following description in terms of the Schur complements, we restrict ourselves to the case $d_+ < a_+$.

Theorem 1.10.12 *Suppose that $\mathcal{A}$ is $\mathcal{J}$-self-adjoint. Let $\Delta = [\alpha, \beta] > \mu$ and let Γ_Δ be a simply closed Jordan curve that surrounds Δ, but no point of $\sigma(\mathcal{A}) \setminus \Delta$, and intersects $\mathbb{R}$ orthogonally in α and β. Define*

$$Q_\Delta := -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\Delta}' S_1(z)^{-1}\,\mathrm{d}z,$$

where $\int'$ *denotes the Cauchy principal value at* $\mathbb{R}$*. Then the range of* Q_Δ *is given by* $R\big(Q_\Delta\big) = \mathcal{H}_1^\Delta$*.*

Proof. The proof follows the lines of the proof of Theorem 1.8.7 if we use the local spectral function $E_\mathcal{A}$ of $\mathcal{A}$ according to Theorem 1.10.9, introduce

$$\widehat{E}_\mathcal{A}(\Delta) := \frac{1}{2}\Big(E_\mathcal{A}(\Delta) + E_\mathcal{A}(\Delta^\circ)\Big) = -\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_\Delta}' (\mathcal{A}-z)^{-1}\,\mathrm{d}z$$

instead of $\widehat{P}_\Delta(\mathcal{A})$ and use the symmetries *e.g.* of $E_\mathcal{A}(\Delta)$ with respect to the indefinite inner product $[\cdot,\cdot]$ (see [LLMT05, Theorem 3.3]). □

The next theorem is the $\mathcal{J}$-self-adjoint analogue of Theorem 1.8.8.

Theorem 1.10.13 *Let* $\mathcal{A}$ *be* $\mathcal{J}$*-self-adjoint with* $d_+ < a_+$ *and let* $\Delta = [\alpha,\beta] > \mu$ *be an interval such that* $\alpha,\beta \in \rho(\mathcal{A})$*. Then*

$$\mathcal{H}_1 = \mathcal{L}_{(-\infty,0)}\big(S_1(\alpha)\big) \dotplus \mathcal{H}_1^\Delta \dotplus \mathcal{L}_{(0,\infty)}\big(S_1(\beta)\big).$$

Proof. The proof is similar to the proof of Theorem 1.8.8 if we observe that, although the Schur complement need not be decreasing in the $\mathcal{J}$-self-adjoint case, the following weaker statement can be proved: If $\mu < \lambda_1 < \lambda_2$ and $\big(S_1(\lambda_2)x,x\big) \geq 0$ for some $x \in \mathcal{H}_1$, $x \neq 0$, then $\big(S_1(\lambda_1)x,x\big) > 0$. For more details we refer to [LLMT05, Section 4]. □

We conclude this section by investigating the corners of the zones Λ_- and Λ_+ of the quadratic numerical range. If $a_- \neq d_-$ and $a_+ \neq d_+$, respectively, then the outer corners $\inf \Lambda_-$ and $\sup \Lambda_+$ are corners of $W^2(\mathcal{A})$ and hence, by Theorems 1.5.8 and 1.5.2, they belong to $\sigma(\mathcal{A})$, or even to $\sigma_\mathrm{p}(\mathcal{A})$ if $\inf \Lambda_- = \min \Lambda_-$ and $\sup \Lambda_+ = \max \Lambda_+$, respectively.

In the following theorem, we prove analogous statements for the interior corner $\sup \Lambda_-$ of Λ_-; the formulation of the analogue for $\inf \Lambda_+$ is obvious.

Theorem 1.10.14 *Let* $\mathcal{A}$ *be* $\mathcal{J}$*-self-adjoint. Suppose that* $\sup \Lambda_- > d_+$ *and that there exists a neighbourhood of* $\sup \Lambda_-$ *containing no non-real points of* $W^2(\mathcal{A})$*. Then* $\sup \Lambda_- \in \sigma(\mathcal{A})$*.*

If, in addition, $\sup \Lambda_- = \max \Lambda_-$*, then* $\sup \Lambda_- \in \sigma_\mathrm{p}(\mathcal{A})$*. In the latter case, if* $\sup \Lambda_- = \lambda_-(x_0,y_0)$ *for some* $x_0 \in \mathcal{H}_1$*,* $y_0 \in \mathcal{H}_2$*,* $x_0, y_0 \neq 0$*, then there is a* $\gamma \in \mathbb{C}$ *so that* $(x_0\ \ \gamma y_0)^\mathrm{t}$ *is an eigenvector of* $\mathcal{A}$ *corresponding to* $\sup \Lambda_-$*.*

Proof. The proof is very similar to the proofs of Theorems 1.5.2 and 1.5.8; for details we refer the reader to the proof of [LLMT05, Theorem 2.7]. □

1.11 The block numerical range

The concept of quadratic numerical range for 2×2 block operator matrices has an obvious generalization to $n \times n$ block operator matrices. For this so-called block numerical range, we prove results on spectral inclusion, estimates of the resolvent, and inclusion theorems between block numerical ranges under refinements of the decomposition of the space (see [TW03]).

Let $n \in \mathbb{N}$, let $\mathcal{H}_1, \dots, \mathcal{H}_n$ be complex Hilbert spaces, and consider $\mathcal{H} = \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n$. With respect to this decomposition, every operator $\mathcal{A} \in L(\mathcal{H})$ has an $n \times n$ block operator matrix representation

$$\mathcal{A} = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \tag{1.11.1}$$

with entries $A_{ij} \in L(\mathcal{H}_j, \mathcal{H}_i)$, $i, j = 1, \dots, n$. In the following we denote by

$\mathcal{S}_{\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n} := S_{\mathcal{H}_1} \times \dots \times S_{\mathcal{H}_n} = \big\{ (x_1 \dots x_n)^{\rm t} \in \mathcal{H} : \|x_i\| = 1,\ i = 1, 2, \dots, n \big\}$

the product of the unit spheres $S_{\mathcal{H}_i}$ in $\mathcal{H}_i$; we also write $\mathcal{S}^n$ or $\mathcal{S}_{\mathcal{H}}$ instead of $\mathcal{S}_{\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n}$ if the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n$ is clear (note the slight difference in notation between $\mathcal{S}_{\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n}$ and the unit sphere $S_{\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n}$ in $\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n$).

Definition 1.11.1 For $x = (x_1 \dots x_n)^{\rm t} \in \mathcal{S}_{\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n}$ we introduce the $n \times n$ matrix

$$\mathcal{A}_x := \begin{pmatrix} (A_{11}x_1, x_1) & \cdots & (A_{1n}x_n, x_1) \\ \vdots & & \vdots \\ (A_{n1}x_1, x_n) & \cdots & (A_{nn}x_n, x_n) \end{pmatrix} \in M_n(\mathbb{C}), \tag{1.11.2}$$

that is, $(\mathcal{A}_x)_{ij} := \big(A_{ij}x_j, x_i\big)$, $i, j = 1, \dots, n$. Then the set

$$W_{\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n}(\mathcal{A}) := \bigcup_{x \in \mathcal{S}^n} \sigma_{\rm p}(\mathcal{A}_x) \tag{1.11.3}$$

is called *block numerical range* of $\mathcal{A}$ (with respect to the block operator matrix representation (1.11.1)). For a fixed decomposition of $\mathcal{H}$, we also write

$$W^n(\mathcal{A}) = W_{\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n}(\mathcal{A}).$$

Clearly, since $\sigma_{\rm p}(\mathcal{A}_x) = \big\{ \lambda \in \mathbb{C} : \det(\mathcal{A}_x - \lambda) = 0 \big\}$ for all $x \in \mathcal{S}^n$, $W^n(\mathcal{A})$ has the equivalent representation

$$W^n(\mathcal{A}) = \big\{ \lambda \in \mathbb{C} : \exists\, x \in \mathcal{S}^n \ \det(\mathcal{A}_x - \lambda) = 0 \big\}. \tag{1.11.4}$$

Remark 1.11.2 For $n = 1$ the block numerical range is just the usual numerical range, for $n = 2$ it is the quadratic numerical range introduced in Section 1.1. For $n = 3$, the block numerical range is also called *cubic numerical range* and for $n = 4$ *quartic numerical range*. If $\mathcal{A} \in M_n(\mathbb{C})$ is an $n \times n$ matrix, then $W^n(\mathcal{A})$ coincides with the set of eigenvalues of $\mathcal{A}$.

Like the numerical range and the quadratic numerical range, the block numerical range of a bounded block operator matrix $\mathcal{A}$ is bounded,

$$W^n(\mathcal{A}) \subset \big\{\lambda \in \mathbb{C} : \ |\lambda| \le \|\mathcal{A}\|\big\},$$

and closed if $\dim \mathcal{H} < \infty$. The former follows if we let $x = (x_1 \dots x_n)^{\mathrm{t}} \in \mathcal{S}^n$, $z = (z_1 \dots z_n)^{\mathrm{t}} \in \mathbb{C}^n$, $\|z\| = 1$, set $y_j := z_j x_j$, $j = 1, \dots, n$, $y := (y_1 \dots y_n)^{\mathrm{t}}$ so that $\|y\| = 1$, and observe that

$$\|\mathcal{A}_x z\|^2 = \sum_{i=1}^{n} \Big| \sum_{j=1}^{n} (A_{ij} x_j, x_i) z_j \Big|^2 \le \sum_{i=1}^{n} \Big|\Big| \sum_{j=1}^{n} A_{ij} y_j \Big|\Big|^2 ||x_i||^2 = \|\mathcal{A}y\|^2 \le \|\mathcal{A}\|^2.$$

If $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ is decomposed into n components, then the corresponding block numerical range consists of at most n (connected) components; as in the case $n = 2$, this follows from the fact that the set of all matrices $\mathcal{A}_x$, $x \in \mathcal{S}^n$, is connected and from a continuity argument for the eigenvalues of matrices (see [Kat95, Theorem II.5.14] and [Wag00]). If, for example, $\mathcal{A}$ is upper or lower block triangular, then

$$W^n(\mathcal{A}) = W(A_{11}) \cup \cdots \cup W(A_{nn}).$$

This shows that, like the quadratic numerical range, $W^n(\mathcal{A})$ need not be convex; the next example shows that its components need not be so either.

