Chapter 1

Mathematical Preliminaries

1.1 An Introduction to the Laplace Transform

Much of this chapter is devoted to describing and deriving some of the properties of the one-sided Laplace transform. The Laplace transform is the engineer's most important tool for analyzing the stability of linear, time-invariant, continuous-time systems. The Laplace transform is defined as:

$$\mathcal{L}(f(t))(s) \equiv \int_0^\infty e^{-st} f(t)\, dt.$$

We often write $F(s)$ for the Laplace transform of $f(t)$. It is customary to use lower-case letters for functions of time, t, and to use the same letter—but in its upper-case form—for the Laplace transform of the function; throughout this book, we follow this practice.

We assume that the functions $f(t)$ are of *exponential type*—that they satisfy an inequality of the form $|f(t)| \leq Ce^{\alpha t}, C \in \mathcal{R}$. If the real part of s, $\Re(s)$, satisfies $\Re(s) < -\alpha$, then the integral that defines the Laplace transform converges. The Laplace transform's usefulness comes largely from the fact that it allows us to convert differential and integro-differential equations into algebraic equations.

We now calculate the Laplace transform of some functions. We start with the unit step function (also known as the Heaviside [1] function):

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}.$$

[1] After Oliver Heaviside (1850-1925) who between 1880 and 1887 invented the "operational calculus" [OR]. His operational calculus was widely used in its time. The Laplace transform that is used today is a "cousin" of Heaviside's operational calculus[Dea97].

From the definition of the Laplace transform, we find that:

$$\begin{aligned}U(s) &= \mathcal{L}(u(t))(s)\\ &= \int_0^\infty e^{-st}\cdot 1\,dt\\ &= \left.\frac{e^{-st}}{-s}\right|_0^\infty\\ &= \lim_{t\to\infty}\frac{e^{-st}}{-s}-\frac{1}{-s}.\end{aligned}$$

Denote the real part of s by α and its imaginary part by β. Continuing our calculation, we find that:

$$\begin{aligned}U(s) &= \lim_{t\to\infty} e^{-\alpha t}\frac{e^{-j\beta t}}{-s}+\frac{1}{s}\\ &= 0+\frac{1}{s}=\frac{1}{s}.\end{aligned}$$

This holds as long as $\alpha > 0$. In this case the first term in the limit:

$$\lim_{t\to\infty} e^{-\alpha t}\frac{e^{-j\beta t}}{-s}$$

is approaching zero while the second term—though oscillatory—is bounded. In general, we assume that s is chosen so that integrals and limits that must converge do. For our purposes, the region of convergence (in terms of s) of the integral is not terribly important.

Next we consider $\mathcal{L}(e^{at})(s)$. We find that:

$$\begin{aligned}\mathcal{L}(e^{at})(s) &= \int_0^\infty e^{-st}e^{at}\,dt\\ &= \left.\frac{e^{(a-s)t}}{a-s}\right|_0^\infty\\ &= \frac{1}{s-a}.\end{aligned}$$

1.2 Properties of the Laplace Transform

The first property of the Laplace transform is its *linearity*.

Theorem 1

$$\mathcal{L}\left(\alpha f(t)+\beta g(t)\right)(s) = \alpha F(s)+\beta G(s).$$

Simply put, "the Laplace transform of a linear combination is the linear combination of the Laplace transforms."

PROOF: Making use of the properties of the integral, we find that:

$$\begin{aligned}\mathcal{L}\left(\alpha f(t)+\beta g(t)\right)(s) &= \int_0^\infty e^{-st}\left(\alpha f(t)+\beta g(t)\right)\,dt \\ &= \alpha\int_0^\infty e^{-st} f(t)\,dt + \beta\int_0^\infty e^{-st} g(t)\,dt \\ &= \alpha F(s)+\beta G(s).\end{aligned}$$

We see that the linearity of the Laplace transform is part of its "inheritance" from the integral which defines it.

