CHAPTER 1

Observer-Based Solution to the Strictly Positive Real Problem

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We study the extension of the class of linear time-invariant plants that may be transformed into SPR systems introducing an observer. It is shown that for open-loop stable systems a cascaded observer achieves the result. For open-loop unstable systems an observer-based feedback is required to succeed. In general any stabilizable and observable system may be transformed into an SPR system using cascade or cascade and feedback controllers. This overcomes the old conditions of minimum phase and relative degree one. The result is illustrated with some examples.

1. Introduction

The celebrated Kalman–Yakubovich–Popov (KYP) Lemma, also called Positive Real Lemma, gives algebraic equations for a square transfer matrix $Z(s)$ to be Strictly Positive Real (SPR). These algebraic equations are equivalent to an analytic condition in the frequency domain which is not easy to test. The solution of these algebraic equations provides a practical way to verify that a given transfer function is SPR.

The statement of the KYP Lemma is due to Popov who proposed the problem [18], although he did not propose a solution. Popov also introduced a Lyapunov function composed of the sum of a quadratic term and an integral term to solve the absolute stability problem for a class of nonlinear systems. Later, Yakubovich [23] established the equations and Kalman [9] further elaborated these results. Anderson [3] established the MIMO version of the KYP Lemma.

The standard assumptions on the state-space representation for the KYP Lemma are: a) minimality, b) relative degree one, c) minimum phase and d) the system must be square, *i.e.* the number of inputs and outputs should be the same.

It had been long recognized that the minimality condition may be weakened to simply stabilizability and observability. Indeed, Meyer [14] pointed at this relaxation of the minimality assumption explicitly, but they did not provide a proof for that statement. Implicitly, Rantzer [19], in his novel proof based on convexity properties and linear algebra, does not require minimality of the state space representation of $Z(s)$. It was not until recently [6] that the minimality relaxation was explicitly proved in an algebraic approach. It has also been pointed out that a state-space representation can have uncontrollable modes and still satisfy the conditions to be SPR provided that the uncontrollable modes are stable. Previous reviewers of this work pointed out that a proof was available in Yakubovich *et al.* [24], but in English it was published very recently. Other interesting properties of SPR systems and comparisons with other related results are presented in [2,10,13,22].

This chapter presents a technique to render SPR any stabilizable and observable linear time-invariant system. The technique is based on a state observer and a feedback control law using the estimate of the state. Molander and Willems [15] solved the problem using state feedback under the assumption that the original system has relative degree one and is minimum-phase. In the context of nonlinear systems, Byrnes *et al.* [5] presented a solution to the problem using smooth state feedback provided that the system has relative degree one and is (weakly) minimum-phase. Furthermore the works of Kokotović *et al.* [11,12], address the problem associated with the stabilization of a linear system in cascade with a globally asymptotically stable nonlinear system. The proposed solutions also require the system to be weakly minimum-phase and have relative degree one. Another interesting solution has been presented by Sun *et al.* [21] based on output feedback. They establish conditions to render the system Extended SPR (ESPR). This definition requires relative degree zero which means $D + D^T > 0$.

Some approaches have been proposed to overcome the condition on relative degree one. Barkana introduced a "parallel feedforward" in the context of adaptive control [4]. Another related idea is passification by means of shunting introduced by Fradkov and coworkers [1]. These two approaches represent derivation of a loop-transfer function with SPR properties for a control object without SPR properties by means of dynamic extensions or observers.

The present chapter addresses the problem of designing an observer and a controller so that the modified system becomes SPR. Since any LTI system with a state observer is not minimal, few studies have been made in the past [8] to define an output for nonminimal system in attempts to obtain SPR systems. The proposed method is described for both stable plants and unstable plants. In the case of stable plants the method reduces to an observer and defining a new output as a function of the estimated state. The new output has to satisfy an algebraic equation. In the unstable case we have in addition to introduce an estimated state feedback controller. The proposed approach does not require the original system to be minimum-phase nor to have relative degree one.

The chapter is organized as follows: Section 2 presents some definitions, Section 3 deals with the open-loop stable case, while the open-loop unstable case is addressed in Section 4. Some illustrative examples are given in Section 5 and concluding remarks are presented in Section 6.

2. Preliminaries

Consider a linear time-invariant m-inputs p-outputs system with transfer matrix $Z(s)$ and with a minimal realization given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $m \leq n$, $p \leq n$, and A, B, C are matrices of appropriate dimensions. Denote by $\mathbb{C}, \mathbb{C}_-$ and $\overset{\circ}{\mathbb{C}}_-$, the complex plane, the closed left complex plane and the open left complex plane, respectively. Denote by $\sigma(T)$ the set of eigenvalues of the square matrix T and let $\mathbb{R}_+$ represent the set of positive real numbers.

