# Chapter 1

# Complex Numbers

## 1.1 Informal Introduction

What is a complex number? It is any number of the form $z = x + iy$, where $x$ and $y$ are real numbers and $i$ obeys $i^2 = -1$. Of course, there is no real number whose square is negative, and so $i$ is not a real number. Accordingly, $x$ is called the real part of $z$, denoted $\operatorname{Re} z$, and $y = \operatorname{Im} z$ is called the imaginary part. (Notice that $\operatorname{Im} z$ is $y$ and not $iy$.)

Complex numbers are declared equal if and only if they have the same real and imaginary parts; if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then $z_1 = z_2$ if and only if both $x_1 = x_2$ and $y_1 = y_2$. We write 0 for $0 + i0$.

Addition and multiplication are as one would expect;

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2) + i\,(y_1 + y_2) \\ z_1\, z_2 &= (x_1 + iy_1)\,(x_2 + iy_2) = x_1x_2 + iy_1x_2 + iy_1iy_2 + x_1iy_2 \\ &= (x_1x_2 - y_1y_2) + i\,(y_1x_2 + x_1y_2)\,. \end{aligned}$$

If $z = x + iy$, then $-z = -x - iy$.

Suppose that $z = x + iy$, and $z \neq 0$. Then at least one of $x$ or $y$ is non-zero. In fact, $z \neq 0$ if and only if $x^2 + y^2 > 0$. We have

$$\begin{aligned} \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{(x + iy)(x - iy)} &= \frac{x - iy}{x^2 + y^2} \\ &= \frac{x}{x^2 + y^2} - i\,\frac{y}{x^2 + y^2}\,, \end{aligned}$$

so that $\operatorname{Re} \dfrac{1}{z} = \dfrac{x}{x^2 + y^2}$ and $\operatorname{Im} \dfrac{1}{z} = \dfrac{-y}{x^2 + y^2}$.

By definition, complex conjugation changes the sign of the imaginary part, that is, the complex conjugate of $z = x + iy$ is defined to be the complex number $\overline{z} = x - iy$. Notice that $z\,\overline{z} = x^2 + y^2$ and that

$$\operatorname{Re} z = x = \frac{z + \overline{z}}{2} \quad \text{and} \quad \operatorname{Im} z = y = \frac{z - \overline{z}}{2i}.$$

**Proposition 1.1** *For any complex numbers $z_1$, $z_2$, we have*

(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$,
(ii) $\overline{z_1\, z_2} = \overline{z_1}\;\overline{z_2}$,
(iii) $\overline{\overline{z_1}} = z_1$.
(iv) *If* $z_2 \neq 0$, *then* $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ .

***Proof.*** This is just straightforward computation. □

## 1.2 Complex Plane

There is a natural correspondence between complex numbers and points in the plane, as follows. To any given complex number $z = x + iy$, we associate the point $(x, y)$ in the plane and, conversely, to any point $(x, y)$ in the plane, we associate the complex number $z = x + iy$

$$z = x + iy \quad \longleftrightarrow \quad (x, y).$$

This is evidently a one-one correspondence.

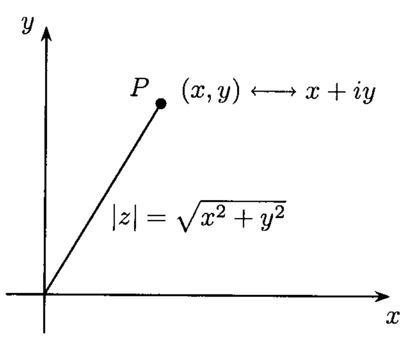


Fig. 1.1 Complex numbers as points in the Argand diagram.

The complex plane (also called the Argand diagram or Gauss plane) is just the set of complex numbers thought of as points in the plane in this way. It is very helpful to be able to picture complex numbers like this. The $x$-axis is called the real axis and the $y$-axis is called the imaginary axis.

If $P$ is the point $(x, y)$, corresponding to $z = x+iy$, then the (Euclidean) distance of $P$ from the origin is equal to $\sqrt{x^2+y^2}$. This value is written $|z|$, the modulus (or absolute value) of $z$. Thus, $|z|$ is the length of the two-dimensional vector $(x, y)$. If $z$ is real, then $y = 0$ and so $|z| = \sqrt{x^2} = |x|$, the usual value of the modulus of a real number.

For any complex numbers $z_1$ and $z_2$, $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$ so that $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ which is the distance between the points $z_1$ and $z_2$ thought of as points in the plane. It makes perfectly good sense to talk about complex numbers being "close together"—this simply means that the distance between them, namely $|z_1 - z_2|$, is "small".