Example 1.11.3 Consider the 4×4 matrix

$$\mathcal{A}_7 = \left(\begin{array}{cc|c|c} 2 & 0 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ \hline \mathrm{i} & \mathrm{i} & 1 & 0 \\ \hline \mathrm{i} & \mathrm{i} & 0 & -1 \end{array}\right)$$

with respect to $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C} \oplus \mathbb{C}$. The corresponding cubic numerical range has 3 components and none of them is convex (see Fig. 1.8).

Proposition 1.11.4 *If $\mathcal{A}^*$ denotes the adjoint of $\mathcal{A}$, then*

i) $W^n(\mathcal{A}^*) = \big\{\overline{\lambda} \in \mathbb{C} : \lambda \in W^n(\mathcal{A})\big\} =: W^n(\mathcal{A})^*$.
ii) $\mathcal{A} = \mathcal{A}^* \implies W^n(\mathcal{A}) \subset \mathbb{R}$.

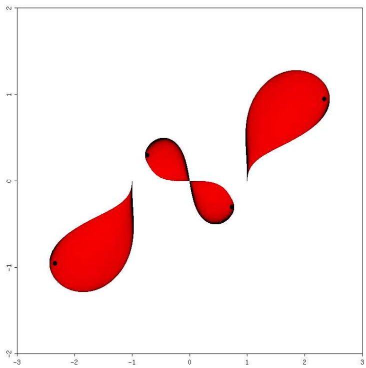

Figure 1.8 Cubic numerical range $W^3(\mathcal{A}_7) = W_{\mathbb{C}^2\oplus\mathbb{C}\oplus\mathbb{C}}(\mathcal{A}_7)$ of $\mathcal{A}_7$.

Proof. For assertion i) we observe that $(\mathcal{A}_x)^* = (\mathcal{A}^*)_x$; assertion ii) is obvious since in this case all matrices $\mathcal{A}_x$ are symmetric. □

In Fig. 1.8 the eigenvalues of $\mathcal{A}_7$, which are marked by black dots, are obviously contained in $W^3(\mathcal{A}_7)$. In order to prove a general spectral inclusion theorem, we need the following generalization of Lemma 1.3.2.

Lemma 1.11.5 *Let $\mathcal{M} \in M_n(\mathbb{C})$. If $\mathcal{M}$ is invertible, then*

$$\|\mathcal{M}^{-1}\| \leq \frac{\|\mathcal{M}\|^{n-1}}{|\det \mathcal{M}|}. \tag{1.11.5}$$

For all $x \in \mathbb{C}^n$, $\|x\| = 1$, we have

$$\operatorname{dist}\big(0, \sigma(\mathcal{M})\big) \leq \sqrt[n]{\|\mathcal{M}\|^{n-1}\|\mathcal{M}x\|}.$$

Proof. The first estimate was proved in [Kat60, Lemma 1] (see also [Kat95, Section I.4.2, (4.12)] and note that $\mathbb{C}^n$ is a unitary space). The second statement is trivial if $\mathcal{M}$ is not invertible. If $\mathcal{M}$ is invertible and $x \in \mathbb{C}^n$, $\|x\| = 1$, then $\|\mathcal{M}x\| \geq \|\mathcal{M}^{-1}\|^{-1} > 0$. Denoting by $\lambda_1, \dots, \lambda_n$ the eigenvalues of $\mathcal{M}$ and using (1.11.5), we obtain

$$\begin{aligned}\big(\operatorname{dist}\big(0, \sigma(\mathcal{M})\big)\big)^n &= \Big(\min_{i=1}^{n} |\lambda_i|\Big)^n \leq |\lambda_1 \cdots \lambda_n| = |\det \mathcal{M}| \\ &\leq \frac{\|\mathcal{M}\|^{n-1}}{\|\mathcal{M}^{-1}\|} \leq \|\mathcal{M}\|^{n-1}\|\mathcal{M}x\|. \end{aligned}$$ □

The next theorem generalizes the spectral inclusion property of the numerical range ($n = 1$, see (1.1.1)) and of the quadratic numerical range ($n = 2$, see Theorem 1.3.1).

Theorem 1.11.6 $\sigma_{\mathrm{p}}(\mathcal{A}) \subset W^n(\mathcal{A}),\ \sigma(\mathcal{A}) \subset \overline{W^n(\mathcal{A})}$.

Proof. First let $\lambda \in \sigma_{\mathrm{p}}(\mathcal{A})$. Then there exists $x = (x_1 \dots x_n)^{\mathrm{t}} \in \mathcal{H}$, $x \neq 0$, such that $\mathcal{A}x - \lambda x = 0$. If we write $x_i = \|x_i\|\, \widehat{x}_i$ with $\widehat{x}_i \in \mathcal{H}_i$, $\|\widehat{x}_i\| = 1$, $i = 1, \dots, n$, then $\widehat{x} := (\widehat{x}_1 \dots \widehat{x}_n)^{\mathrm{t}} \in \mathcal{S}^n$, $(x_i, \widehat{x}_i) = \|x_i\|$, $i = 1, \dots, n$, and

$$\big(\mathcal{A}_{\widehat{x}} - \lambda\big)\begin{pmatrix} \|x_1\| \\ \vdots \\ \|x_n\| \end{pmatrix} = \begin{pmatrix} (A_{11}\widehat{x}_1, \widehat{x}_1) - \lambda & \cdots & (A_{1n}\widehat{x}_n, \widehat{x}_1) \\ \vdots & & \vdots \\ (A_{n1}\widehat{x}_1, \widehat{x}_n) & \cdots & (A_{nn}\widehat{x}_n, \widehat{x}_n) - \lambda \end{pmatrix}\begin{pmatrix} \|x_1\| \\ \vdots \\ \|x_n\| \end{pmatrix}$$

$$= \begin{pmatrix} (A_{11}x_1, \widehat{x}_1) + \dots + (A_{1n}x_n, \widehat{x}_1) - \lambda(x_1, \widehat{x}_1) \\ \vdots \\ (A_{n1}x_1, \widehat{x}_n) + \dots + (A_{nn}x_n, \widehat{x}_n) - \lambda(x_n, \widehat{x}_n) \end{pmatrix}$$

$$= \begin{pmatrix} \Big(\sum_{j=1}^n A_{1j}x_j - \lambda x_1, \widehat{x}_1\Big) \\ \vdots \\ \Big(\sum_{j=1}^n A_{nj}x_j - \lambda x_n, \widehat{x}_n\Big) \end{pmatrix} = \big(\mathcal{A}x - \lambda x, \widehat{x}\big) = 0.$$

Hence $\lambda \in \sigma_{\mathrm{p}}(\mathcal{A}_{\widehat{x}}) \subset W^n(\mathcal{A})$ by definition (1.11.3).

Now let $\lambda \in \sigma(\mathcal{A})$. Then we either have $\lambda \in \sigma_{\mathrm{p}}(\mathcal{A}^*)^*$ or $\lambda \in \sigma_{\mathrm{app}}(\mathcal{A})$ (the approximate point spectrum of $\mathcal{A}$, see (1.3.4)). If $\lambda \in \sigma_{\mathrm{p}}(\mathcal{A}^*)^*$, then the inclusion already proved and Proposition 1.11.4 i) yield $\overline{\lambda} \in \sigma_{\mathrm{p}}(\mathcal{A}^*) \subset W^n(\mathcal{A}^*) = W^n(\mathcal{A})^*$ and hence $\lambda \in W^n(\mathcal{A})$. If $\lambda \in \sigma_{\mathrm{app}}(\mathcal{A})$, then there is a sequence $\big(x^{(\nu)}\big)_1^\infty \subset \mathcal{H}$, $\|x^{(\nu)}\| = 1$, so that $\mathcal{A}x^{(\nu)} - \lambda x^{(\nu)} \to 0$, $\nu \to \infty$. If we write $x_i^{(\nu)} = \|x_i^{(\nu)}\|\, \widehat{x}_i^{(\nu)}$ with $\widehat{x}_i^{(\nu)} \in \mathcal{H}_i$, $\|\widehat{x}_i^{(\nu)}\| = 1$, $i = 1, \dots, n$, $\nu = 1, 2, \dots$, then $\widehat{x}^{(\nu)} := \big(\widehat{x}_1^{(\nu)} \dots \widehat{x}_n^{(\nu)}\big)^{\mathrm{t}} \in \mathcal{S}^n$ and, in a similar way as above, we obtain

$$\big(\mathcal{A}_{\widehat{x}^{(\nu)}} - \lambda\big)\begin{pmatrix} \|x_1^{(\nu)}\| \\ \vdots \\ \|x_n^{(\nu)}\| \end{pmatrix} = \Big(\mathcal{A}x^{(\nu)} - \lambda x^{(\nu)}, \widehat{x}^{(\nu)}\Big) \longrightarrow 0, \quad \nu \to \infty;$$

thus we have $\varepsilon_\nu := \Big\|\big(\mathcal{A}_{\widehat{x}^{(\nu)}} - \lambda\big)\big(\|x_1^{(\nu)}\| \dots \|x_n^{(\nu)}\|\big)^{\mathrm{t}}\Big\| \to 0$, $\nu \to \infty$. Since $\Big\|\big(\|x_1^{(\nu)}\| \dots \|x_n^{(\nu)}\|\big)^{\mathrm{t}}\Big\| = \big\|x^{(\nu)}\big\| = 1$, Lemma 1.11.5 implies that

$$\begin{aligned} \operatorname{dist}\big(\lambda, \sigma\big(\mathcal{A}_{x^{(\nu)}}\big)\big) &= \operatorname{dist}\big(0, \sigma\big(\mathcal{A}_{x^{(\nu)}} - \lambda\big)\big) \leq \sqrt[n]{\|\mathcal{A}_{x^{(\nu)}} - \lambda\|^{n-1}\varepsilon_\nu} \\ &\leq \sqrt[n]{\big(\|\mathcal{A}\| + |\lambda|\big)^{n-1}\varepsilon_\nu} \longrightarrow 0, \quad \nu \to \infty, \end{aligned}$$

and therefore

$$\lambda \in \overline{\bigcup_{\nu \in \mathbb{N}} \sigma\big(\mathcal{A}_{x^{(\nu)}}\big)} \subset \overline{\bigcup_{x \in \mathcal{S}^n} \sigma(\mathcal{A}_x)} = \overline{W^n(\mathcal{A})}. \qquad \square$$

Example 1.11.7 As an illustration of Theorem 1.11.6, we consider the block operator matrices

$$\mathcal{A}_8 = \left(\begin{array}{c|cc|c} 2 & \mathrm{i} & 1 & 0 \\ \hline \mathrm{i} & 2 & 0 & 1 \\ 1 & 0 & -2 & \mathrm{i} \\ \hline 0 & 1 & \mathrm{i} & -2 \end{array}\right), \quad \mathcal{A}_9 = \left(\begin{array}{c|c|cc} -2 & -1 & 1 & 0 \\ \hline -1 & -2 & 0 & 1 \\ \hline -2 & -1 & 0 & -3\mathrm{i} \\ -1 & -2 & 3\mathrm{i} & 0 \end{array}\right).$$

Figure 1.9 shows their eigenvalues marked by black dots and their cubic numerical ranges. Note that the horizontal line in the right picture is part of the cubic numerical range.