The Laplace Transform of $\sin(t)$ I—An Example

Following the engineering convention that $j \equiv \sqrt{-1}$, we write:

$$\sin(t) = \frac{e^{jt}-e^{-jt}}{2j}.$$

By linearity we find that:

$$\mathcal{L}(\sin(t))(s) = \frac{1}{2j}\left(\mathcal{L}(e^{jt})(s) - \mathcal{L}(e^{-jt})(s)\right).$$

Making use of the fact that we know what the Laplace transform of an exponential is, we find that:

$$\mathcal{L}(\sin(t))(s) = \frac{1}{2j}\left(\frac{1}{s-j}-\frac{1}{s+j}\right) = \frac{1}{s^2+1}.$$

The next property we consider is the property that makes the Laplace transform transform so useful. As we shall see, it is possible to calculate the Laplace transform of the solution of a constant-coefficient ordinary differential equation (ODE) *without solving the ODE.*

Theorem 2 *Assume that $f(t)$ is has a well defined limit as t approaches zero from the right*[2]. *Then we find that:*

$$\mathcal{L}(f'(t))(s) = sF(s) - f(0^+).$$

PROOF: This result is proved by making use of integration by parts. We see that:

$$\mathcal{L}(f'(t))(s) = \int_0^\infty e^{-st} f'(t)\, dt.$$

Let $u = e^{-st}$ and $dv = f'(t)dt$. Then $du = -se^{-st}$ and $v = f(t)$. Assuming that $\alpha = \mathcal{R}(s) > 0$, we find that:

$$\begin{aligned}\int_0^\infty e^{-st} f'(t)\, dt &= -\int_0^\infty \frac{d}{dt} e^{-st} f(t)\, dt + e^{-st} f(t)|_{0+}^{\infty} \\ &= s\int_0^\infty e^{-st} f(t)\, dt + \lim_{t\to\infty} e^{-st} f(t) - f(0^+) \\ &= sF(s) + 0 - f(0^+) \\ &= sF(s) - f(0^+).\end{aligned}$$

We take the limit of $f(t)$ as $t \to 0^+$ because the integral itself deals only with positive values of t. Often we dispense with the added generality that the limit from the right gives us, and we write $f(0)$.

We can use this theorem to find the Laplace transform of the second (or higher) derivative of a function. To find the Laplace transform of the second derivative of a function, one applies the theorem twice. I.e.:

$$\begin{aligned}\mathcal{L}(f''(t))(s) &= s\mathcal{L}(f'(t))(s) - f'(0) \\ &= s(sF(s) - f(0)) - f'(0) \\ &= s^2F(s) - sf'(0) - f(0).\end{aligned}$$

The Laplace Transform of sin(t) II—An Example

[2]The limit of $f(t)$ as t tends to zero from the right is the value to which $f(t)$ tends as t approaches zero through the positive numbers. In many cases, we assume that $f(t) = 0$ for $t \leq 0$. Sometimes there is a jump in the value of the function at $t = 0$. As the zero value for $t \leq 0$ is often something we do not want to relate to, we sometimes consider only the limit from the right. The limit as one approaches a number, a, from the right is denoted by a^+. By convention $f(0^+) \equiv \lim_{t\to 0^+} f(t)$. Of course, if $f(t)$ is continuous at 0, then $f(0^+) = f(0)$.

We now calculate the Laplace transform of $\sin(t)$ a second way. Let $f(t) = \sin(t)$. Note that $f''(t) = -f(t)$ and that $f(0) = 0, f'(0) = 1$. We find that:

$$\begin{aligned}
\mathcal{L}(-\sin(t))(s) &= s^2\mathcal{L}(\sin(t))(s) - s0 - 1 \Leftrightarrow \\
-\mathcal{L}(\sin(t))(s) &= s^2\mathcal{L}(\sin(t))(s) - 1 \Leftrightarrow \\
(s^2+1)\mathcal{L}(\sin(t))(s) &= 1 \Leftrightarrow \\
\mathcal{L}(\sin(t))(s) &= \frac{1}{s^2+1}.
\end{aligned}$$

The Laplace Transform of $\cos(t)$—An Example

From the fact that $\cos(t) = (\sin(t))'$ and that $\sin(0) = 0$, we see that:

$$\mathcal{L}(\cos(t))(s) = s\mathcal{L}(\sin(t))(s) - 0 = \frac{s}{s^2+1}.$$

An easy corollary of Theorem 2 is:

Corollary 3 $\mathcal{L}\left(\int_0^t f(y)\,dy\right)(s) = \frac{F(s)}{s}$.

PROOF: Let $g(t) = \int_0^t f(y)\,dy$. Clearly, $g(0) = 0$, and $g'(t) = f(t)$. From Theorem 2 we see that $\mathcal{L}(g'(t))(s) = s\mathcal{L}(g(t))(s) - 0 = \mathcal{L}(f(t))(s)$. We find that $\mathcal{L}(\int_0^t f(y)\,dy) = F(s)/s$.