**Examples 1.1**

(1) What is the set $S = \{\, z : |z - \zeta| = r \,\}$, where $r > 0$ and $\zeta$ is fixed? The complex number $z$ belongs this set if (and only if) its distance from $\zeta$ is equal to $r$. We conclude that $S$ is the circle in the complex plane with centre $\zeta$ and radius $r$. In terms of cartesian coordinates, we see that $z = x + iy$ belongs to $S$ if and only if

$$r^2 = |z - \zeta|^2 = (x - \xi)^2 + (y - \eta)^2$$

where $\zeta = \xi + i\eta$. This is the equation of a circle in $\mathbb{R}^2$ with centre at the point $(\xi, \eta)$ and radius $r$.
By considering values $r < R$, we see that $\{\, z : |z - \zeta| < R \,\}$ is the disc in the complex plane formed by all those complex numbers whose distance from $\zeta$ is strictly less than $R$.
Note that $\{\, z : |z| = 1 \,\}$ is the circle with radius 1 and centre at the origin. The set $\{\, z : |z| < 1 \,\}$ is the disc with centre at the origin and radius 1 but *not* including the perimeter $\{\, z : |z| = 1 \,\}$.

(2) What is the set $A = \{\, z : |z - i| = |z - 3| \,\}$? We see that $z \in A$ if and only if its distance from the complex number $i$ is the same as its distance from 3. It follows that $A$ is a straight line—the perpendicular bisector of the line between $i$ and 3 in the complex plane. We can see this in terms of cartesian coordinates. If $z = x + iy$, then $z - i = x + i(y - 1)$ and $z - 3 = x - 3 + iy$, so that $z$ belongs to $A$ if and only if

$$x^2 + (y - 1)^2 = (x - 3)^2 + y^2 .$$

Simplifying, this becomes $y = 3x - 4$, the equation of a straight line.

(3) The set $\{ z : \operatorname{Im} z > 0 \}$ is the set of those complex numbers $z = x + iy$ such that $\operatorname{Im} z = y > 0$. This is just the set of all points in the upper half-plane—those points lying *strictly above* the $x$-axis.

**Proposition 1.2** *For any complex number $z$,*

$$|\operatorname{Re} z| \leq |z|$$

$$|\operatorname{Im} z| \leq |z|$$

$$\tfrac{1}{\sqrt{2}}\left(|x| + |y|\right) \leq |z| \leq |x| + |y|$$

*where $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$.*

***Proof.*** The first two inequalities are direct consequences of the inequality $a^2 \leq a^2 + b^2$, valid for any real numbers $a$ and $b$ (take positive square roots).

Furthermore, $|z|^2 = x^2 + y^2 \leq x^2 + y^2 + 2\,|x|\,|y| = (|x| + |y|)^2$. Taking positive square roots gives $|z| \leq |x| + |y|$.

Finally, the inequality $(a - b)^2 \geq 0$, for any real numbers $a$ and $b$, can be rewritten as $2ab \leq a^2 + b^2$. Using this, we have

$$\left(|x| + |y|\right)^2 = |x|^2 + |y|^2 + 2\,|x|\;|y| \leq 2\,|x|^2 + 2\,|y|^2 .$$

Taking the positive square root completes the proof. □

## 1.3 Properties of the Modulus

Further properties of the modulus are as follows.

**Proposition 1.3** *For any complex numbers $z$, $\zeta$, we have*

(i) $|z| = 0$ *if and only if* $z = 0$ ;

(ii) $|z|^2 = z\,\overline{z}$ ;

(iii) $|z| = |\,\overline{z}\,|$ ;

(iv) $|z\,\zeta| = |z|\;|\zeta|$ ;

(v) *if* $\zeta \neq 0$, *then* $\left|\dfrac{z}{\zeta}\right| = \dfrac{|z|}{|\zeta|}$ ;

(vi) $|z + \zeta| \leq |z| + |\zeta|$ (the triangle inequality) ;

(vii) $|z_1 + \cdots + z_m| \leq |z_1| + \cdots + |z_m|$ *for any* $z_1, z_2, \ldots, z_m \in \mathbb{C}$.

***Proof.*** Parts (i), (ii) and (iii) are straightforward. To prove (iv), we observe that $|z\,\zeta|^2 = z\zeta\,\overline{z\zeta}$, by (ii), $= z\,\zeta\,\overline{z}\,\overline{\zeta} = |z|^2\,|\zeta|^2$, again by (ii).

Taking positive square roots gives (iv). Similarly, if $\zeta \neq 0$, then

$$\left|\frac{z}{\zeta}\right|^2 = \frac{z}{\zeta}\overline{\left(\frac{z}{\zeta}\right)} = \frac{z\,\overline{z}}{\zeta\,\overline{\zeta}} = \frac{|z|^2}{|\zeta|^2}$$

and (v) follows.

It is possible to prove part (vi) by substituting in the real and imaginary parts and doing a bit of algebra. However, we can give a slick and relatively painless proof as follows:

$$\begin{aligned}
|z+\zeta|^2 &= \overline{(z+\zeta)}\,(z+\zeta) \\
&= (\overline{z}+\overline{\zeta})\,(z+\zeta) \\
&= \overline{z}z + \overline{\zeta}z + \overline{z}\,\zeta + \overline{\zeta}\,\zeta \\
&= |z|^2 + |\zeta|^2 + z\,\overline{\zeta} + \overline{z\,\overline{\zeta}} \\
&= |z|^2 + |\zeta|^2 + 2\,\mathrm{Re}(z\,\overline{\zeta}) \\
&\leq |z|^2 + |\zeta|^2 + 2\,|z\,\overline{\zeta}| \\
&= |z|^2 + |\zeta|^2 + 2\,|z|\,|\zeta| \\
&= (|z| + |\zeta|)^2 .
\end{aligned}$$

Taking positive square roots completes the proof.