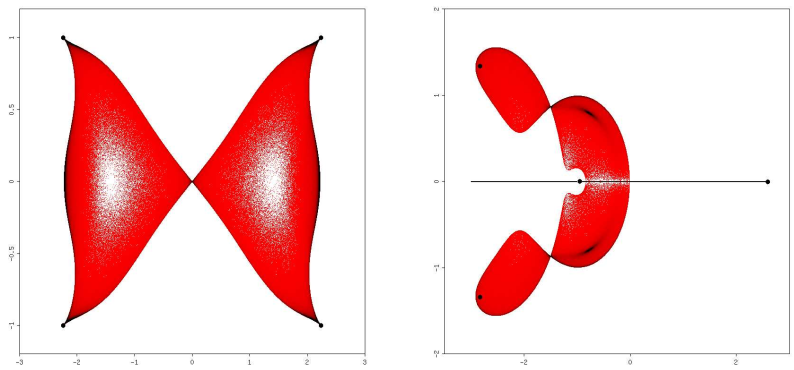

Figure 1.9 Cubic numerical ranges $W_{\mathbb{C}\oplus\mathbb{C}^2\oplus\mathbb{C}}(\mathcal{A}_8)$, $W_{\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}^2}(\mathcal{A}_9)$, and eigenvalues.

The next theorem generalizes Theorem 1.1.9; it shows that the block numerical range of a principal minor of an $n \times n$ block operator matrix $\mathcal{A}$ is contained in $W^n(\mathcal{A})$ if a certain dimension condition holds.

Theorem 1.11.8 *Let $k \in \mathbb{N}$, $1 \le k \le n$, $1 \le i_1 < \cdots < i_k \le n$, and denote by $P : \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n \to \mathcal{H}_{i_1} \oplus \cdots \oplus \mathcal{H}_{i_k}$ the projection onto the components $i_1, \dots, i_k$ of $\mathcal{H}$.*
If there exists an enumeration $i'_1, \dots, i'_{n-k}$ of the elements of the set $\{1, \dots, n\} \setminus \{i_1, \dots, i_k\}$ with $\dim \mathcal{H}_{i'_j} \ge n-(j-1)$, $j = 1, \dots, n-k$, then

$$W_{\mathcal{H}_{i_1}\oplus\cdots\oplus\mathcal{H}_{i_k}}(P\mathcal{A}P) \subset W_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_n}(\mathcal{A}).$$

Proof. For $k = n$, the statement is trivial. For $k = n-1$ there is an $i \in \{1, \dots, n\}$ such that $\{i_1, \dots, i_k\} \cup \{i\} = \{1, \dots, n\}$. If we denote $\mathcal{H}'_i := \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{i-1} \oplus \mathcal{H}_{i+1} \oplus \cdots \oplus \mathcal{H}_n$ and $\mathcal{A}'_i := P\mathcal{A}P$, then

$$\mathcal{A}_i' = \begin{pmatrix} A_{11} & \cdots & A_{1,i-1} & A_{1,i+1} & \cdots & A_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{i-1,1} & \cdots & A_{i-1,i-1} & A_{i-1,i+1} & \cdots & A_{i-1,n} \\ A_{i+1,1} & \cdots & A_{i+1,i-1} & A_{i+1,i+1} & \cdots & A_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{n1} & \cdots & A_{n,i-1} & A_{n,i+1} & \cdots & A_{nn} \end{pmatrix}.$$

Now let $\lambda \in W_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_{i-1}\oplus\mathcal{H}_{i+1}\oplus\cdots\oplus\mathcal{H}_n}(\mathcal{A}_i')$. Then there exists an element $x' = (x_1 \dots x_{i-1}\, x_{i+1} \dots x_n)^{\mathrm{t}} \in \mathcal{S}_{\mathcal{H}_i'}$ with $\det\big((\mathcal{A}_i')_{x'} - \lambda\big) = 0$. Since

$$\dim \operatorname{span}\big\{A_{i1}x_1, \dots, A_{i,i-1}x_{i-1}, A_{i,i+1}x_{i+1}, \dots, A_{in}x_n\big\} \le n-1 < \dim \mathcal{H}_i$$

by assumption, there is an $x_i \in \mathcal{H}_i$, $\|x_i\| = 1$, with $(\mathcal{A}_x)_{ij} = (A_{ij}x_j, x_i) = 0$ for $j = 1, \dots, i-1, i+1, \dots, n$. Then we have $x := (x_1 \dots x_n)^{\mathrm{t}} \in \mathcal{S}_{\mathcal{H}}$ and

$$\mathcal{A}_x = \begin{pmatrix} (\mathcal{A}_x)_{11} & \cdots & (\mathcal{A}_x)_{1,i-1} & (\mathcal{A}_x)_{1i} & (\mathcal{A}_x)_{1,i+1} & \cdots & (\mathcal{A}_x)_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \\ (\mathcal{A}_x)_{i-1,1} & \cdots & (\mathcal{A}_x)_{i-1,i-1} & (\mathcal{A}_x)_{i-1,i} & (\mathcal{A}_x)_{i-1,i+1} & \cdots & (\mathcal{A}_x)_{i-1,n} \\ 0 & \cdots & 0 & (\mathcal{A}_x)_{ii} & 0 & \cdots & 0 \\ (\mathcal{A}_x)_{i+1,1} & \cdots & (\mathcal{A}_x)_{i+1,i-1} & (\mathcal{A}_x)_{i+1,i} & (\mathcal{A}_x)_{i+1,i+1} & \cdots & (\mathcal{A}_x)_{i+1,n} \\ \vdots & & \vdots & \vdots & \dots & & \\ (\mathcal{A}_x)_{n1} & \cdots & (\mathcal{A}_x)_{n,i-1} & (\mathcal{A}_x)_{ni} & (\mathcal{A}_x)_{n,i+1} & \cdots & (\mathcal{A}_x)_{nn} \end{pmatrix}.$$

Thus $\det(\mathcal{A}_x - \lambda) = \big((\mathcal{A}_x)_{ii} - \lambda\big) \det\big((\mathcal{A}_i')_{x'} - \lambda\big) = 0$ and, consequently, $\lambda \in W_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_n}(\mathcal{A})$. The case $k < n-1$ follows by induction. □

The particular case $k = 1$ of Theorem 1.11.8 shows that, under a certain dimension condition, the numerical ranges of the diagonal entries A_{ii} of $\mathcal{A}$ are contained in the block numerical range of $\mathcal{A}$ (compare Theorem 1.1.9 and Corollary 1.1.10 for the case $n = 2$).

Corollary 1.11.9 *Let $i_0 \in \mathbb{N}$. If there exists an enumeration $i_1', \dots, i_{n-1}'$ of $\{1, \dots, i_0-1, i_0+1, \dots, n\}$ with $\dim \mathcal{H}_{i_j'} \ge n-(j-1)$, $j = 1, \dots, n-1$, then*

$$W(A_{i_0 i_0}) \subset W^n(\mathcal{A});$$

in particular, if $\dim \mathcal{H}_i \ge n$ for $i = 1, \dots, n$, then

$$W(A_{ii}) \subset W^n(\mathcal{A}), \quad i = 1, \dots, n.$$

Corollary 1.11.10 *Suppose that $\dim \mathcal{H}_i \ge n$, $i = 1, \dots, n$, and that $W^n(\mathcal{A}) = \mathcal{F}_1 \dot\cup \cdots \dot\cup \mathcal{F}_n$ consists of n disjoint components. Then there exists a permutation π of $\{1, \dots, n\}$ such that $W(A_{ii}) \subset \mathcal{F}_{\pi(i)}$, $i = 1, \dots, n$.*

Proof. Let $x_1 \in S_{\mathcal{H}_1}$ be arbitrary. Then, by the dimension condition, we can recursively choose $x_k \in S_{\mathcal{H}_k}$, $k = 2, \dots, n$, in such a way that $x_k \perp \{A_{k1}x_1, \dots, A_{k,k-1}x_{k-1}\}$. Since $W^n(\mathcal{A}) = \mathcal{F}_1 \dot\cup \cdots \dot\cup \mathcal{F}_n$, every matrix $\mathcal{A}_x$, $x \in \mathcal{S}_{\mathcal{H}}$, has exactly one eigenvalue in each component of $W^n(\mathcal{A})$. In particular, if we let $x := (x_1 \dots x_n)^{\mathrm{t}} \in \mathcal{S}_{\mathcal{H}}$, then

$$\mathcal{A}_x = \begin{pmatrix} (A_{11}x_1, x_1) & \cdots & (A_{1n}x_n, x_1) \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (A_{nn}x_n, x_n) \end{pmatrix};$$

hence there exists a permutation π of $\{1, \dots, n\}$ with $(A_{ii}x_i, x_i) \in \mathcal{F}_{\pi(i)}$ for $i = 1, \dots, n$. By Corollary 1.11.9 we have $W(A_{ii}) \subset W^n(\mathcal{A})$ for $i = 1, \dots, n$; since $W(A_{ii})$ is convex, the assertion follows. □

The dimension condition in Theorem 1.11.8 cannot be dropped; this can be seen from the following example.