Definition 1: [2,16] The transfer matrix $Z(s)$ is said to be PR if:

(i) All elements of $Z(s)$ are analytical in $\mathrm{Re}[s] > 0$, and
(ii) $Z(s) + Z^T(-s) \geq 0$ for all $\mathrm{Re}[s] > 0$.

Z(s) is said to be SPR if $Z(s - \varepsilon)$ is PR for some $\varepsilon > 0$.

For the scalar case, $m = p = 1$, the classical interpretation of $Z(s)$ being PR (SPR) is that its Nyquist plot lies entirely in the right complex plane (open right complex plane).

We will need in the sequel the following version of the KYP Lemma for strictly proper systems.

Lemma 1: *Let $Z(s) = C(sI - A)^{-1}B$ be a $m \times m$ transfer matrix such that $Z(s) + Z^T(-s)$ has normal rank m, where A is Hurwitz, (A, B) is stabilizable, and (C, A) is observable. Then $Z(s)$ is Strictly Positive Real (SPR) if and only if there exist symmetric positive definite matrices P and Q such that*

$$\begin{aligned} PA + A^T P &= -Q \\ PB &= C^T. \end{aligned}$$

3. Stable Case

Let us consider a linear time-invariant system described in standard state-space equations [20] as:

$$\Sigma_1 \begin{cases} \dot{x} = Ax + Bu \\ y = Cx. \end{cases} \tag{1.2}$$

Assumption 1: The A matrix is stable [10,20], *i.e.* $\sigma(A) \subset \mathring{\mathbb{C}}_-$.

A full-order observer for the system Σ_1 is given by

$$\widehat{\Sigma} \begin{cases} \dot{\widehat{x}} = A\widehat{x} + Bu + LC\,(x - \widehat{x}) \\ z = M\widehat{x} \end{cases} \tag{1.3}$$

where the observer gain matrix L is such that

$$\sigma(A - LC) \subset \mathring{\mathbb{C}}_-. \tag{1.4}$$

The system (1.2) and the observer (1.3) may be written compactly as:

$$\Sigma_{1+\text{obs}} \left\{ \begin{bmatrix} \dot{x} \\ \dot{\widehat{x}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ LC & A - LC \end{bmatrix} \begin{bmatrix} x \\ \widehat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} u. \right. \tag{1.5}$$

Introducing the state estimation error as $\widetilde{x} \triangleq \widehat{x} - x$, the system $\Sigma_{1+\text{obs}}$ may be expressed as

$$\Sigma_0 \left\{ \begin{bmatrix} \dot{x} \\ \dot{\widetilde{x}} \end{bmatrix} = A_0 \begin{bmatrix} x \\ \widetilde{x} \end{bmatrix} + B_0 u \right. \tag{1.6}$$

where

$$A_0 \triangleq \begin{bmatrix} A & 0 \\ 0 & A - LC \end{bmatrix} \tag{1.7a}$$

and

$$B_0 \triangleq \begin{bmatrix} B \\ 0 \end{bmatrix}. \tag{1.7b}$$

Remark 1: The system Σ_0 is not minimal: all the modes associated to the block $(A - LC)$ are uncontrollable.

Since A and $A_L \triangleq A - LC$ are stable, then for all positive definite matrices Q_{11} and Q_{22}, there exist positive definite matrices P and P_L solving the Lyapunov equations

$$\begin{aligned} A^T P + PA &= -Q_{11} \\ A_L^T P_L + P_L A_L &= -Q_{22}. \end{aligned} \tag{1.8}$$

Now define

$$P_0 \triangleq \begin{bmatrix} P & P \\ P & \mu P_L \end{bmatrix} \tag{1.9}$$

where $\mu > 0$ will be determined later. Then

$$\begin{aligned} A_0^T P_0 + P_0 A_0 &= \begin{bmatrix} A & 0 \\ 0 & A - LC \end{bmatrix}^T \begin{bmatrix} P & P \\ P & \mu P_L \end{bmatrix} + \begin{bmatrix} P & P \\ P & \mu P_L \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & A - LC \end{bmatrix} \\ &= -Q_0. \end{aligned} \tag{1.10}$$

Note that block (1,1) corresponds to the first equation of (1.8), block (2,2) is a μ-scaled version of the second equation of (1.8) and the cross term is

$$A^T P + P(A - LC) = A^T P + PA - PLC = -Q_{11} - PLC.$$

Then

$$Q_0 = \begin{bmatrix} Q_{11} & Q_{11} + PLC \\ Q_{11} + C^T L^T P & \mu Q_{22} \end{bmatrix}. \tag{1.11}$$

The composite system Σ_0 (1.6) will satisfy the first equation of the KYP Lemma if $Q_0 > 0$ and $P_0 > 0$. Using the Schur complement formula [7], we obtain the following conditions for positive definiteness of Q_0 and P_0.