Part (vii) is the generalized triangle inequality and follows directly from part (vi) by induction. Indeed, for each $m \in \mathbb{N}$, let $P(m)$ be the statement that $|z_1 + \cdots + z_m| \leq |z_1| + \cdots + |z_m|$ for any $z_1, z_2, \ldots, z_m \in \mathbb{C}$. Clearly, $P(1)$ is true.

We suppose that $P(n)$ is true and show that this implies that $P(n+1)$ is true. Indeed, for any $z_1, \ldots, z_{n+1}$ in $\mathbb{C}$, let $\zeta = z_n + z_{n+1}$. Then, we have

$$\begin{aligned}
|z_1 + \cdots + z_n + z_{n+1}| &= |z_1 + \cdots + z_{n-1} + \zeta| \\
&\leq |z_1| + \cdots + |z_{n-1}| + |\zeta|
\end{aligned}$$

by the induction hypothesis (namely, that $P(n)$ is true)

$$\leq |z_1| + \cdots + |z_{n-1}| + |z_n| + |z_{n+1}| \,,$$

by part (vi), and therefore $P(n+1)$ is true, as claimed. Hence, by induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

**Remark 1.1** Replacing $\zeta$ by $-\zeta$, the triangle inequality becomes

$$|z-\zeta| \le |z| + |\zeta|.$$

Now, $|z| = |z-0|$ and $|\zeta| = |\zeta - 0|$, and so the above inequality tells us that the distance between the pair of complex numbers $z$ and $\zeta$ is no greater than the the sum of the distances of each of $z$ and $\zeta$ from the origin. This is just the statement that if we form the triangle with vertices 0, $z$ and $\zeta$, then the length of the side joining $z$ and $\zeta$ is never longer than the sum of the other two sides—hence the name "triangle inequality".

For any complex numbers $u$, $v$, $w$, we see that

$$|u-w| = |(u-v)+(v-w)| \le |u-v| + |v-w|,$$

so there is nothing special about the origin in the above discussion—it works for any triangle.

Evidently, part (vi) is just a special but important case of part (vii), the generalized triangle inequality.

**Remark 1.2** For any $z, \zeta \in C$, we have $|\zeta| = |\zeta - z + z| \le |\zeta - z| + |z|$, giving $|\zeta| - |z| \le |z-\zeta|$. Interchanging $z$ and $\zeta$, we obtain the inequality $|z| - |\zeta| \le |\zeta - z| = |z-\zeta|$. Thus $-|z-\zeta| \le |z| - |\zeta| \le |z-\zeta|$, which can be written as

$$\big|\,|z| - |\zeta|\,\big| \le |z-\zeta|.$$

From this, we see that if two complex numbers $z$ and $\zeta$ are close (i.e., the distance between them, $|z-\zeta|$, is small) then they have nearly the same modulus. The converse, however, need not be true, for example, $i$ and $-i$ have the same modulus, namely 1, but $|\,i-(-i)\,| = |2i| = 2$.

**Example 1.2** If the complex numbers $u$ and $v$ are proportional, with positive constant of proportionality, then $u+v = u+ru = (1+r)u$ for some $r>0$. Evidently, $|u+v| = (1+r)\,|u| = |u| + |v|$. Geometrically, this is clear. The complex number $u+v$ is got by putting the vector $v$ onto the end of the vector $u$ in the complex plane. If $v = ru$, then $u$ and $v$ "line up" and the triangle with vertices 0, $u$ and $u+v$ collapses to a straight line.

Furthermore, if $v_1, \dots, v_m$ are each of the form $v_j = r_j u$, for some $r_j > 0$, then the vector $u + v_1 + v_2 + \cdots + v_m$ is got by placing parallel vectors end to end and so its length will be the sum of the parts,

$$|u + v_1 + v_2 + \cdots + v_m| = |u| + |v_1| + \cdots + |v_m|.$$

We shall see that the converse is also true, as one might expect.

First, we shall show that the equality $|z_1 + z_2| = |z_1| + |z_2|$ holds for non-zero complex numbers, $z_1$ and $z_2$, only if they are proportional (with positive constant of proportionality), that is, if and only if $z_2 = rz_1$ for some real number $r > 0$.

Indeed, as we have just discussed, if $z_2 = rz_1$, with $r > 0$, then the claimed equality holds.

Conversely, suppose that $|z_1 + z_2| = |z_1| + |z_2|$. Then

$$\big(|z_1| + |z_2|\big)^2 = |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\overline{z}_2)$$

and so $\mathrm{Re}(z_1\overline{z}_2) = |z_1|\,|z_2| = |z_1\overline{z}_2|$. It follows that $z_1\overline{z}_2$ has no imaginary part and so $z_1\overline{z}_2 = \mathrm{Re}(z_1\overline{z}_2) = |z_1\overline{z}_2|$. From this we see that $z_2 = r\,z_1$ where $r = |z_1\overline{z}_2| \,/\, |z_1|^2$.