Example 1.11.11 We reconsider the matrix $\mathcal{A}_5$ from Example 1.5.5, now with the 3×3 block decomposition in $\mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}$, and its principal minor $\mathcal{A}_5'$ given by

$$\mathcal{A}_5 = \left(\begin{array}{c|cc|c} 1 & 3+\mathrm{i} & 2 & \mathrm{i} \\ \hline 3+\mathrm{i} & 1 & \mathrm{i} & 2 \\ -2 & \mathrm{i} & 1 & 3+\mathrm{i} \\ \hline \mathrm{i} & -2 & 3+\mathrm{i} & 1 \end{array}\right), \quad \mathcal{A}_5' = \left(\begin{array}{cc|c} 1 & \mathrm{i} & 2 \\ \mathrm{i} & 1 & 3+\mathrm{i} \\ \hline -2 & 3+\mathrm{i} & 1 \end{array}\right).$$

Figure 1.10 shows that the quadratic numerical range of $\mathcal{A}_5'$ is not contained in the cubic numerical range of $\mathcal{A}_5$; here $n=3$, $k=2$, $i_1=2$, $i_2=3$, $i_1'=1$ and so the dimension condition $\dim \mathcal{H}_1 \geq 3$ of Theorem 1.11.8 is violated.

Next we consider the behaviour of the block numerical range under refinements of the decomposition $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ of $\mathcal{H}$. Theorem 1.11.13 below is a generalization of the fact that the quadratic numerical range is contained in the numerical range (see Theorem 1.1.8).

Definition 1.11.12 Let $n, \widetilde{n} \in \mathbb{N}$ and $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n = \widetilde{\mathcal{H}}_1 \oplus \cdots \oplus \widetilde{\mathcal{H}}_{\widetilde{n}}$ with Hilbert spaces $\mathcal{H}_1, \dots, \mathcal{H}_n$ and $\widetilde{\mathcal{H}}_1, \dots, \widetilde{\mathcal{H}}_{\widetilde{n}}$. Then $\widetilde{\mathcal{H}}_1 \oplus \cdots \oplus \widetilde{\mathcal{H}}_{\widetilde{n}}$ is called a *refinement* of $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ if $n \leq \widetilde{n}$ and there exist integers $0 = i_0 < \cdots < i_n = \widetilde{n}$ with $\mathcal{H}_k = \widetilde{\mathcal{H}}_{i_{k-1}+1} \oplus \cdots \oplus \widetilde{\mathcal{H}}_{i_k}$ for all $k = 1, \dots, n$.

Theorem 1.11.13 *If $\widetilde{\mathcal{H}}_1 \oplus \cdots \oplus \widetilde{\mathcal{H}}_{\widetilde{n}}$ is a refinement of $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$, then*

$$W_{\widetilde{\mathcal{H}}_1 \oplus \cdots \oplus \widetilde{\mathcal{H}}_{\widetilde{n}}}(\mathcal{A}) \subset W_{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n}(\mathcal{A}),$$

or, briefly,

$$W^{\widetilde{n}}(\mathcal{A}) \subset W^n(\mathcal{A}), \quad \widetilde{n} \geq n.$$

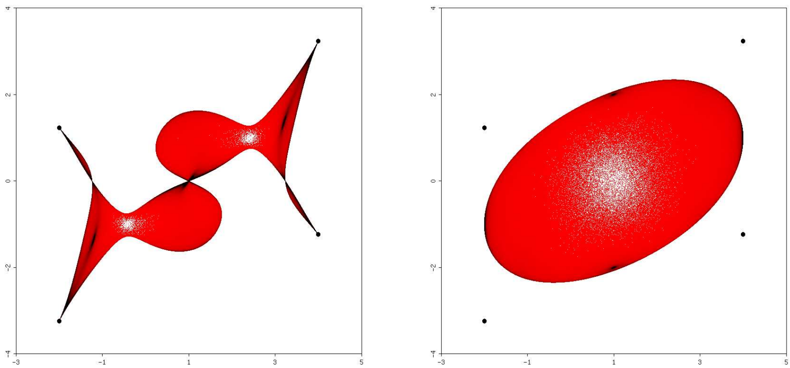

Figure 1.10 $W_{\mathbb{C}\oplus\mathbb{C}^2\oplus\mathbb{C}}(\mathcal{A}_5)$ and $W_{\mathbb{C}^2\oplus\mathbb{C}}(\mathcal{A}_5')$

Proof. It is sufficient to consider the case $\widetilde{n} = n+1$; the general case easily follows by induction. If $\widetilde{n} = n+1$, there exists a $k \in \{1,\dots,n\}$ such that the refinement $\widetilde{\mathcal{H}}_1 \oplus \cdots \oplus \widetilde{\mathcal{H}}_{\widetilde{n}}$ of $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ is of the form $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{k-1} \oplus \mathcal{H}_k^1 \oplus \mathcal{H}_k^2 \oplus \mathcal{H}_{k+1} \oplus \cdots \oplus \mathcal{H}_n$ where $\mathcal{H}_k = \mathcal{H}_k^1 \oplus \mathcal{H}_k^2$. With respect to this refined decomposition, $\mathcal{A}$ has the representation

$$\mathcal{A} = \begin{pmatrix} A_{11} & \cdots & A_{1k}^1 & A_{1k}^2 & \cdots & A_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{k1}^1 & \cdots & A_{kk}^{11} & A_{kk}^{12} & \cdots & A_{kn}^1 \\ A_{k1}^2 & \cdots & A_{kk}^{21} & A_{kk}^{22} & \cdots & A_{kn}^2 \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{n1} & \cdots & A_{nk}^1 & A_{nk}^2 & \cdots & A_{nn} \end{pmatrix}$$

with $A_{kk}^{st} \in L(\mathcal{H}_k^t, \mathcal{H}_k^s)$, $A_{ki}^t \in L(\mathcal{H}_i, \mathcal{H}_k^t)$, $A_{jk}^s \in L(\mathcal{H}_k^s, \mathcal{H}_j)$, $k,i,j = 1,\dots,n$, $s,t = 1,2$. For the entries $A_{ij} \in L(\mathcal{H}_j, \mathcal{H}_i)$, $i,j = 1,\dots,n$, of the representation (1.11.1) of $\mathcal{A}$ with respect to $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$, we have

$$A_{kk} = \begin{pmatrix} A_{kk}^{11} & A_{kk}^{12} \\ A_{kk}^{21} & A_{kk}^{22} \end{pmatrix}, \quad A_{ki} = \begin{pmatrix} A_{ki}^1 \\ A_{ki}^2 \end{pmatrix}, \quad A_{jk} = \begin{pmatrix} A_{jk}^1 & A_{jk}^2 \end{pmatrix}.$$

By Theorem 1.11.6 about the spectral inclusion, we conclude that

$$\begin{aligned} & W_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_k^1\oplus\mathcal{H}_k^2\oplus\cdots\oplus\mathcal{H}_n}(\mathcal{A}) \\ & = \bigcup \left\{ \sigma(\mathcal{A}_x) : x \in \mathcal{S}_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_k^1\oplus\mathcal{H}_k^2\oplus\cdots\oplus\mathcal{H}_n} \right\} \\ & \subset \bigcup \left\{ W_{\mathbb{C}\oplus\cdots\oplus\mathbb{C}^2\oplus\cdots\oplus\mathbb{C}}(\mathcal{A}_x) : x \in \mathcal{S}_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_k^1\oplus\mathcal{H}_k^2\oplus\cdots\oplus\mathcal{H}_n} \right\}. \end{aligned}$$

The theorem is proved if we show that, for $x \in \mathcal{S}_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_k^1\oplus\mathcal{H}_k^2\oplus\cdots\oplus\mathcal{H}_n}$,

$$W_{\mathbb{C}\times\cdots\times\mathbb{C}^2\times\cdots\times\mathbb{C}}(\mathcal{A}_x) \subset \bigcup \left\{\sigma(\mathcal{A}_y) : y \in \mathcal{S}_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_n}\right\} = W_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_n}(\mathcal{A}).$$

To this end, let $x \in \mathcal{S}_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_k^1\oplus\mathcal{H}_k^2\oplus\cdots\oplus\mathcal{H}_n}$, $x = (x_1 \dots x_k^1\, x_k^2 \dots x_n)^{\mathrm{t}} \in \mathcal{H}$ with $\|x_1\| = \cdots = \|x_k^1\| = \|x_k^2\| = \cdots = \|x_n\| = 1$. Then

$$\mathcal{A}_x = \left(\begin{array}{c|c|cc|c|c}
(A_{11}x_1,x_1) & \cdots & (A_{1k}^1x_k^1,x_1) & (A_{1k}^2x_k^2,x_1) & \cdots & (A_{1n}x_n,x_1)\\ \hline
\vdots & & \vdots & \vdots & & \vdots\\ \hline
(A_{k1}^1x_1,x_k^1) & \cdots & (A_{kk}^{11}x_k^1,x_k^1) & (A_{kk}^{12}x_k^2,x_k^1) & \cdots & (A_{kn}^1x_n,x_k^1)\\
(A_{k1}^2x_1,x_k^2) & \cdots & (A_{kk}^{21}x_k^1,x_k^2) & (A_{kk}^{22}x_k^2,x_k^2) & \cdots & (A_{kn}^2x_n,x_k^2)\\ \hline
\vdots & & \vdots & \vdots & & \vdots\\ \hline
(A_{n1}x_1,x_n) & \cdots & (A_{nk}^1x_k^1,x_n) & (A_{nk}^2x_k^2,x_n) & \cdots & (A_{nn}x_n,x_n)
\end{array}\right)$$

$$=: \begin{pmatrix} B_{11} & \cdots & B_{1k} & \cdots & B_{1n}\\ \vdots & & \vdots & & \vdots\\ B_{k1} & \cdots & B_{kk} & \cdots & B_{kn}\\ \vdots & & \vdots & & \vdots\\ B_{n1} & \cdots & B_{nk} & \cdots & B_{nn}\end{pmatrix} =: \mathcal{B} \in L\big(\mathbb{C}\oplus\cdots\oplus\mathbb{C}^2\oplus\cdots\oplus\mathbb{C}\big).$$