I) Conditions that guarantee that P_0 is positive definite are:

$$P_o > 0 \Longleftrightarrow \begin{cases} \textbf{I.1} & P > 0 \\ \textbf{I.2} & \mu P_L - P > 0. \end{cases} \tag{1.12}$$

Condition **I.1** in (1.12) is satisfied in view of the Lyapunov Eqs. (1.8). Using the fact that for any given $R = R^T$ and $W = W^T$, not necessarily of the same dimensions, the following relation holds:

$$\begin{bmatrix} I & 0 \\ V^T R^{-1} & I \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & W - V^T R^{-1} V \end{bmatrix} \begin{bmatrix} I & R^{-1} V \\ 0 & I \end{bmatrix} = \begin{bmatrix} R & V \\ V^T & W \end{bmatrix}, \tag{1.13}$$

condition **I.2** in (1.12) is obtained by selecting in the above $R = V = P$ and $W = \mu P_L$. Condition **I.2** can also be expressed as $\mu P_L > P$. Further reduction is possible[a] if we use the following theorem.

Theorem 1: [17] Let H_1, H_2 be Hermitian matrices of the same dimensions, at least one of them being positive definite, say $H_1 > 0$. Then there exists a nonsingular matrix M such that

(i) $M^* H_1 M = I$, and
(ii) $M^* H_2 M = \mathrm{diag}(\rho_1, \rho_2, \ldots, \rho_n)$

where $\rho_i \in \mathbb{R}$ are eigenvalues of $H_1^{-1} H_2$.

Remark 2: The previous result says that given two Hermitian matrices of the same dimensions with one positive definite, they may be simultaneously diagonalized by means of a congruence transformation.

Applying Theorem 1 for $H_1 = \mu P_L$ and $H_2 = P$, the inequality $\mu P_L > P$ becomes $\mu I = M^* (\mu P_L) M > M^* P M = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_i \in \mathbb{R}_+$ are eigenvalues of $P_L^{-1} P$. The condition **I.2** becomes

$$\mu I > \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{1.14}$$

this condition being satisfied if

$$\mu > \mu_1 \triangleq \max_i \lambda_i. \tag{1.15}$$

II) Conditions that guarantee positive definiteness of Q_0 are:

$$Q_0 > 0 \iff \begin{cases} \textbf{II.1} \;\; Q_{11} > 0 \\ \textbf{II.2} \;\; \mu Q_{22} > \left(Q_{11} + C^T L^T P\right) Q_{11}^{-1} \left(Q_{11} + PLC\right). \end{cases} \tag{1.16}$$

Using Theorem 1, condition **II.2** may be reduced to a similar form as condition **I.2**. If we define

$$F \triangleq Q_{22}^{-1} \left(Q_{11} + C^T L^T P\right) Q_{11}^{-1} \left(Q_{11} + PLC\right)$$

having spectrum

$$\sigma(Q_{22}^{-1} \left(Q_{11} + C^T L^T P\right) Q_{11}^{-1} \left(Q_{11} + PLC\right)) = \{\nu_1, \nu_2, \ldots, \nu_n\} \in \mathbb{R}_+$$

then **II.2** is equivalent to

$$\mu > \mu_2 \triangleq \max_i \nu_i. \tag{1.17}$$

[a] Acknowledgment to Prof. V. Kharitonov who pointed out this simplification.

Combining conditions **I** and **II**, P_0 and Q_0 are positive definite if (see Eqs. (1.15) and (1.17))

$$\mu > \mu^* \triangleq \max\{\mu_1, \mu_2\}. \tag{1.18}$$

Remark 3: μ can always be chosen to satisfy the above inequality. Also note that the bound is tight, *i.e.* if $\mu = \mu^*$ then we can only guarantee $P_0 > 0$ or $Q_0 > 0$, but not both conditions.

We have proved the first part of the following theorem.