Now, suppose that $z_k \neq 0$, for all $k = 1, \dots, n$, and that

$$\sum_{k=1}^{n} |z_k| = \Big|\sum_{k=1}^{n} z_k\Big|. \tag{$*$}$$

We wish to show that there are positive real numbers $r_2, \dots, r_n$ such that $z_j = r_j z_1$ for $j = 2, \dots, n$. Now, for any partition of the set $\{1, 2, \dots, n\}$ into two subsets $I$ and $J$, the equality $(*)$ implies that

$$\begin{aligned}
\sum_{k=1}^{n} |z_k| = \Big|\sum_{k=1}^{n} z_k\Big| &= \Big|\sum_{k\in I} z_k + \sum_{k\in J} z_k\Big| \\
&\le \Big|\sum_{k\in I} z_k\Big| + \Big|\sum_{k\in J} z_k\Big| \\
&\le \Big|\sum_{k\in I} z_k\Big| + \sum_{k\in J} |z_k| \\
&\le \sum_{k\in I} |z_k| + \sum_{k\in J} |z_k| = \sum_{k=1}^{n} |z_k|
\end{aligned}$$

and so $\big|\sum_{k\in I} z_k\big| = \sum_{k\in I} |z_k|$. Taking $I = \{1, j\}$ and applying the first part, the result follows.

Let $\zeta_0 = 0$ and $\zeta_j = \zeta_{j-1} + z_j$, for $j = 1, \dots, n$ and let $P$ be the polygon $[\zeta_0, \zeta_1] \cup \cdots \cup [\zeta_{n-1}, \zeta_n]$, where $[w, z]$ denotes the straight line segment from $w$ to $z$. Then the equality in question is the statement that the distance between the initial and final points of $P$ is equal to the sum of the lengths of its segments. This can only happen if the polygon stretches out into a straight line (and does not turn back on itself).

**Example 1.3** For given $z_1, z_2, z_3 \in \mathbb{C}$, what is the set

$$\{\, w \in \mathbb{C} : w = \alpha z_1 + \beta z_2 + \gamma z_3, \text{ some } \alpha, \beta, \gamma \in [0,1] \text{ with } \alpha + \beta + \gamma = 1 \,\}?$$

In fact, this set is the triangle (including its interior) with vertices $z_1, z_2, z_3$. To see this, first notice that we can write

$$w = \alpha z_1 + \beta z_2 + \gamma z_3 = (1-\gamma)\big((1-\mu)z_1 + \mu z_2\big) + \gamma z_3$$

where $\mu = \beta/(\alpha+\beta)$ (assuming that the denominator is not zero).] Now, the set $\{\, \zeta : \zeta = (1-\mu)z_1 + \mu z_2,\ 0 \le \mu \le 1 \,\}$ is just the line segment from $z_1$ to $z_2$. Let $\zeta_\mu = (1-\mu)z_1 + \mu z_2$ be some point on this line segment. The set $L_\mu = \{\, w : w = (1-\gamma)\zeta_\mu + \gamma z_3,\ 0 \le \gamma \le 1 \,\}$ is the line segment from $\zeta_\mu$ to $z_3$. As $\mu$ varies between 0 and 1, so $\zeta_\mu$ varies along the line segment from $z_1$ to $z_2$ and the $L_\mu$s fill out the triangle.

## 1.4 The Argument of a Complex Number

We have agreed that a complex number can be usefully pictured as a point in the plane. Now, we can use polar coordinates rather than cartesian coordinates, giving the correspondences (assuming $z \ne 0$)

$$z = x + iy \quad \longleftrightarrow \quad (x,y) \quad \longleftrightarrow \quad (r,\theta),$$

where $r = \sqrt{x^2+y^2} = |z|$, and where $\theta$ is given by the pair of equations

$$\cos\theta = \frac{x}{r} = \frac{x}{|z|} = \frac{\operatorname{Re} z}{|z|}$$

and

$$\sin\theta = \frac{y}{r} = \frac{y}{|z|} = \frac{\operatorname{Im} z}{|z|}\,.$$

The value of $\theta$ is determined only to within additive multiples of $2\pi$, that is, if $\theta$ satisfies both $\cos\theta = x/r$ and $\sin\theta = y/r$ then so does $\theta + 2k\pi$, for any $k \in \mathbb{Z}$. Moreover, these are the only possibilities: if $\psi$ also satisfies $\cos\psi = x/r$ and $\sin\psi = y/r$ then $\psi = \theta + 2n\pi$ for some suitable $n \in \mathbb{Z}$.

The angle $\theta$ is called the argument of $z$, denoted $\arg z$. Note that according to the above discussion, $\arg z$ is not well-defined. One could call $\arg z$ *an* argument of $z$, i.e., a solution to $\cos\theta = x/r$ and $\sin\theta = y/r$, or one could define $\arg z$ to be the *set* of all such solutions, $\arg z = \{\, \theta, \theta \pm 2\pi, \theta \pm 4\pi, \cdots \}$,

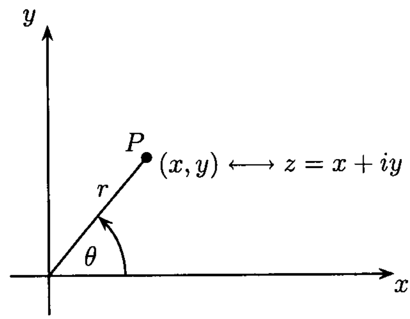


Fig. 1.2 Cartesian and polar coordinates.

where $\theta$ is any solution. We prefer the first idea, even though it is something of a nuisance.