Now let $z \in \mathcal{S}_{\mathbb{C}\oplus\cdots\oplus\mathbb{C}^2\oplus\cdots\oplus\mathbb{C}}$ be arbitrary. If we find a $y \in \mathcal{S}_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_n}$ with $\mathcal{B}_z = \mathcal{A}_y$, then $\sigma\big((\mathcal{A}_x)_z\big) = \sigma(\mathcal{B}_z) = \sigma(\mathcal{A}_y)$ and hence

$$\begin{aligned} W_{\mathbb{C}\oplus\cdots\oplus\mathbb{C}^2\oplus\cdots\oplus\mathbb{C}}(\mathcal{A}_x) &= \bigcup \left\{\sigma\big((\mathcal{A}_x)_z\big) : z \in \mathcal{S}_{\mathbb{C}\oplus\cdots\oplus\mathbb{C}^2\oplus\cdots\oplus\mathbb{C}}\right\}\\ &\subset \bigcup \left\{\sigma(\mathcal{A}_y) : y \in \mathcal{S}_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_n}\right\}, \end{aligned}$$

as required. To this end, let $z = (z_1 \dots z_k \dots z_n)^{\mathrm{t}} \in \mathbb{C}\oplus\cdots\oplus\mathbb{C}^2\oplus\cdots\oplus\mathbb{C}$, $z_k = \big(z_k^1\ z_k^2\big)^{\mathrm{t}} \in \mathbb{C}^2$, with $|z_1|^2 = \cdots = \|z_k\|^2 = \cdots = |z_n|^2 = 1$. Then

$$\mathcal{B}_z = \begin{pmatrix} (B_{11}z_1,z_1) & \cdots & (B_{1k}z_k,z_1) & \cdots & (B_{1n}z_n,z_1)\\ \vdots & & \vdots & & \vdots\\ (B_{k1}z_1,z_k) & \cdots & (B_{kk}z_k,z_k) & \cdots & (B_{kn}z_n,z_k)\\ \vdots & & \vdots & & \vdots\\ (B_{n1}z_1,z_n) & \cdots & (B_{nk}z_k,z_n) & \cdots & (B_{nn}z_n,z_n)\end{pmatrix}.$$

Set

$$y_i := z_i x_i, \quad i = 1,\dots,n,\ i \neq k, \quad y_k := (y_k^1\ y_k^2)^{\mathrm{t}} := \big(z_k^1 x_k^1\ z_k^2 x_k^2\big).$$

Then it is not difficult to check that $\|y_i\| = 1$, $i = 1,\dots,n$, and hence $y := (y_1 \dots y_k \dots y_n)^{\mathrm{t}} \in \mathcal{S}_{\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_n}$. With this choice of y, we obtain the desired equality $\mathcal{B}_z = \mathcal{A}_y$. For example, for $k = 1, \dots, n$, the k-th diagonal elements $(B_z)_{kk}$ of B_z and $(\mathcal{A}_y)_{kk}$ of $\mathcal{A}_y$ coincide since

$$\begin{aligned}(B_z)_{kk} &= \left(\begin{pmatrix} (A_{kk}^{11}x_k^1, x_k^1)z_k^1 + (A_{kk}^{12}x_k^2, x_k^1)z_k^2 \\ (A_{kk}^{21}x_k^1, x_k^2)z_k^1 + (A_{kk}^{22}x_k^2, x_k^2)z_k^2 \end{pmatrix}, \begin{pmatrix} z_k^1 \\ z_k^2 \end{pmatrix} \right) \\ &= \big((A_{kk}^{11}y_k^1, x_k^1) + (A_{kk}^{12}y_k^2, x_k^1)\big)\overline{z_k^1} + \big((A_{kk}^{21}y_k^1, x_k^2) + (A_{kk}^{22}y_k^2, x_k^2)\big)\overline{z_k^2} \\ &= \big(A_{kk}^{11}y_k^1 + A_{kk}^{12}y_k^2, y_k^1\big) + \big(A_{kk}^{21}y_k^1 + A_{kk}^{22}y_k^2, y_k^2\big) \\ &= \left(\begin{pmatrix} A_{kk}^{11}y_k^1 + A_{kk}^{12}y_k^2 \\ A_{kk}^{21}y_k^1 + A_{kk}^{22}y_k^2 \end{pmatrix}, \begin{pmatrix} y_k^1 \\ y_k^2 \end{pmatrix} \right) = (A_{kk}y_k, y_k) = (\mathcal{A}_y)_{kk};\end{aligned}$$

the proof for the other entries is similar. □

Example 1.11.14 As an illustration for Theorem 1.11.13, we reconsider the matrix $\mathcal{A}_3$ from Example 1.3.3:

$$\mathcal{A}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -1 & \mathrm{i} & 5\mathrm{i} \\ -1 & -2 & -5\mathrm{i} & \mathrm{i} \end{pmatrix}. \tag{1.11.6}$$

Its block numerical ranges with respect to the four successively refined decompositions $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2 = \mathbb{C}^2 \oplus \mathbb{C} \oplus \mathbb{C} = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ (the first one being the numerical range and the last one the spectrum) are displayed in Fig. 1.11 below.

Remark 1.11.15 In [FH08], K.-H. Förster and N. Hartanto considered the block numerical range of (entrywise) nonnegative matrices. They developed a Perron-Frobenius theory for it, thus generalizing corresponding results for the spectrum and the numerical range.

The estimate for the resolvent of a block operator matrix in terms of the quadratic numerical range also generalizes to the block numerical range. For the proof we need the following generalization of Lemma 1.4.2.

Lemma 1.11.16 *Let $\mathcal{A}_{(\cdot)} : \mathcal{S}^n \to M_n(\mathbb{C})$ be uniformly bounded from below, i.e. assume there exists a $\delta > 0$ such that for all $x \in \mathcal{S}^n$*

$$\|\mathcal{A}_x \alpha\| \geq \delta \, \|\alpha\|, \quad \alpha \in \mathbb{C}^n. \tag{1.11.7}$$

Then

$$\|\mathcal{A}y\| \geq \delta \, \|y\|, \quad y \in \mathcal{H};$$

if, in addition, $\mathcal{A}$ is boundedly invertible, then $\|\mathcal{A}^{-1}\| \leq \delta^{-1}$.

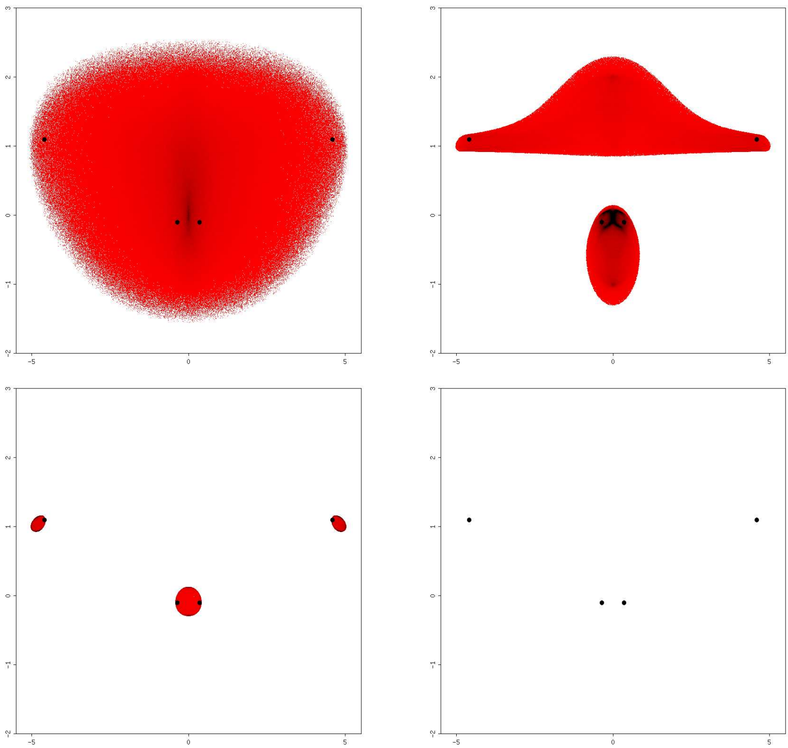

Figure 1.11 $W_{\mathbb{C}^4}(\mathcal{A}_3)$, $W_{\mathbb{C}^2\oplus\mathbb{C}^2}(\mathcal{A}_3)$, $W_{\mathbb{C}^2\oplus\mathbb{C}\oplus\mathbb{C}}(\mathcal{A}_3)$, and $W_{\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}}(\mathcal{A}_3)$.

Proof. Let $y = (y_1 \dots y_n)^{\mathrm{t}} \in \mathcal{H}$ be arbitrary, write $y_i = \|y_i\|\,\widehat{y}_i$ with $\widehat{y}_i \in \mathcal{H}_i$, $\|\widehat{y}_i\| = 1$, $i = 1, \dots, n$, and set $\alpha := \big(\|y_1\| \dots \|y_n\|\big)^{\mathrm{t}} \in \mathbb{C}^n$. Then $\widehat{y} := (\widehat{y}_1 \dots \widehat{y}_n)^{\mathrm{t}} \in \mathcal{S}^n$ and hence, by assumption (1.11.7), we have $\|\mathcal{A}_{\widehat{y}}\,\alpha\|^2 \geq \delta^2 \|\alpha\|^2 = \delta^2 \|y\|^2$. Together with the equalities

$$\|\mathcal{A}_{\widehat{y}}\,\alpha\|^2 = \left\| \begin{pmatrix} (A_{11}\widehat{y}_1, \widehat{y}_1)\|y_1\| + \dots + (A_{1n}\widehat{y}_n, \widehat{y}_1)\|y_n\| \\ \vdots \\ (A_{n1}\widehat{y}_1, \widehat{y}_n)\|y_1\| + \dots + (A_{nn}\widehat{y}_n, \widehat{y}_n)\|y_n\| \end{pmatrix} \right\|^2$$

$$= \sum_{i=1}^n \Big| \Big(\sum_{j=1}^n A_{ij} y_j, \widehat{y}_i \Big) \Big|^2 \leq \sum_{i=1}^n \Big\| \sum_{j=1}^n A_{ij} y_j \Big\|^2 \|\widehat{y}_i\|^2$$

$$= \sum_{i=1}^n \Big\| \sum_{j=1}^n A_{ij} y_j \Big\|^2 = \|\mathcal{A} y\|^2,$$

the desired estimate follows. The last claim is obvious. □

The following theorem generalizes the resolvent estimate in terms of the numerical range ($n = 1$, see (1.1.2)) and in terms of the quadratic numerical range ($n = 2$, see Theorem 1.4.1).