Theorem 2: Consider the stable transfer matrix $Z(s)$ with m-inputs and p-outputs and its state-space realization

$$\Sigma_1 \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

where A is stable, the pair (A, B) is stabilizable and the pair (C, A) is observable. Then there exists an observer gain L given in (1.3) satisfying (1.4) such that the transfer matrix between u and the new output

$$z = M_0 \begin{bmatrix} x \\ \widetilde{x} \end{bmatrix} = M\widehat{x}, \qquad M = B^T P$$

is characterized by a state-space representation (A_0, B_0, M_0) which is SPR.

Proof: The proof of the first equation of the KYP Lemma is already done provided that $\mu > \mu^*$. If the new output z is defined as

$$\begin{aligned} z &= M_0 \begin{bmatrix} x \\ \widetilde{x} \end{bmatrix} = B_0^T P_0 \begin{bmatrix} x \\ \widetilde{x} \end{bmatrix} \\ &= \begin{bmatrix} B^T & 0 \end{bmatrix} \begin{bmatrix} P & P \\ P & \mu P_L \end{bmatrix} \begin{bmatrix} x \\ \widehat{x} - x \end{bmatrix} \\ &= B^T P x + B^T P (\widehat{x} - x) \\ &= B^T P \widehat{x} = M\widehat{x}, \end{aligned}$$

then the composed system (A_0, B_0, M_0), which is not minimal, satisfies the KYP Lemma equations, *i.e.*

$$A_0^T P_0 + P_0 A_0 = -Q_0 \quad \text{and} \quad M_0 = B_0^T P_0 = \begin{bmatrix} B^T P & B^T P \end{bmatrix}.$$

Not only the equation

$$z = M_0 \begin{bmatrix} x \\ \widetilde{x} \end{bmatrix} = \underbrace{B^T P}_{M} \widehat{x}$$

satisfies the second equation of the KYP Lemma, but also provides the output z as a function of the estimated state $\widehat{x}$, which is required to be implementable. □

The system (1.2) and the observer (1.3) can be combined to obtain the composite system (1.6). Note that A_0 in Eqs. (1.6) and (1.7a) satisfies the Lyapunov Eq. (1.10) where P_0 and Q_0 are positive definite. Therefore, if the new output z is defined as $z = M\widehat{x}$, the transfer function from u to z is SPR.

The diagram in Figure 1.1 shows the cascade compensator for the stable case.

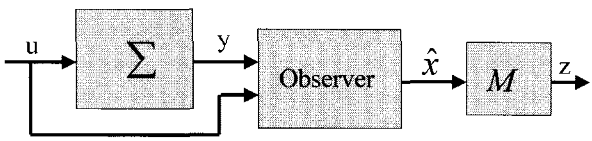

Figure 1.1. SPR transformation for open-loop stable systems.

4. Unstable Case

In this section we will remove Assumption 1 concerning the stability of the system. We will assume that system (1.2) is not stable, *i.e.* $\sigma(A) \nsubseteq \overset{\circ}{\mathbb{C}}_-$. Introduce a stabilizing control law based on the observer (1.3)

$$u = -K\widehat{x} + v \tag{1.19}$$

where v is a new input signal. The composed system becomes

$$\Sigma_K \left\{ \begin{bmatrix} \dot{x} \\ \dot{\widetilde{x}} \end{bmatrix} = \underbrace{\begin{bmatrix} A - BK & -BK \\ 0 & A - LC \end{bmatrix}}_{A_0} \begin{bmatrix} x \\ \widetilde{x} \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B_0} v. \right.$$

Introduce the short hand notation $A_K = A - BK$ and $A_L = A - LC$. Again K and L are such that $\sigma(A - BK) \subset \overset{\circ}{\mathbb{C}}_-$ and $\sigma(A - LC) \subset \overset{\circ}{\mathbb{C}}_-$. Then for every positive definite Q_K and Q_L there exist $P_K > 0$ and $P_L > 0$ solving the Lyapunov equations

$$\begin{aligned} A_K^T P_K + P_K A_K &= -Q_K \\ A_L^T P_L + P_L A_L &= -Q_L. \end{aligned} \tag{1.20}$$

Define P_0 as in the stable case, *i.e.*

$$P_0 \triangleq \begin{bmatrix} P_K & P_K \\ P_K & \mu P_L \end{bmatrix}.$$

Then the block diagonal elements of the equation

$$A_0^T P_0 + P_0 A_0 = -Q_0$$

correspond to Eqs. (1.20). The off-diagonal block is

$$\begin{aligned} [-Q_0]_{1,2} &= -P_K BK + P_K A_L + A_K^T P_K \\ &= P_K (A - BK) + (A - BK)^T P_K - P_K LC \\ &= -Q_K - P_K LC. \end{aligned}$$

Obviously $[-Q_0]_{2,1} = [-Q_0]_{1,2}^T$. For stability of the feedback system it is required that $P_0 > 0$ and $Q_0 > 0$.