By convention, we can pick on a particular choice. There is a unique solution $\theta$ satisfying $-\pi < \theta \leq \pi$; this choice of $\theta$ is called the principal value of the argument of the complex number $z$ and is denoted by $\operatorname{Arg} z$. Thus, for any $z \neq 0$, $\operatorname{Arg} z$ *is* well-defined and is uniquely determined by the requirement that $\operatorname{Arg} z = \theta \in (-\pi, \pi]$ and $\cos\theta = x/r$ and $\sin\theta = y/r$. For example, $\operatorname{Arg} x = 0$ for any real number $x$ with $x > 0$. If $x$ is real and $x < 0$, then $\operatorname{Arg} x = \pi$. Also $\operatorname{Arg} i = \pi/2$, $\operatorname{Arg}(-1) = \pi$, $\operatorname{Arg}(-i) = -\pi/2$.

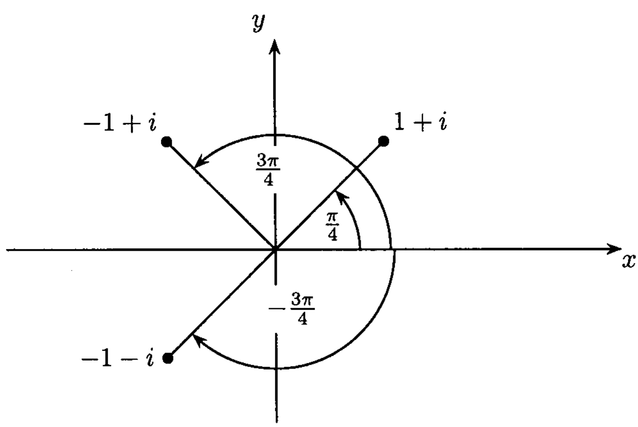


Fig. 1.3 $\operatorname{Arg}(1+i) = \frac{\pi}{4}$, $\operatorname{Arg}(-1+i) = \frac{3\pi}{4}$, $\operatorname{Arg}(-1-i) = -\frac{3\pi}{4}$.

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then, using the standard trigonometric formulae, we find that

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

and so $\theta_1 + \theta_2$ is a possible choice of argument for the product $z_1 z_2$, for any choices $\theta_1$ and $\theta_2$ of arguments for $z_1$ and $z_2$, respectively. In general,

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 + 2k\pi, \quad \text{for some } k \in \mathbb{Z},$$

where $\arg z_1$, $\arg z_2$ and $\arg(z_1 z_2)$ denote any particular choices of the arguments. Of course, different choices will lead to different values for $k$.

In particular, by induction, we obtain De Moivre's formula

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta\,.$$

**Remark 1.3** It is *not always* true that

$$\operatorname{Arg} z_1 z_2 = \operatorname{Arg} z_1 + \operatorname{Arg} z_2.$$

For example, $\operatorname{Arg}(-1) = \pi$ but $\operatorname{Arg}\big((-1)(-1)\big) = \operatorname{Arg} 1 = 0 \neq \pi + \pi$.

The ambiguity of the argument of a complex number reappears when we try to set up the notion of the logarithm of a complex number.

**Remark 1.4** Suppose that $z$ has polar coordinates $(r, \theta)$, $z \leftrightarrow (r, \theta)$. Then $z^n \leftrightarrow (r^n, n\theta)$. Multiplying complex numbers amounts to multiplying their moduli and adding their arguments. Furthermore, $1/z = \overline{z}/(z\overline{z}) = \overline{z}/|z|^2$, so that $1/z \leftrightarrow (1/r, \varphi)$, where $\varphi$ is an argument of $\overline{z}$. But $\overline{z} = r\cos\theta - ir\sin\theta = r(\cos(-\theta) + i\sin(-\theta))$ giving $\overline{z} \leftrightarrow (r, -\theta)$. Hence $1/z \leftrightarrow (1/r, -\theta)$. Dividing by a complex number amounts to dividing by the modulus and subtracting the angle; $z/w \leftrightarrow (r/\rho, \theta - \alpha)$, where $z \leftrightarrow (r, \theta)$ and $w \leftrightarrow (\rho, \alpha)$.

**Example 1.4** For any $w \neq 1$,

$$1 + w + w^2 + \cdots + w^{n-1} = \frac{1 - w^n}{1 - w}$$

(because $(1 - w)(1 + w + \cdots + w^{n-1}) = 1 - w^n$). Setting $w = \cos\theta + i\sin\theta$ (so that $w^k = \cos k\theta + i\sin k\theta \leftrightarrow (1, k\theta)$), we find that

$$S_n \equiv \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{\sin((n + \frac{1}{2})\theta) - \sin(\frac{\theta}{2})}{2\sin(\frac{\theta}{2})}$$

and that

$$T_n \equiv \sin\theta + \sin 2\theta + \cdots + \sin n\theta = \frac{\cos(\frac{\theta}{2}) - \cos((n+\frac{1}{2})\theta)}{2\sin(\frac{\theta}{2})}$$

provided $\theta \neq 2k\pi$ for any $k \in \mathbb{Z}$.