Theorem 1.11.17 *The resolvent of $\mathcal{A}$ admits the estimate*

$$\left\|(\mathcal{A}-\lambda)^{-1}\right\| \leq \frac{\big(\|\mathcal{A}\|+|\lambda|\big)^{n-1}}{\operatorname{dist}\big(\lambda, W^n(\mathcal{A})\big)^n}, \qquad \lambda \notin \overline{W^n(\mathcal{A})}. \tag{1.11.8}$$

More exactly, if $\mathcal{F}_1, \dots, \mathcal{F}_s$ are the components of $\overline{W^n(\mathcal{A})}$, then there are integers n_j, $j = 1, \dots, s$, with $\sum_{j=1}^s n_j = n$ such that

$$\left\|(\mathcal{A}-\lambda)^{-1}\right\| \leq \frac{\|\mathcal{A}-\lambda\|^{n-1}}{\prod_{j=1}^s \operatorname{dist}\big(\lambda, \mathcal{F}_j\big)^{n_j}}, \quad \lambda \notin \overline{W^n(\mathcal{A})}; \tag{1.11.9}$$

in particular, if $W^n(\mathcal{A})$ consists of n components, then

$$\left\|(\mathcal{A}-\lambda)^{-1}\right\| \leq \frac{\|\mathcal{A}-\lambda\|^{n-1}}{\prod_{j=1}^n \operatorname{dist}\left(\lambda, \mathcal{F}_j\right)}, \qquad \lambda \notin \overline{W^n(\mathcal{A})}.$$

Proof. Let $\lambda \notin \overline{W^n(\mathcal{A})}$. If $\mathcal{F}_1, \dots, \mathcal{F}_s$ are the components of $\overline{W^n(\mathcal{A})}$, then there are integers n_j, $j = 1, \dots, s$, with $\sum_{j=1}^s n_j = n$ such that each matrix $\mathcal{A}_x$, $x \in \mathcal{S}^n$, has exactly n_j eigenvalues in $\mathcal{F}_j$ for all $j = 1, \dots, s$. Now let $x \in \mathcal{S}^n$ and let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $\mathcal{A}_x$. Then there exists a partition $I_1 \,\dot\cup \cdots \dot\cup\, I_s = \{1, \dots, n\}$ so that $\lambda_i \in \mathcal{F}_j$ if and only if $i \in I_j$. Then $n_j = \# I_j$, $j = 1, \dots, s$, and

$$\left|\det(\mathcal{A}_x-\lambda)\right| = |\lambda-\lambda_1| \cdots |\lambda-\lambda_n| = \prod_{j=1}^s \prod_{i\in I_j} |\lambda-\lambda_i| \geq \prod_{j=1}^s \operatorname{dist}\left(\lambda, \mathcal{F}_j\right)^{n_j} > 0$$

for $x \in \mathcal{S}^n$ since $\lambda \notin \overline{W^n(\mathcal{A})}$; in particular, $\mathcal{A}_x - \lambda$ is invertible. This and Lemma 1.11.5 now imply that

$$\left\|(\mathcal{A}-\lambda)_x^{-1}\right\| \leq \frac{\|(\mathcal{A}-\lambda)_x\|^{n-1}}{|\det(\mathcal{A}_x-\lambda)|} \leq \frac{\|\mathcal{A}-\lambda\|^{n-1}}{\prod_{j=1}^s \operatorname{dist}\left(\lambda, \mathcal{F}_j\right)^{n_j}} \tag{1.11.10}$$

for all $x \in \mathcal{S}^n$. Since $\lambda \notin \overline{W^n(\mathcal{A})}$, we have $\lambda \in \rho(A)$ by Theorem 1.11.6 and hence $\mathcal{A} - \lambda$ is invertible. Using this and (1.11.10), we obtain the second assertion of the theorem from Lemma 1.11.16. The first and the third estimate are immediate consequences of the second inequality. □

As for the numerical range and the quadratic numerical range, the estimate of the resolvent yields an upper bound for the length of Jordan chains at boundary points of the block numerical range (compare Corollary 1.4.8).

Proposition 1.11.18 *Let $\lambda_0 \in \sigma_{\mathrm{p}}(\mathcal{A})$. If $\lambda_0 \in \partial W^n(\mathcal{A})$ has the exterior cone property, then the length of a Jordan chain at λ_0 is at most n.*

More exactly, if $\overline{W^n(\mathcal{A})} = \mathcal{F}_1 \dot\cup \ldots \dot\cup \mathcal{F}_s$ *consists of* s *disjoint components, such that* $\lambda_0 \in \mathcal{F}_{j_0}$*, and the integers* n_j*,* $j = 1, \ldots, s$*, are as in the proof of Theorem* 1.11.17*, then the length of a Jordan chain at* λ_0 *is at most* n_{j_0}*. In particular, if* $\overline{W^n(\mathcal{A})}$ *consists of* n *components, then the length of a Jordan chain at* λ_0 *is at most one, i.e. there are no associated vectors at* λ_0*.*

Proof. The proof is completely analogous to the proof of Corollary 1.4.8 for the quadratic numerical range (see [TW03, Proposition 4.4]). □

Remark 1.11.19 The block diagonalization theorem (see Theorem 1.7.1 and Corollary 1.7.2) was generalized recently to the $n \times n$ case by M. Wagenhofer (see the PhD thesis [Wag07]). He did not only assume that $W^n(\mathcal{A})$ has n disjoint components, but that they are separated in some stronger sense.

1.12 Numerical ranges of operator polynomials

A special class of $n \times n$ block operator matrices, so-called *companion operators*, arises as linearizations of operator polynomials of degree n. Here we study the relation between the block numerical range of a companion operator and the numerical range of the corresponding operator polynomial.

Let $\mathcal{H}_0$ be a complex Hilbert space, let $A_i \in L(\mathcal{H}_0)$, $i = 0, \ldots, n-1$, and set $A := (A_0, \ldots, A_{n-1})$. Consider the operator polynomial P_A given by

$$P_A(\lambda) := \lambda^n I + \lambda^{n-1} A_{n-1} + \cdots + \lambda A_1 + A_0, \quad \lambda \in \mathbb{C}.$$

The companion operator $\mathcal{C}^A$ of P_A is the $n \times n$ block operator matrix in the Hilbert space $\mathcal{H} = \mathcal{H}_0^n = \mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_0$ given by

$$\mathcal{C}^A := \begin{pmatrix} 0 & I & \cdots & \cdots & 0 \\ \vdots & 0 & I & & \\ \vdots & & \ddots & \ddots & \\ \vdots & & & 0 & I \\ -A_0 & -A_1 & \cdots & -A_{n-2} & -A_{n-1} \end{pmatrix}.$$

It is well-known that the spectral properties of P_A and its companion operator $\mathcal{C}^A$ are intimately related (see [Mül56], [Mar88, § 12.1]); in particular, $\sigma(P_A) = \sigma(\mathcal{C}^A)$ and $\sigma_p(P_A) = \sigma_p(\mathcal{C}^A)$.

The numerical range of the operator polynomial P_A is given by $W(P_A) := \{\lambda \in \mathbb{C} : \exists\, f \in \mathcal{H},\, f \neq 0,\ (P_A(\lambda)f, f) = 0\}$ (see (1.6.1)). It is not difficult to check that $W(P_A) \subset W(\mathcal{C}^A)$; in fact, if $\big(P_A(\lambda)f, f\big) = 0$, then $\big((\mathcal{C}^A - \lambda)\mathbf{x}, \mathbf{x}\big) = -\lambda^{n-1}\big(P_A(\lambda)f, f\big) = 0$ for $\mathbf{x} := (f, \lambda f, \dots, \lambda^{n-1} f)^{\mathrm{t}}$.

The next theorem shows that $W(P_A)$ is even contained in the block numerical range of its companion operator $\mathcal{C}^A$:

Theorem 1.12.1 $W(P_A) \subset W^n\big(\mathcal{C}^A\big)$.

Proof. Let $\lambda_0 \in W(P_A)$. Then there exists an $x \in \mathcal{H}_0$, $\|x\| = 1$, such that λ_0 is a zero of the scalar polynomial

$$\big(P_A(\lambda)x, x\big) = \lambda^n + \lambda^{n-1}(A_{n-1}x, x) + \cdots + \lambda(A_1 x, x) + (A_0 x, x) = 0.$$

The companion operator of the scalar polynomial $\big(P_A(\lambda)x, x\big)$ is the $n \times n$ matrix $\mathcal{C}^A_{(x,\dots,x)}$. Since the zeroes of $\big(P_A(\lambda)x, x\big)$ coincide with the eigenvalues of $\mathcal{C}^A_{(x,\dots,x)}$, it follows that $\lambda_0 \in \sigma_{\mathrm{p}}\big(\mathcal{C}^A_{(x,\dots,x)}\big) \subset W^n\big(\mathcal{C}^A\big)$. □

Example 1.12.2 To illustrate Theorem 1.12.1, we reconsider the matrix $\mathcal{A}_3$ in Example 1.3.3. It is the companion operator of the quadratic matrix polynomial

$$P_3(\lambda) := \lambda^2 I_2 + \lambda \begin{pmatrix} -\mathrm{i} & -5\mathrm{i} \\ 5\mathrm{i} & -\mathrm{i} \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \lambda \in \mathbb{C}.$$

Figure 1.12 shows the numerical range of P_3 on the left. It is contained in the quadratic numerical range of $\mathcal{A}_3$ with respect to the decomposition $\mathbb{C}^2 \oplus \mathbb{C}^2$ on the right.