III) Conditions for positive definiteness of P_0:

$$P_0 > 0 \Longleftrightarrow \begin{cases} \textbf{III.1} \;\; P_K > 0 \\ \textbf{III.2} \;\; \mu P_L - P_K > 0. \end{cases}$$

Condition **III.1** is satisfied in view of the Lyapunov Eqs. (1.20). Let the spectrum of $P_L^{-1} P_K$ be $\{\eta_1, \eta_2, \ldots, \eta_n\} \in \mathbb{R}_+$. We then have that condition **III.2** is equivalent to

$$\mu > \mu_3 \triangleq \max_i \eta_i. \tag{1.21}$$

IV) For positive definiteness of Q_0 we require:

$$Q_0 > 0 \Longleftrightarrow \begin{cases} \textbf{IV.1} \;\; Q_{11} > 0 \\ \textbf{IV.2} \;\; \mu Q_L - \left(Q_K + C^T L^T P_K\right) Q_K^{-1} \left(Q_K + P_K LC\right) > 0. \end{cases}$$

Again let $\sigma(Q_L^{-1} \left(Q_K + C^T L^T P_K\right) Q_K^{-1} \left(Q_K + P_K LC\right)) = \{\rho_1, \ldots, \rho_n\} \in \mathbb{R}_+$, then condition **IV.2** is equivalent to

$$\mu > \mu_4 \triangleq \max_i \rho_i. \tag{1.22}$$

Combining (1.21) and (1.22), $P_0 > 0$ and $Q_0 > 0$ if and only if

$$\mu > \mu^* \triangleq \max \{\mu_3, \; \mu_4\}.$$

Now we may state the main result of this section.

Theorem 3: Given the strictly proper transfer matrix $Z(s)$ of dimensions $p \times m$ not identically zero, construct a stabilizable and observable realization (A, B, C). There exists a gain observer matrix L as in (1.3), an estimated state feedback K ($K = 0$ if A is Hurwitz), and a matrix M_0 which defines a new output such that the transfer matrix from v of Eq. (1.19) to the new output z is SPR.

Proof: The first equation of the KYP Lemma is just proved for sufficiently large μ, the second part is similar to the proof of Theorem 2 for stable systems. □

Remark 4: Notice that in either, stable or unstable cases, the Lyapunov equation $A_0^T P_0 + P_0 A_0 = -Q_0$ does not have a positive definite solution P_0 of the form given in (1.9) for all $Q_0 > 0$ of the form given in (1.11). We are imposing the structure on the matrix P_0, therefore there exist positive definite P_0 and Q_0 only for μ sufficiently large.

Figure 1.2 shows the structure of the compensator proposed for the unstable case.

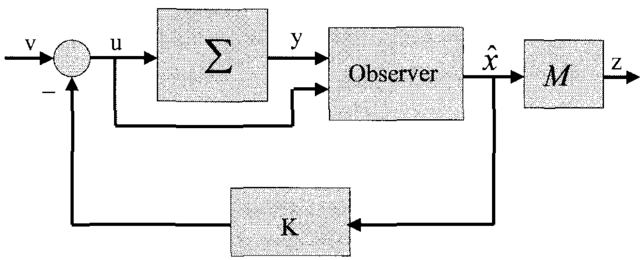

Figure 1.2. SPR transformation for open-loop unstable systems.

5. Illustrative Examples

In this section we will present three detailed examples. The first example is a relative degree two stable system, the second one presents a relative degree three unstable system and the third example is a nonminimum-phase unstable system with a nonminimal state-space representation. We construct a compensator which renders the new system SPR in the three cases.

Example 1: Consider the following transfer function

$$h(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)}$$

which has a minimal state-space representation

$$\Sigma_1 \begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x. \end{cases}$$

A full-order observer for Σ_1 with eigenvalues at $\{-3, -4\}$ is

$$\widehat{\Sigma}_1 \left\{ \dot{\widehat{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \widehat{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \underbrace{\begin{bmatrix} 4 \\ -2 \end{bmatrix}}_{L} \begin{bmatrix} 1 & 0 \end{bmatrix} (x - \widehat{x}) . \right.$$

If we choose $Q_{11} = 2I$, then the solution of the Lyapunov equation $A^T P + PA = -Q_{11}$ is

$$P = \begin{bmatrix} 2.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} > 0.$$