Indeed, if we let $\zeta = \cos(\frac{\theta}{2}) - i\sin(\frac{\theta}{2}) \leftrightarrow (1, -\frac{\theta}{2})$, then we have

$$\begin{aligned} S_n + iT_n &= \cos\theta + i\sin\theta + \cos 2\theta + i\sin 2\theta + \cdots + \cos n\theta + i\sin n\theta \\ &= w + w^2 + \cdots + w^n \\ &= \frac{w - w^{n+1}}{1-w} \\ &= \frac{(w^{n+1} - w)\,\zeta}{(w-1)\,\zeta} \\ &= \frac{w^{n+1}\,\zeta - w\,\zeta}{w\,\zeta - \zeta} \\ &= \frac{\cos((n+\frac{1}{2})\theta) + i\sin((n+\frac{1}{2})\theta) - \cos(\frac{\theta}{2}) - i\sin(\frac{\theta}{2})}{2\,i\sin(\frac{\theta}{2})}\,. \end{aligned}$$

Equating real and imaginary parts gives the required formulae.

Returning to the general discussion, we have declared a complex number to be one of the form $z = x + iy$, where $i^2 = -1$, and have so far accepted this uncritically. The question is "is this legitimate?" What is this magical number which we denote by $i$? It is certainly not a real number, so what is it? Now watch closely:

$$\begin{aligned} & \frac{(-1)}{1} = -1 = \frac{1}{(-1)} \\ \Longrightarrow\quad & \sqrt{\frac{-1}{1}} = \sqrt{\frac{1}{-1}} \\ \Longrightarrow\quad & \frac{\sqrt{-1}}{\sqrt{1}} = \frac{\sqrt{1}}{\sqrt{-1}} \\ \Longrightarrow\quad & \sqrt{-1}\,\sqrt{-1} = \sqrt{1}\,\sqrt{1} \\ \Longrightarrow\quad & -1 = 1, \end{aligned}$$

which is some cause for concern.

A similar but somewhat less picturesque "observation" is that

$$\sqrt{10} = \sqrt{(-5)(-2)} = \sqrt{-5}\,\sqrt{-2} = i\,\sqrt{5}\;\, i\,\sqrt{2} = i^2\,\sqrt{10} = -\sqrt{10}.$$

Maybe we should take a little more care in setting up the notion of a complex number. Happily, this can be done, as we will now see (but the paradoxes above must wait until we discuss complex powers).

## 1.5 Formal Construction of Complex Numbers

We define $\mathbb{C}$ to be the set of ordered pairs $(x, y)$, with $x, y \in \mathbb{R}$, together with the binary operations of "addition" (denoted $+$) and "multiplication" (denoted $\cdot$ ) given, respectively, by

$$(a, b) + (c, d) = (a + c, b + d)$$

and

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc).$$

(Secretly, we think of $(a, b)$ and $(c, d)$ as being $a + ib$ and $c + id$. Then the addition and multiplication laws above are the obvious ones.)

**Proposition 1.4** *The set $\mathbb{C}$ equipped with these operations is a field in which $(0, 0)$ is the identity for addition and $(1, 0)$ is the identity for multiplication.*

***Proof.*** It is clear from the definitions above that $(a, b) + (0, 0) = (a, b) = (0, 0) + (a, b)$ and that $(a, b) \cdot (1, 0) = (a, b) = (1, 0) \cdot (a, b)$. Also, $(-a, -b)$ is an additive inverse for $(a, b)$. Furthermore, provided $(a, b) \neq (0, 0)$, we see that $(c, d)$ is a multiplicative inverse for $(a, b)$ where $c = a/\sqrt{a^2 + b^2}$ and $d = -b/\sqrt{a^2 + b^2}$. (These are what we would expect if $(a, b)$ is to somehow be a rigorous realization of the expression $a + ib$).

Straightforward computations, using the definitions of $+$ and $\cdot$ , show that, for any $(a, b)$, $(c, d)$, $(e, f) \in \mathbb{C}$,

$$\begin{aligned}
&(a, b) + (c, d) = (c, d) + (a, b), \quad (+ \text{ is commutative}),\\
&(a, b) + \big((c, d) + (e, f)\big) = \big((a, b) + (c, d)\big) + (e, f), \quad (+ \text{ is associative}),\\
&(a, b) \cdot (c, d) = (c, d) \cdot (a, b), \quad (\cdot \text{ is commutative}),\\
&(a, b) \cdot \big((c, d) \cdot (e, f)\big) = \big((a, b) \cdot (c, d)\big) \cdot (e, f), \quad (\cdot \text{ is associative}),\\
&(a, b) \cdot \big((c, d) + (e, f)\big) = (a, b) \cdot (c, d) + (a, b) \cdot (e, f),\\
&\qquad\qquad\qquad\qquad (\cdot \text{ is distributive over } +)
\end{aligned}$$