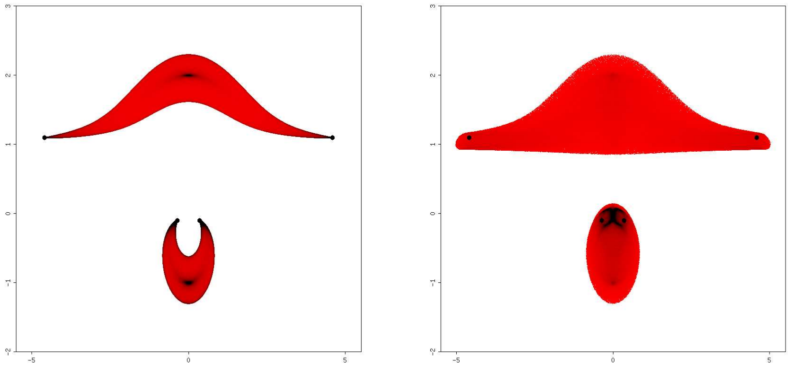

Figure 1.12 $W(P_3)$ and $W_{\mathbb{C}^2 \oplus \mathbb{C}^2}(\mathcal{A}_3)$.

Next we show that if $\mathcal{H}_0$ is finite-dimensional, $\mathcal{H}_0 = \mathbb{C}^k$, then, up to the point 0, the numerical range of P_A coincides with a higher degree block numerical range of its companion operator. To this end, we consider $\mathcal{C}^A$ with respect to a refined decomposition of $\mathcal{H} = \mathcal{H}_0^n = \mathbb{C}^{nk}$.

Theorem 1.12.3 *If we consider the companion operator $\mathcal{C}^A$ with respect to the decomposition*

$$\mathbb{C}^{nk} = \overbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}^{(n-1)k} \oplus \mathbb{C}^k, \tag{1.12.1}$$

then

$$W_{\mathbb{C}\oplus\cdots\oplus\mathbb{C}\oplus\mathbb{C}^k}\big(\mathcal{C}^A\big) = W^{(n-1)k+1}\big(\mathcal{C}^A\big) = \begin{cases} W(P_A), & n=1, \\ W(P_A)\cup\{0\}, & n>1. \end{cases}$$

Proof. For $n=1$ the assertion is immediate. For $n>1$, with respect to the decomposition (1.12.1), $\mathcal{C}^A$ has the block operator representation

$$\left(\begin{array}{cccccccccccccc}
0 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0 & 0_{1,k} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \cdots & \vdots & & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & & 0 & \cdots & 0 & 0_{1,k} \\
0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & & 0 & \cdots & 0 & 0_{1,k} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \cdots & \vdots & & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & & 0 & \cdots & 0 & 0_{1,k} \\
 & \vdots & & & \vdots & & & \vdots & & \ddots & & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & & 1 & \cdots & 0 & 0_{1,k} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \cdots & \vdots & & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots & 1 & 0_{1,k} \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0 & e_1 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \cdots & \vdots & & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0 & e_k \\
-A_0^{(1)} & \cdots & -A_0^{(k)} & -A_1^{(1)} & \cdots & -A_1^{(k)} & -A_2^{(1)} & \cdots & -A_2^{(k)} & \cdots & -A_{n-2}^{(1)} & \cdots & -A_{n-2}^{(k)} & -A_{n-1}
\end{array}\right).$$

Here $0_{1,k} = (0\cdots0) \in L(\mathbb{C}^k,\mathbb{C})$ is the zero vector, $e_j = (0\cdots1\cdots0) \in L(\mathbb{C}^k,\mathbb{C})$ is the j-th row unit vector, $j=1,\dots,k$, and

$$A_i^{(j)} = \begin{pmatrix} a_{1j}^{(i)} \\ \vdots \\ a_{kj}^{(i)} \end{pmatrix} \in L(\mathbb{C},\mathbb{C}^k)$$

is the j-th column of $A_i = \big(a_{st}^{(i)}\big)_{s,t=1}^k$, $i=0,\dots,n-1$, $j=1,\dots,k$. Now let

$$x := \Big(x_0^{(1)} \dots\, x_0^{(k)} \dots\, x_{n-2}^{(1)} \dots\, x_{n-2}^{(k)}\ (\xi_1 \dots \xi_k)\Big)^{\mathrm{t}} \in \mathbb{C}\oplus\cdots\oplus\mathbb{C}\oplus\mathbb{C}^k$$

with $\big|x_0^{(1)}\big| = \cdots = \big|x_{n-2}^{(k)}\big| = \|\xi\| = 1$ where $\xi := (\xi_1 \dots \xi_k)^{\mathrm{t}}$. By similar manipulations of determinants as in the proof of the previous theorem, one can show that, for $\lambda \neq 0$,

$$\det\big(\mathcal{C}_x^A - \lambda\big) = (-1)^n \lambda^{(n-1)(k-1)}\big(P_A(\lambda)\xi,\xi\big).$$

This implies $W^{(n-1)k+1}(\mathcal{C}^A)\setminus\{0\} = W(P_A)\setminus\{0\}$. Finally, it is not difficult to see that $0 \in W^{(n-1)k+1}(\mathcal{C}^A)$ for $n > 1$. □

Example 1.12.4 As an example for Theorem 1.12.3, we consider the quadratic matrix polynomial (compare [LMZ98])

$$P_{11}(\lambda) := \lambda^2 I_2 + \lambda \begin{pmatrix} 0 & 2.8\mathrm{i} \\ -2.8\mathrm{i} & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \lambda \in \mathbb{C},$$

with its companion operator $\mathcal{A}_{11}$ decomposed as

$$\mathcal{A}_{11} = \left(\begin{array}{c|c|cc} 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline -2 & -1 & 0 & -2.8\mathrm{i} \\ -1 & -2 & 2.8\mathrm{i} & 0 \end{array}\right).$$

In Fig. 1.13 the cubic numerical range of $\mathcal{A}_{11}$ with respect to this decomposition and the numerical range of the operator polynomial P_{11} are displayed. A closer look shows that the numerical range of P_{11} on the right does not contain the point 0, whereas the cubic numerical range on the left does.

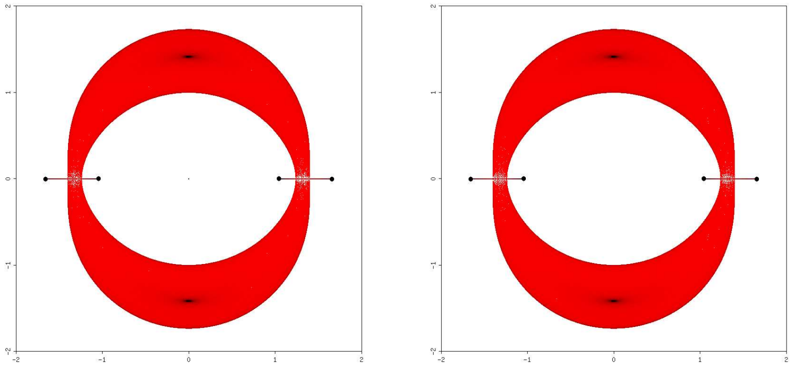

Figure 1.13 $W_{\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}^2}(\mathcal{A}_{11})$ and $W(P_{11})$.

Remark 1.12.5 In [Lin03] H. Linden applied the quadratic numerical range to derive enclosures for the zeroes of monic polynomials in $\mathbb{C}$ of degree $n \geq 3$. He enclosed the quadratic numerical range of the companion matrix $\mathcal{C}_A$ with respect to the decompositions $\mathbb{C}^n = \mathbb{C}^{n-1} \oplus \mathbb{C}$ and $\mathbb{C}^n = \mathbb{C}^{n-2} \oplus \mathbb{C}^2$ (and thus the zeroes) in two circles of equal radius. Examples show that the enclosures are tighter than those obtained from the numerical range of $\mathcal{C}_A$.

1.13 Gershgorin's theorem for block operator matrices

Gershgorin's circle theorem is a valuable tool to enclose the spectrum of matrices (see [Ger31], [Bra58], [Var04]). Its generalization to partitioned matrices and bounded block operator matrices (see [FV62], [Sal99]) is straightforward. In general, there is no inclusion between the quadratic or block numerical range and the Gershgorin sets; depending on the particular situation, one or the other may give a better spectral enclosure. However, the quadratic or, more generally, block numerical range has the advantage of not using norms of inverses.

The following Gershgorin theorem for bounded block operator matrices is due to H. Salas (see [Sal99]). Its proof generalizes Householder's proof of Gershgorin's theorem in the matrix case; it may even be generalized to unbounded diagonally dominant block operator matrices.

Theorem 1.13.1 *Let $n \in \mathbb{N}$, let $\mathcal{H}_1, \dots, \mathcal{H}_n$ be complex Hilbert spaces, $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$, and let $\mathcal{A} \in L(\mathcal{H})$,*

$$\mathcal{A} = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}$$

with $A_{ij} \in L(\mathcal{H}_j, \mathcal{H}_i)$, $i, j = 1, \dots, n$. If we define

$$\mathcal{G}_i := \sigma(A_{ii}) \cup \Big\{ \lambda \in \rho(A_{ii}) : \big\| (A_{ii} - \lambda)^{-1} \big\|^{-1} \le \sum_{\substack{j=1 \\ j \ne i}}^{n} \big\| A_{ij} \big\| \Big\} \qquad (1.13.1)$$

for $i = 1, \dots, n$, then

$$\sigma(\mathcal{A}) \subset \bigcup_{i=1}^{n} \mathcal{G}_i.$$

Proof. Suppose that $\lambda \notin \bigcup_{i=1}^{n} \sigma(A_{ii})$. Then we can write

$$\mathcal{A} - \lambda = \begin{pmatrix} A_{11} - \lambda & & 0 \\ & \ddots & \\ 0 & & A_{nn} - \lambda \end{pmatrix} \big(I + M(\lambda) \big) \qquad (1.13.2)$$

where

$$M(\lambda) := \begin{pmatrix} 0 & (A_{11}-\lambda)^{-1} A_{12} & \cdots & (A_{11}-\lambda)^{-1} A_{1n} \\ (A_{22}-\lambda)^{-1} A_{21} & 0 & & (A_{22}-\lambda)^{-1} A_{2n} \\ \vdots & & \ddots & \vdots \\ (A_{nn}-\lambda)^{-1} A_{n1} & \cdots & \cdots & 0 \end{pmatrix}.$$