For $Q_{22} = 2I$ the solution of the Lyapunov equation $A_L^T P_L + P_L A_L = -Q_{22}$ is

$$P_L = \begin{bmatrix} 0.25 & 0.0357 \\ 0.0357 & 0.3452 \end{bmatrix} > 0.$$

For the computation of μ^* we require the values:

$$\sigma(P_L^{-1} P) = \{10.0272, 1.1729\}$$

$$\sigma(Q_{22}^{-1} \left(Q_{11} + C^T L^T P\right) Q_{11}^{-1} \left(Q_{11} + PLC\right)) = \{44.5517, 0.9483\}$$

$$\begin{aligned} \mu^* &= \max \{\mu_1, \mu_2\} \\ &= \max \left\{ \max_i \lambda_i, \max_i \nu_i \right\} \\ &= \max \{10.0272, 44.5517\} \\ &= 44.5517. \end{aligned}$$

If $\mu = 45 > \mu^*$, then

$$P_0 = \left[\begin{array}{cc|cc} 2.5 & 0.5 & 2.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ \hline 2.5 & 0.5 & 112.5 & 22.5 \\ 0.5 & 0.5 & 22.5 & 22.5 \end{array}\right]$$

$$\sigma(P_0) = \{0.3733, 2.5586, 17.1971, 117.871\}$$

$$Q_0 = \left[\begin{array}{cc|cc} 2 & 0 & 13 & 0 \\ 0 & 2 & 3 & 2 \\ \hline 13 & 3 & 90 & 0 \\ 0 & 2 & 0 & 90 \end{array}\right]$$

$$\sigma(Q_0) = \{0.0195, 1.9569, 90.0431, 91.9805\}.$$

The output system matrix becomes $M_0 = B_0^T P_0 = \begin{bmatrix} 0.5 \; 0.5 \; 0.5 \; 0.5 \end{bmatrix}$ which gives us $M = \begin{bmatrix} 0.5 \; 0.5 \end{bmatrix}$ and the transfer function between z and u is

$$H_{zu}(s) = \frac{0.5s + 0.5}{s^2 + 3s + 2} = \frac{0.5}{s+2}.$$

In this particular case, there was a cancellation in the final transfer function $H_{zu}(s)$, but this is not always the case. If we change the matrices Q_{11} and Q_{22}, we will get generically a second-order system in $H_{zu}(s)$.

Example 2: Consider the following transfer function

$$h(s) = \frac{1}{s^3 + 2s^2 - s - 2} = \frac{1}{(s+1)(s-1)(s+2)}$$

which has a minimal state-space representation

$$\Sigma_2 \begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 \; 0 \; 0 \end{bmatrix} x. \end{cases}$$

A full-order observer for Σ_2 with eigenvalues at $\{-2, -3, -4\}$ is

$$\widehat{\Sigma}_2 \left\{ \dot{\widehat{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} \widehat{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \underbrace{\begin{bmatrix} 7 \\ 13 \\ 7 \end{bmatrix}}_{L} \begin{bmatrix} 1 \; 0 \; 0 \end{bmatrix} (x - \widehat{x}). \right.$$

If we assign the closed-loop eigenvalues at $\{-1, -1 \pm j\}$ we get $K = \begin{bmatrix} 4 \; 5 \; 1 \end{bmatrix}$. Choosing $Q_K = 2I$, then the solution of the Lyapunov equation $A_K^T P_K + P_K A_K = -Q_K$ is

$$P_K = \begin{bmatrix} 3.9 & 2.8 & 0.5 \\ 2.8 & 4.95 & 0.95 \\ 0.5 & 0.95 & 0.65 \end{bmatrix} > 0.$$

For $Q_L = 2I$ the solution of the Lyapunov equation $A_L^T P_L + P_L A_L = -Q_L$ is

$$P_L = \begin{bmatrix} 3.0958 & -1.2583 & -0.8625 \\ -1.2583 & 0.7500 & 0.2583 \\ -0.8625 & 0.2583 & 0.6292 \end{bmatrix} > 0.$$