We have seen how to calculate the transform of the derivative of a function; the transform of the derivative is s times the transform of the original function less a constant. We now show that the derivative of the transform of a function is the transform of $-t$ times the original function. By linearity this is identical to:

Theorem 4

$$\mathcal{L}(tf(t))(s) = -\frac{d}{ds}F(s)$$

PROOF:

$$\begin{aligned}
-\frac{d}{ds}F(s) &= -\frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt \\
&= -\int_0^\infty \frac{d}{ds}e^{-st}f(t)\,dt \\
&= -\int_0^\infty (-tf(t))\,dt \\
&= \mathcal{L}(tf(t)).
\end{aligned}$$

The Transforms of $t\sin(t)$ and te^{-t}—An Example
Using Theorem 4, we find that:

$$\mathcal{L}(t\sin(t)) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}.$$

Similarly, we find that:

$$\mathcal{L}(te^{-t}) = -\frac{d}{ds}\left(\frac{1}{s+1}\right) = \frac{1}{(s+1)^2}.$$

We see that there is a connection between transforms whose denominators have repeated roots and functions that have been multiplied by powers of t.

As many equations have solutions of the form $y(t) = e^{-at}f(t)$, it will prove useful to know how to calculate $\mathcal{L}\left(e^{-at}f(t)\right)$. We find that:

Theorem 5

$$\mathcal{L}\left(e^{-at}f(t)\right)(s) = F(s+a).$$

PROOF:

$$\begin{aligned}
\mathcal{L}\left(e^{-at}f(t)\right)(s) &= \int_0^\infty e^{-st}e^{-at}f(t)\,dt \\
&= \int_0^\infty e^{-(s+a)t}f(t)\,dt \\
&= F(s+a).
\end{aligned}$$

The Laplace Transform of $te^{-t}\sin(t)$—An Example

Consider $\mathcal{L}\left(e^{-t}\sin(t)\right)(s)$. As we know that $\mathcal{L}\left(\sin(t)\right)(s) = \frac{1}{s^2+1}$, using our theorem we find that:

$$\mathcal{L}\left(e^{-t}\sin(t)\right)(s) = \frac{1}{(s+1)^2+1}.$$

As we know that multiplication in the time domain by t is equivalent to taking minus the derivative of the Laplace transform we find that:

$$\mathcal{L}\left(te^{-t}\sin(t)\right)(s) = -\frac{d}{ds}\left(\frac{1}{s^2+2s+2}\right).$$

Using the chain rule[3] we find that:

$$(1/f(s))' = (-1/f^2(s))f'(s).$$

Thus we find that $\mathcal{L}(te^{-t}\sin(t))(s)$ is equal to:

$$\begin{aligned} -\frac{d}{ds}\left(\frac{1}{s^2+2s+2}\right) &= -\frac{-1}{(s^2+2s+2)^2}\frac{d}{ds}\left(s^2+2s+2\right) \\ &= \frac{2s+2}{(s^2+2s+2)^2}. \end{aligned}$$

Often we need to calculate the Laplace transform of a function $g(t) = f(at), a > 0$. It is important to understand the effect that this "dilation" of the time variable has on the Laplace transform. We find that:

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$$\mathcal{L}(f(at))(s) = \frac{1}{a}F(s/a), a > 0.$$

PROOF:

$$\begin{aligned} \mathcal{L}(f(at))(s) &= \int_0^\infty e^{-st}f(at)\,dt \\ &\overset{u=at}{=} \frac{1}{a}\int_0^\infty e^{-(s/a)u}f(u)\,du \\ &= \frac{1}{a}F(s/a). \end{aligned}$$

[3]The chain rule states that:

$$\frac{d}{dt}f(g(t)) = \left.\frac{df(y)}{dy}\right|_{y=g(t)} g'(t).$$

$\sin(\omega t)$—An Example

Suppose $f(t) = \sin(\omega t)$. What is $\mathcal{L}(f(t))(s)$? Using Theorem 6, we find that:

$$\mathcal{L}(\sin(\omega t))(s) = \frac{1}{\omega}\frac{1}{(s/\omega)^2+1} = \frac{\omega}{s^2+\omega^2}.$$

Because we often need to model the effects of a delay on a system, it is important for us to understand how the addition of a delay to a function affects the function's Laplace transform. We find that:

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$$\mathcal{L}(f(t-T)u(t-T))(s) = e^{-Ts}F(s), T \geq 0.$$