If $\big\|(A_{ii}-\lambda)^{-1}\big\|^{-1} > \sum_{\substack{j=1\\ j\neq i}}^{n}\big\|A_{ij}\big\|$ for $i=1,\dots,n$, then $\|M(\lambda)\|<1$. Hence both factors in (1.13.2) are boundedly invertible and so $\lambda\in\rho(\mathcal{A})$. □

Remark 1.13.2 For a self-adjoint diagonal entry A_{ii}, the norms of the inverses in (1.13.1) are known explicitly and we have

$$\mathcal{G}_i = \sigma(A_{ii}) \cup \Big\{\lambda\in\rho(A_{ii}) : \operatorname{dist}\big(\lambda,\sigma(A_{ii})\big) \le \sum_{\substack{j=1\\ j\neq i}}^{n}\big\|A_{ij}\big\|\Big\};$$

for non-self-adjoint A_{ii}, only the estimate in terms of the numerical range is available and thus, in general, we only have the inclusion

$$\mathcal{G}_i \subset \sigma(A_{ii}) \cup \Big\{\lambda\in\rho(A_{ii}) : \operatorname{dist}\big(\lambda,W(A_{ii})\big) \le \sum_{\substack{j=1\\ j\neq i}}^{n}\big\|A_{ij}\big\|\Big\}.$$

In the remaining part of this section we consider the case $n=2$. For a 2×2 block operator matrix $\mathcal{A}$, the spectrum of $\mathcal{A}$ can be described in terms of the spectra of the Schur complements as (see Proposition 1.6.2)

$$\sigma(\mathcal{A})\setminus\sigma(A_{ii}) = \sigma(S_i), \quad i=1,2. \tag{1.13.3}$$

This description and a spectral enclosure for the Schur complements allows us to tighten the spectral enclosure (1.13.1) by the Gershgorin sets:

Proposition 1.13.3 *Let $n=2$ and define*

$$\mathcal{N}_1 := \sigma(A_{11})\cup\Big\{\lambda\in\rho(A_{11})\cap\rho(A_{22}) : \big\|(A_{11}-\lambda)^{-1}A_{12}(A_{22}-\lambda)^{-1}A_{21}\big\|\ge 1\Big\},$$
$$\mathcal{N}_2 := \sigma(A_{22})\cup\Big\{\lambda\in\rho(A_{11})\cap\rho(A_{22}) : \big\|(A_{22}-\lambda)^{-1}A_{21}(A_{11}-\lambda)^{-1}A_{12}\big\|\ge 1\Big\}.$$

Then

$$\sigma(\mathcal{A}) \subset \big(\sigma(A_{11})\cup\sigma(S_1)\big)\cup\big(\sigma(A_{22})\cup\sigma(S_2)\big) \subset \mathcal{N}_1\cup\mathcal{N}_2 \subset \mathcal{G}_1\cup\mathcal{G}_2. \tag{1.13.4}$$

Proof. The first inclusion in (1.13.4) is obvious from (1.13.3). For the second inclusion, we observe that we have $\lambda\notin\mathcal{N}_1\cup\mathcal{N}_2$ if and only if $\lambda\in\rho(A_{11})\cap\rho(A_{22})$ and the two inequalities

$$\big\|(A_{11}-\lambda)^{-1}A_{12}(A_{22}-\lambda)^{-1}A_{21}\big\| < 1, \tag{1.13.5}$$
$$\big\|(A_{22}-\lambda)^{-1}A_{21}(A_{11}-\lambda)^{-1}A_{12}\big\| < 1 \tag{1.13.6}$$

hold. Since, for $\lambda\in\rho(A_{11})\cap\rho(A_{22})$, we can write

$$S_1(\lambda) = (A_{11}-\lambda)\big(I-(A_{11}-\lambda)^{-1}A_{12}(A_{22}-\lambda)^{-1}A_{21}\big),$$
$$S_2(\lambda) = (A_{22}-\lambda)\big(I-(A_{22}-\lambda)^{-1}A_{21}(A_{11}-\lambda)^{-1}A_{12}\big),$$

we conclude that $\lambda \notin \mathcal{N}_1 \cup \mathcal{N}_2$ implies $\lambda \in \rho(S_1) \cap \rho(S_2)$. Because S_i is defined on $\mathbb{C} \setminus \sigma(A_{ii})$, we have $\rho(S_i) \,\dot\cup\, \sigma(S_i) = \mathbb{C} \setminus \sigma(A_{ii})$, $i = 1, 2$. Hence $\lambda \in \rho(S_1) \cap \rho(S_2)$ is equivalent to $\lambda \notin \big(\sigma(A_{11}) \cup \sigma(S_1)\big) \cup \big(\sigma(A_{22}) \cup \sigma(S_2)\big)$.

For the third inclusion in (1.13.4), we note that $\lambda \notin \mathcal{G}_1 \cup \mathcal{G}_2$ if and only if $\lambda \notin \sigma(A_{11}) \cup \sigma(A_{22})$ and the two inequalities

$$\big\|(A_{11} - \lambda)^{-1}\big\| \, \|A_{12}\| < 1, \quad \big\|(A_{22} - \lambda)^{-1}\big\| \, \|A_{21}\| < 1$$

hold; the latter imply the two inequalities (1.13.5), (1.13.6). Therefore $\lambda \notin \mathcal{G}_1 \cup \mathcal{G}_2$ implies $\lambda \notin \mathcal{N}_1 \cup \mathcal{N}_2$. □

To conclude this section, we compare the spectral enclosures by the Gershgorin sets to those by the quadratic numerical range for self-adjoint and $\mathcal{J}$-self-adjoint 2×2 block operator matrices. In particular, we consider situations where the quadratic numerical range yields estimates of the spectrum that are independent of the size of the off-diagonal entries.

Remark 1.13.4 Let $n = 2$, $A_{11} = A_{11}^*$, $A_{22} = A_{22}^*$, and suppose that either $A_{21} = A_{12}^*$ or $A_{21} = -A_{12}^*$.

i) The Gershgorin type Theorem 1.13.1 yields the inclusion

$$\sigma(\mathcal{A}) \subset \big\{\lambda \in \mathbb{C} : \operatorname{dist}\big(\lambda, \sigma(A_{11}) \cup \sigma(A_{22})\big) \le \|A_{12}\|\big\}.$$

ii) If $\operatorname{dist}\big(\sigma(A_{11}), \sigma(A_{22})\big) > 0$ and $A_{21} = A_{12}^*$, then Theorem 1.3.7 i), which uses the quadratic numerical range, gives the tighter inclusion

$$\sigma(\mathcal{A}) \subset \big\{\lambda \in \mathbb{C} : \operatorname{dist}\big(\lambda, \sigma(A_{11}) \cup \sigma(A_{22})\big) \le \|\delta_{A_{12}}\|\big\},$$

where

$$\delta_{A_{12}} = \|A_{12}\| \tan\left(\frac{1}{2}\arctan\left(\frac{2\|A_{12}\|}{\operatorname{dist}\big(\sigma(A_{11}), \sigma(A_{22})\big)}\right)\right) < \|A_{12}\|;$$

if, in addition, $\max \sigma(A_{22}) < \min \sigma(A_{11})$, then, by Theorem 1.3.6 ii),

$$\sigma(\mathcal{A}) \cap \big(\max \sigma(A_{22}), \min \sigma(A_{11})\big) = \emptyset$$

independently of the size of $\|\mathcal{A}_{12}\|$.

iii) If $A_{21} = -A_{12}^*$, then Proposition 1.3.9 i) shows that

$$\operatorname{Re} \sigma(\mathcal{A}) \subset \big[\min\big\{\min \sigma(A_{11}), \min \sigma(A_{22})\big\}, \max\big\{\max \sigma(A_{11}), \max \sigma(A_{22})\big\}\big]$$

independently of the size of $\|\mathcal{A}_{12}\|$.

If the spectra of the diagonal elements are not disjoint, then the Gershgorin enclosure in Remark 1.13.4 i) still applies, but the estimates in Remark 1.13.4 ii) do not. The following example shows that, in some cases, a change of the decomposition of the space leads to a block operator matrix with diagonal elements having disjoint spectra.

Example 1.13.5 Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$, and let $A_{11} = A_{11}^*$, $A_{22} = A_{11}$, $A_{21} = A_{12}^*$ be such that $\sigma(A_{11}) = \sigma_1 \dot\cup \sigma_2$ with $\sigma_i \neq \emptyset$, $i = 1, 2$, and $\operatorname{dist}(\sigma_1, \sigma_2) > 2\,\|A_{12}\|$. Then the two Gershgorin sets $\mathcal{G}_1, \mathcal{G}_2$ coincide and consist of two components,

$$\mathcal{G}_1 = \mathcal{G}_2 = \Sigma_1 \dot\cup \Sigma_2, \quad \Sigma_i := \big\{\lambda \in \mathbb{C} : \operatorname{dist}(\lambda, \sigma_i) \le \|A_{12}\|\big\}, \quad i = 1, 2,$$

whereas the quadratic numerical range with respect to the given decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$ consists of a single component by Corollary 1.1.10 ii).

The inclusion by the quadratic numerical range may be improved by using another decomposition of $\mathcal{H}$. Since $\sigma(A_{11}) = \sigma_1 \dot\cup \sigma_2$, there exist invariant subspaces $\mathcal{H}_1^1$, $\mathcal{H}_1^2$ of A_{11} such that

$$\sigma\big(A_{11}\big|_{\mathcal{H}_1^1}\big) = \sigma_1, \quad \sigma\big(A_{11}\big|_{\mathcal{H}_1^2}\big) = \sigma_2.$$

We consider the new decomposition $\mathcal{H} = \widetilde{\mathcal{H}}_1 \oplus \widetilde{\mathcal{H}}_2$ with $\widetilde{\mathcal{H}}_i = \mathcal{H}_1^i \oplus \mathcal{H}_1^i$ for $i = 1, 2$. By a standard perturbation argument for self-adjoint operators (see [Kat95, Theorem V.4.10]), the assumption $\operatorname{dist}(\sigma_1, \sigma_2) > 2\,\|A_{12}\|$ implies that the new diagonal elements $\widetilde{A}_{ii}$, $i = 1, 2$, have separated spectra (they are contained *e.g.* in the two disjoint components Σ_i of the Gershgorin sets). Hence Remark 1.13.4 ii) applies to the block operator matrix obtained with respect to this new decomposition.