For the computation of μ^* we require the values:

$$\sigma(P_L^{-1} P_K) = \{0.4098, 1.1196, 44.7432\}$$

$$\sigma(Q_L^{-1}\left(Q_K + C^T L^T P_K\right) Q_K^{-1}\left(Q_K + P_K LC\right)) = \{0.35694, 1, 3353.933\}$$

$$\mu^* = \max\{\mu_3, \mu_4\} = \max\{44.7432, 3353.933\} = 3353.933.$$

Now setting $\mu = 3400 > \mu^*$ yields

$$P_0 = \left[\begin{array}{ccc|ccc} 3.9 & 2.8 & 0.5 & 3.9 & 2.8 & 0.5 \\ 2.8 & 4.95 & 0.95 & 2.8 & 4.95 & 0.95 \\ 0.5 & 0.95 & 0.65 & 0.5 & 0.95 & 0.65 \\ \hline 3.9 & 2.8 & 0.5 & 10525.83 & -4278.33 & -2935.00 \\ 2.8 & 4.95 & 0.95 & -4278.33 & 2550 & 878.33 \\ 0.5 & 0.95 & 0.65 & -2935.00 & 878.33 & 2139.16 \end{array}\right]$$

and its spectrum is $\sigma(P_0) = \{0.4447, 1.6181, 7.3383, 529.6, 1464.4, 13221\}$,

$$Q_0 = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 69.2 & 0 & 0 \\ 0 & 2 & 0 & 90.6 & 2 & 0 \\ 0 & 0 & 2 & 20.4 & 0 & 2 \\ \hline 69.2 & 90.6 & 20.4 & 6800 & 0 & 0 \\ 0 & 2 & 0 & 0 & 6800 & 0 \\ 0 & 0 & 2 & 0 & 0 & 6800 \end{array}\right]$$

and its spectrum is $\sigma(Q_0) = \{0.027, 1.9994, 1.9997, 6800, 6800, 6801.9\}$. The output matrix becomes $M_0 = \begin{bmatrix} 0.5 \; 0.95 \; 0.65 \; 0.5 \; 0.95 \; 0.65 \end{bmatrix}$ and $M = \begin{bmatrix} 0.5 \; 0.95 \; 0.65 \end{bmatrix}$. The transfer function from the new input v to the new output z becomes

$$H_{zv}(s) = \frac{0.65s^2 + 0.95s + 0.5}{s^3 + 3s^2 + 4s + 2} = \frac{0.65\,(s + 0.731 \pm 0.4846j)}{(s+1)\,(s+1 \pm j)}.$$

The corresponding Nyquist diagrams are shown in Figure 1.3.

Example 3: Consider the following transfer function

$$h(s) = \frac{s-2}{s^2 + s - 2} = \frac{s-2}{(s-1)\,(s+2)}$$

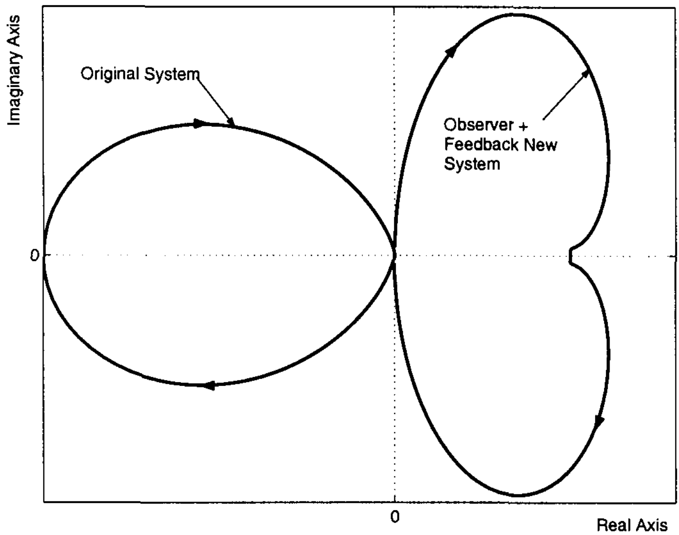

Figure 1.3. Nyquist diagrams for the second example.

which has a nonminimal, but stabilizable and observable state-space representation

$$\Sigma_3 \begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} -2 & 1 & 4 \end{bmatrix} x. \end{cases}$$

A full-order observer for Σ_3 with eigenvalues at $\{-2, -3, -4\}$ is

$$\widehat{\Sigma}_3 \left\{ \dot{\widehat{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix} \widehat{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u + \underbrace{\begin{bmatrix} -13 \\ -7 \\ -3 \end{bmatrix}}_{L} \begin{bmatrix} -2 & 1 & 4 \end{bmatrix} (x - \widehat{x}) . \right.$$

If we assign the closed-loop eigenvalues at $\{-1, -1 \pm j\}$ we get $K = \begin{bmatrix} 4 & 1 & -2 \end{bmatrix}$. Choosing $Q_K = 2I$, then the solution of the Lyapunov equation $A_K^T P_K + P_K A_K = -Q_K$ is

$$P_K = \begin{bmatrix} 2.5 & 0.5 & 0 \\ 0.5 & 0.75 & 0 \\ 0 & 0 & 1 \end{bmatrix} > 0.$$