Thus, $\mathbb{C}$ is a field, as claimed. ☐

**Proposition 1.5** *The set $\mathbb{F} = \{(a,0) : a \in \mathbb{R}\}$ is a subfield of $\mathbb{C}$ and the map $\phi : \mathbb{R} \to \mathbb{C}$ given by $\phi : a \mapsto (a,0)$ is a field isomorphism of $\mathbb{R}$ onto $\mathbb{F}$.*

***Proof.*** From the definitions, we see that $(a,0)+(b,0) = (a+b,0)$ and that $(a,0) \cdot (b,0) = (ab,0)$, for any $(a,0),(b,0) \in \mathbb{F}$. Furthermore, the additive inverse of $(a,0)$ is $(-a,0) \in \mathbb{F}$ and, if $a \neq 0$, the multiplicative inverse is $(1/a,0) \in \mathbb{F}$. It follows that $\mathbb{F}$ is a subfield of $\mathbb{C}$.

Next, we note that $\phi(a+b) = (a+b,0) = (a,0)+(b,0) = \phi(a)+\phi(b)$ and $\phi(ab) = (ab,0) = (a,0) \cdot (b,0) = \phi(a) \cdot \phi(b)$, $\phi(0) = (0,0)$ and $\phi(1) = (1,0)$ and so $\phi$ is a homomorphism with respect to both operations $+$ and $\cdot$.

Finally, we observe that $(a,0) = \phi(a)$ and so $\phi$ maps $\mathbb{R}$ onto $\mathbb{F}$, and if $\phi(a) = \phi(b)$, then $(a,0) = (b,0)$ and therefore $a = b$. Hence $\phi$ maps $\mathbb{R}$ one-one onto $\mathbb{F}$ and is a field isomorphism. □

This means that $\mathbb{F}$ and $\mathbb{R}$ are "the same", that is, $\mathbb{R}$ can be embedded in $\mathbb{C}$ as $\mathbb{F}$. This is just the formal proof that the "real line" is still the "real line" when we consider it as the $x$-axis of the complex plane. This is not an entirely vacuous statement because we are also considering the additive and multiplicative structures involved. (The plane is more naturally considered as a linear space, so that addition is natural but multiplication is a little special. In fact, it can be shown that $\mathbb{R}^n$ (with $n > 1$) can be given a multiplication making it into a field only for $n = 2$, in which case the multiplication is as above.)

Now, any $(a,b) \in \mathbb{C}$ can be written as

$$\begin{aligned}(a,b) &= (a,0) + (0,b) = (a,0) + (0,1) \cdot (b,0) \\ &= \phi(a) + (0,1) \cdot \phi(b) = \phi(a) + i\,\phi(b) \\ &= a + ib\end{aligned}$$

where we have dropped the isomorphism notation $\phi$ by writing $\phi(x)$ as just $x$, for any $x \in \mathbb{R}$. Also, we have set $i = (0,1)$, and we have dropped the $\cdot$, denoting multiplication merely by juxtaposition, as usual. Thus, with this new streamlined notation, any complex number has the form $a+ib$, with $a$, $b \in \mathbb{R}$, and where $i$ satisfies $i^2 = (0,1) \cdot (0,1) = (-1,0) = \phi(-1) = -1$, i.e., $i^2 = -1$. We have therefore given substance to the hopeful but vague idea of "a number of the form $x + iy$, with $x$, $y \in \mathbb{R}$, and with $i^2 = -1$", and have recovered our original formulae for addition and multiplication. The complex numbers are well-defined—they form a field and contain the set of real numbers $\mathbb{R}$ as a subfield.

**What's going on?** We might worry about simply asserting that $i^2 = -1$ without having said precisely what $i$ was in the first place. However, it turns out to be all right. We can just go ahead and write any complex number as $a + ib$, where $i^2 = -1$ and not worry. It can all be justified. (But we still need to sort out square roots.)

**Remark 1.5** The set of real numbers has a notion of order, defined in terms of positivity. For any real number $x$, precisely one of the following three statements is true; $x = 0$, $x > 0$, $-x > 0$. A property of positivity is that if $x > 0$ and $y > 0$, then $xy > 0$. It follows that $1 > 0$. (To see this, first we note that, clearly, $1 \neq 0$ (otherwise, $x = 1\,x$ would be $0$ for all $x \in \mathbb{R}$). Hence either $1 > 0$ or $-1 > 0$, but not both. If $-1 > 0$ were true, then we would have $(-1)(-1) > 0$. But $(-1)(-1) = 1^2 = 1$ so that also $1 > 0$. Both $-1 > 0$ and $1 > 0$ is not allowed, so we conclude that $-1 > 0$ is false and therefore $1 > 0$.)

Is there such a notion for complex numbers which extends that for the real numbers? If this were possible then, for example, we would have either $i > 0$ or $-i > 0$ (since $i \neq 0$). If $i > 0$ were true, we would have $-1 = i^2 > 0$, which is false. Hence $-i > 0$ must be true. But then, again, this would imply $-1 = (-i)^2 > 0$, which is false. We must concede that there is no generalization of positivity extending from the real numbers to the set of complex numbers.

For real numbers $x$ and $y$, the inequality $x > y$ is just a way of writing $x - y > 0$. This latter does not make sense, in general, for complex numbers, so it follows that inequalities, such as $z > \zeta$, do not make sense for complex numbers.