PROOF:

$$\begin{aligned}
\mathcal{L}(f(t-T)u(t-T))(s) &= \int_0^\infty e^{-st}f(t-T)u(t-T)\,dt \\
&= \int_T^\infty e^{-st}f(t-T)\,dt \\
&\stackrel{u=t-T}{=} \int_0^\infty e^{-s(u+T)}f(u)\,du \\
&= e^{-sT}\int_0^\infty e^{-su}f(u)\,du \\
&= e^{-sT}F(s).
\end{aligned}$$

It is important to be able to determine the steady-state output of a system—the output of the system after all the transients have died down. For this purpose one can sometimes make use of the final value theorem:

The Final Value Theorem 8 If $f(t)$ approaches a limit as $t \to \infty$, and if $\int_0^\infty |f'(t)|\,dt$ converges[4] then:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0^+} sF(s).$$

PROOF: Consider $sF(s) = s\int_0^\infty e^{-st} f(t)\,dt$. Note that if $f(t)$ takes a non-zero limit at infinity (and even sometimes when it takes zero as its limit) the Laplace transform is only well defined for $\Re(s) > 0$. In what follows we assume that that s has always been chosen in such a way that $\lim_{t\to\infty} e^{-st} f(t) = 0$—i.e. $\mathcal{R}(s) > 0$. With this in mind, we find that:

$$\begin{aligned} s\int_0^\infty e^{-st} f(t)\,dt &= \int_0^\infty \left(-\frac{d}{dt}e^{-st}\right) f(t)\,dt \\ &= -e^{st} f(t)\Big|_{0^+}^{\infty} + \int_0^\infty e^{-st} f'(t)\,dt \\ &= f(0^+) + \int_0^\infty e^{-st} f'(t)\,dt. \end{aligned}$$

It seems reasonable to hope that:

$$\begin{aligned} \lim_{s\to 0^+} \int_0^\infty e^{-st} f'(t)\,dt &= \int_0^\infty \lim_{s\to 0^+} e^{-st} f'(t)\,dt \\ &= \int_0^\infty 1 \cdot f'(t)\,dt \\ &= \lim_{t\to\infty} f(t) - f(0^+). \end{aligned}$$

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[4] If $f(t)$ approaches a final value in a reasonable fashion this condition will hold. Note that:

$$\int_0^\infty |f'(t)|\,dt = \int_{\{t|f'(t)\geq 0\}} f'(t)\,dt - \int_{\{t|f'(t)<0\}} f'(t)\,dt, \tag{1.1}$$

and recall that:

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Considering (1.1) we see that the first integral must be the total amount that the function increases in all region in which it is increasing while the second integral is just the total amount that the function decreases while the function is decreasing. The difference of the two integrals is just the total amount that function varies as t goes from zero to infinity. This is often referred to as the *total variation* of the function. If the oscillations are exponentially damped (as they generally are in the cases of interest to us), then the total variation is finite and the theorem applies.

It is not too hard too show that as long as $\int_0^\infty |f'(t)|\,dt$ converges, the limit of the integral indeed converges to the integral of the limit. Consider the difference between:

$$I_1(s) \equiv \int_0^\infty e^{-st} f'(t)\,dt$$

and

$$I_2 \equiv \int_0^\infty f'(t)\,dt = \lim_{t\to\infty} f(t) - f(0^+).$$

We would like to show that the difference tends to zero as $s \to \infty$. We find that:

$$\begin{aligned} |I_1(s) - I_2| &= \left|\int_0^\infty e^{-st} f'(t)\,dt - \int_0^\infty f'(t)\,dt\right| \\ &= \left|\int_0^\infty (e^{-st} - 1) f'(t)\,dt\right| \\ &= \left|\int_0^A (e^{-st} - 1) f'(t)\,dt + \int_A^\infty (e^{-st} - 1) f'(t)\,dt\right| \end{aligned}$$

where $A > 0$.