For $Q_L = 2I$ the solution of the Lyapunov equation $A_L^T P_L + P_L A_L = -Q_L$ is

$$P_L = \begin{bmatrix} 3.7571 & -3.1286 & -9.8571 \\ -3.1286 & 3.2024 & 7.8619 \\ -9.8571 & 7.8619 & 27.9238 \end{bmatrix} > 0.$$

For the computation of μ^* we require the values:

$$\sigma(P_L^{-1} P_K) = \{0.0286, 0.6767, 20.1044\}$$

$$\sigma(Q_L^{-1}\left(Q_K + C^T L^T P_K\right) Q_K^{-1}\left(Q_K + P_K LC\right)) = \{0.0827, 1, 7626.24\}$$

$$\mu^* = \max\{\mu_3, \mu_4\} = \max\{20.1044, 7626.24\} = 7626.24.$$

Setting $\mu = 7700$ yields

$$P_0 = \left[\begin{array}{ccc|ccc} 2.5 & 0.5 & 0 & 2.5 & 0.5 & 0 \\ 0.5 & 0.75 & 0 & 0.5 & 0.75 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 2.5 & 0.5 & 0 & 28930 & -24090 & -75900 \\ 0.5 & 0.75 & 0 & -24090 & 24658 & 60536 \\ 0 & 0 & 1 & -75900 & 60536 & 21501 \end{array}\right]$$

and $\sigma(P_0) = \{0.6171, 0.9999, 2.626, 968.03, 7577.86, 260055.76\}$,

$$Q_0 = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 74 & 24 & 6 \\ 0 & 2 & 0 & -36 & -10 & 3 \\ 0 & 0 & 2 & -144 & -47 & -10 \\ \hline 74 & -36 & -144 & 15400 & 0 & 0 \\ 24 & -10 & -47 & 0 & 15400 & 0 \\ 6 & 3 & -10 & 0 & 0 & 15400 \end{array}\right]$$

and $\sigma(Q_0) = \{0.019, 1.9997, 1.9999, 15400, 15400, 15401.9874\}$. The output matrix becomes $M_0 = \begin{bmatrix} 0.5\ 0.75\ 0\ 0.5\ 0.75\ 0 \end{bmatrix}$ and $M = \begin{bmatrix} 0.5\ 0.75\ 0 \end{bmatrix}$. The transfer function from the new input v to the new output z becomes

$$H_{zv}(s) = \frac{0.75s + 0.5}{s^2 + 2s + 2} = \frac{0.75\,(s + 0.6667)}{(s + 1 \pm j)}.$$

The corresponding Nyquist diagrams are shown in Figure 1.4.

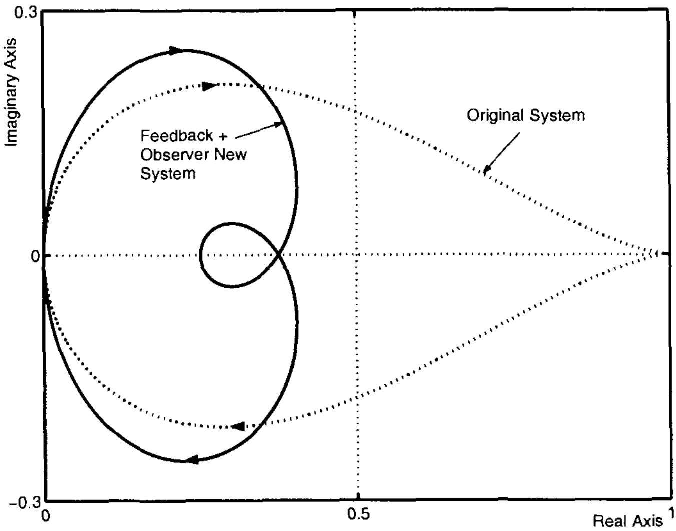

Figure 1.4. Nyquist diagrams for the third example.

6. Concluding Remarks

This chapter has presented a new approach to modify a linear time-invariant system so that the modified system is SPR. The proposed method applies to stable plants as well as unstable systems and does not require the system to be minimum-phase nor to have relative degree one. The original system may have a nonsquare transfer matrix, *i.e.* the number of inputs can be different from the number of outputs. We have proved that the Kalman–Yakubovich–Popov Lemma holds for a series connected system with an observer for stable open-loop systems. In the unstable case, an observer state feedback should be introduced in order to stabilize the system. Some examples failing the relative degree one or minimum phase conditions have been given to illustrate the procedure. Future work in this area includes studying the robustness of the proposed method with respect to parametric uncertainties.

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