**What's going on?** Try as we might, we cannot make (useful) sense of inequalities between complex numbers.

## 1.6 The Riemann Sphere and the Extended Complex Plane

Let $S^2$ denote the sphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ in $\mathbb{R}^3$ and let $N$ denote the point $(0, 0, 1)$, the "north pole" of $S^2$. Think of $\mathbb{C}$ as the plane $\{(x, y, z) : z = 0\}$, containing the equator of $S^2$. Then given any point $P$ in this plane, the straight line through $P$ and $N$ cuts the sphere $S^2$ in a unique point, $P'$, say. As $P$ varies over the plane, the corresponding point $P'$ varies over $S^2 \setminus \{N\}$. This sets up a one-one correspondence between $\mathbb{C}$ and $S^2 \setminus \{N\}$.

We note that points far from the origin in $\mathbb{C}$ are mapped into points near the north pole (and points close to the origin are mapped into points close to the south pole of $S^2$, i.e., the point $(0,0,-1)$). Notice too that if $(P_n)$ is a sequence of points in $\mathbb{C}$ which converges to some point $P$ in $\mathbb{C}$, then the images $P'_n$ of $P_n$ converge in $S^2$ to the image $P'$ of $P$. We also see that if $(z_n)$ is a sequence in $\mathbb{C}$ such that $|z_n| \to \infty$, then the sequence $(P'_n)$ of their images converges to $N$ in $S^2$ (and vice versa). The point $N$ is called the "point at infinity".

The extended complex plane, $\mathbb{C}_\infty$, is defined to be $\mathbb{C}$ together with one additional element, that is, $\mathbb{C}_\infty \equiv \mathbb{C} \cup \{\infty\}$, where $\{\infty\}$ is a singleton set with $\infty \notin \mathbb{C}$. It does not matter what $\infty$ actually is, as long as it is not already a member of the set $\mathbb{C}$. For example, we could take $\infty$ to be $\varnothing$, which is certainly not a complex number. (Note that $a$ and $\{a\}$ are different mathematical objects, so, in particular, $\varnothing$ is not the same as $\{\varnothing\}$. Indeed, the objects $\{\varnothing\}$, $\{\varnothing, \{\varnothing\}\}$ and $\{\varnothing, \{\varnothing, \{\varnothing\}\}\}$ are different, as different as the numbers $1, 2, 3$.)

**What's going on?** The issue is how to augment a given set to give it just one new element. That is, given a set $A$, how does one construct a new set $B$ such that $B \setminus A$ is a singleton set? In the case above, $A = \mathbb{C}$ and $B = \mathbb{C}_\infty$ is the set we seek. If it does not matter what the new element is, as in the case here, then the explicit construction above is just one of many possibilities.

There is then a one-one correspondence between $\mathbb{C}_\infty$ and $S^2$ given by $\infty \longleftrightarrow N$ together with the correspondence between $\mathbb{C}$ and $S^2 \setminus \{N\}$, as introduced above. The extended complex plane, $\mathbb{C}_\infty$, viewed in this way is referred to as the Riemann sphere.

This gives a sensible realization of "infinity". For example, the mapping $z \mapsto 1/z$ is not *a priori* defined at $z = 0 \in \mathbb{C}$. However, if we consider the extended complex plane, or the Riemann sphere, then in addition to the mapping $z \mapsto 1/z$ for $z \in \mathbb{C} \setminus \{0\}$, we can define $0 \mapsto N$ and $N \mapsto 0$, (or, in more suggestive notation, $0 \mapsto \infty$ $(1/0 \equiv \infty)$ and $\infty \mapsto 0$ $(1/\infty \equiv 0)$). This defines the map $z \mapsto 1/z$ as a mapping from $\mathbb{C}_\infty \to \mathbb{C}_\infty$. This construction is reasonable in that if $z_n \to z$, then $1/z_n \to 1/z$ even if $z$ or any $z_n$ is equal to $0$ or to $\infty$.

The point here is to notice that by studying $\mathbb{C}_\infty$, rather than just $\mathbb{C}$, we can sometimes handle singularities just as ordinary points—after all, one point on a sphere is much the same as any other.

In real analysis, one considers the limits $x \to \infty$ and $x \to -\infty$. Whilst it must be stressed at the outset that this is just shorthand symbolism, nevertheless, it does invoke a kind of image of two infinities—one positive and the other negative. In view of the picture of complex numbers as points in the plane, one might wonder if it might be worth considering some kind of collection of "complex infinities", each being somewhere off in some given direction (perhaps corresponding to some "end of the rainbow" at the "end" of the ray $r(\cos\theta + i\sin\theta)$ as $r$ becomes very large). The view of $\mathbb{C}$ as being wrapped around a sphere, as developed above, suggests that we can bundle all these "infinities" into just a single "point at infinity", namely, the north pole.

It should be stressed that whilst $\mathbb{C}$ is a field (so one can do arithmetic), this is no longer true of $\mathbb{C}_\infty$. There is no attempt to assign any meaning whatsoever to expressions such as $\infty + \infty$ or $0 \times \infty$. The operations of addition and multiplication are simply not directly applicable when $\infty$ is involved.