In order to deal with the two integrals separately, we make use of the triangle inequality:

$$|a + b| \le |a| + |b|$$

which says that the absolute value of the sum of two numbers is less than or equal to the sum of the absolute value of the numbers considered separately. We will also use the generalized triangle inequality:

$$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt, \quad b \ge a$$

which says that the absolute value of the integral of a a function is less than or equal to the integral of the absolute value of the function. Using the triangle inequality and the generalized triangle inequality, we find that:

$$\begin{aligned}
|I_1(s) - I_2| &\leq \left|\int_0^A (e^{-st} - 1)f'(t)\,dt\right| \\
&\quad + \left|\int_A^\infty (e^{-st} - 1)f'(t)\,dt\right|, \quad A > 0 \\
&\leq \int_0^A |(e^{-st} - 1)f'(t)|\,dt \\
&\quad + \int_A^\infty |(e^{-st} - 1)f'(t)|\,dt, \quad A > 0 \\
&= \int_0^A |(e^{-st} - 1)||f'(t)|\,dt \\
&\quad + \int_A^\infty |(e^{-st} - 1)||f'(t)|\,dt, \quad A > 0 \\
&\overset{|e^{-st}-1|<2}{\leq} \int_0^A |(e^{-st} - 1)||f'(t)|\,dt \\
&\quad + \int_A^\infty 2|f'(t)|\,dt, \quad A > 0.
\end{aligned}$$

As we know that I_2 converges, we find that for any ϵ we can find a B such that for all $A > B$ we have:

$$\int_A^\infty 2|f'(t)|\,dt < \epsilon/2.$$

For any such fixed value of A it is clear that so long as $s > 0$ is close enough to zero we find that for all $t \in [0, A]$, for all t in the interval $[0, A]$:

$$|e^{-st} - 1| < \frac{\epsilon/2}{\int_0^\infty |f'(t)|\,dt}$$

We see that for any choice of $\epsilon > 0$ as long as s is sufficiently close to 0 we have:

$$\begin{aligned}
|I_1(s) - I_2| &\leq \epsilon/2 + \int_0^A |(e^{-st} - 1)||f'(t)|\,dt \\
&\leq \epsilon/2 + \int_0^A \frac{\epsilon/2}{\int_0^\infty |f'(t)|\,dt}|f'(t)|\,dt \\
&\leq \epsilon/2 + \frac{\epsilon/2}{\int_0^\infty |f'(t)|\,dt}\int_0^A |f'(t)|\,dt \\
&\leq \epsilon/2 + \frac{\epsilon/2}{\int_0^\infty |f'(t)|\,dt}\int_0^\infty |f'(t)|\,dt \\
&= \epsilon.
\end{aligned}$$

From the definition of the limit we find that:

$$\lim_{s\to 0+} I_1(s) = I_2$$

just as we needed. Combining all of these steps, we find that:

$$\lim_{s\to 0^+} sF(s) = \lim_{t\to\infty} f(t).$$

A *Mis*application of the Final Value Theorem—An Example

Let us find the final value of $\sin(t)$. Using Theorem 8, we find that:

$$\lim_{t\to\infty} \sin(t) = \lim_{s\to 0} s\frac{1}{s^2+1} = 0.$$

This is ridiculous; the sine function has no final value. What happened? One of the conditions of Theorem 8 is that the function under consideration have a limit as $t \to \infty$. If the function does not have a limit, then the theorem does not apply, and when the theorem does not apply one can get nonsensical results if one uses the theorem.

The Delayed Unit Step Function—An Example

Consider the function $f(t) = u(t-a), a > 0$. Clearly, here a final value does exist; the final value of a delayed unit step function is 1. The Laplace transform of the delayed unit step function is (according to Theorem 7):

$$\mathcal{L}(f(t))(s) = \frac{e^{-as}}{s}.$$

From the final value theorem, we see that:

$$\begin{aligned}\lim_{t\to\infty} u(t-a) &= \lim_{s\to 0+} s\mathcal{L}(u(t-a))(s)\\ &= \lim_{s\to 0+} s\frac{e^{-as}}{s}\\ &= \lim_{s\to 0+} e^{-as}\\ &= 1.\end{aligned}$$

This is just as it should be.

We will also find use for the initial value theorem:

The Initial Value Theorem 9

$$f(0^+) = \lim_{t\to 0+} f(t) = \lim_{\Re(s)\to+\infty} sF(s).$$

PROOF: In the proof of Theorem 8 we found that:

$$\begin{aligned}sF(s) &= s\int_0^\infty e^{-st} f(t)\,dt\\ &= f(0^+) + \int_0^\infty e^{-st} f'(t)\,dt.\end{aligned}$$

As long as $f'(t)$ is of exponential type, we know that:

$$|f'(t)| \leq Ce^{at}.$$

If this is so, then from the generalized triangle inequality we find that:

$$\begin{aligned}\left|\int_0^\infty e^{-st} f'(t)\, dt\right| &\leq \int_0^\infty |e^{-st} f'(t)|\, dt \\ &= \int_0^\infty |e^{-st}||f'(t)|\, dt \\ &\leq \int_0^\infty |e^{-st}| Ce^{at}\, dt \\ &= C\int_0^\infty e^{-\mathcal{R}(s)t} e^{at}\, dt \\ &= C\int_0^\infty e^{-(\mathcal{R}(s)-a)t}\, dt.\end{aligned}$$

It is clear that as $\mathcal{R}(s) \to \infty$ the integral above tends to zero. This completes the proof of the theorem.

The Delayed Cosine—An Example

Let us make use of the initial value theorem to find the value of $\cos(t-\tau)u(t-\tau), \tau > 0$ when $t = 0$. Evaluating the expression we see that it is equal to $\cos(-\tau)u(-\tau) = \cos(\tau) \cdot 0 = 0$. We know that:

$$\mathcal{L}(\cos(t-\tau)u(t-\tau))(s) = e^{-\tau s}\frac{s}{s^2+1}.$$

Using the initial value theorem we find that:

$$\cos(0-\tau)u(0-\tau) = \lim_{\Re(s)\to\infty} se^{-\tau s}\frac{s}{s^2+1} = 0$$

as it must.

We present one final, important, theorem without proof.

Theorem 10 *If $f(t)$ and $g(t)$ are piecewise continuous*[5] *and if $\mathcal{L}(f(t))(s) = \mathcal{L}(g(t))(s)$, then $f(t) = g(t)$ (except, possibly, at the points of discontinuity). I.e., the Laplace transform is (to all intents and purposes) unique.*

[5] Piecewise continuous functions are functions that are "pieced together" from continuous functions. An example is the delayed unit step function $u(t-2)$ which is 0 until 2, and is 1 afterwards. Both 0 and 1 are continuous functions. At the interface of the continuous pieces, at $t = 2$, $u(t-2)$ is not continuous.

Thus, if one recognizes a function $F(s)$ as the Laplace transform of the function $f(t)$ one can state that $F(s)$ does come from $f(t)$. It is not possible that there is a second piecewise continuous function $\tilde{f}(t)$ whose transform is also $F(s)$.

Solving Ordinary Differential Equations—An Example

Let us solve the equation $y'(t) = -2y(t)$, $y(0) = 4$. We start by finding the Laplace transform of both sides of the equation. We find that:

$$sY(s) - 4 = -2Y(s) \Leftrightarrow Y(s) = \frac{4}{s+2}.$$

We know that $\mathcal{L}(e^{-2t})(s) = \frac{1}{s+2}$. From the linearity of the Laplace transform we know that $\mathcal{L}(4e^{-2t})(s) = \frac{4}{s+2}$. From our uniqueness result, we find that $y(t) = 4e^{-2t}$.

1.3 Finding the Inverse Laplace Transform

We will not attempt to find a formula that gives us the inverse Laplace transform of a function. (Such a formula exists, but it is somewhat complicated.) In general one calculates the inverse Laplace transform of a Laplace transform by inspection. That is, one "massages" the transform until one has it in a form that one recognizes. Note that with the exception of Theorem 7 all of the transforms that we have encountered and all of the theorems that we have seen lead to transforms that are rational functions with real coefficients—that are of the form:

$$\frac{P(s)}{Q(s)}, \quad P(s) = a_n s^n + \cdots + a_0, \quad Q(s) = b_m s^m + \cdots + b_0$$

where the coefficients are all real numbers.

From modern algebra we know that all such fractions can be written in the form:

$$\begin{aligned}\frac{P(s)}{Q(s)} &= R(s) + \frac{a_{00}}{s-p_0} + \frac{a_{01}}{(s-p_0)^2} + \cdots \frac{a_{0n_0}}{(s-p_0)^{n_0}} \\ &+ \cdots \\ &+ \frac{a_{l0}}{s-p_l} + \frac{a_{l1}}{(s-p_l)^2} + \cdots + \frac{a_{p_{n_l}}}{(s-p_l)^{n_l}} \\ &+ \frac{b_{00}s + c_{00}}{(s-pc_0)(s-\overline{pc_0})} + \cdots + \frac{b_{0nc_0}s + c_{0nc_0}}{((s-pc_0)(s-\overline{pc_0}))^{nc_0}} \\ &+ \cdots \\ &+ \frac{b_{L0}s + c_{L0}}{(s-pc_L)(s-\overline{pc_L})} + \cdots + \frac{b_{Lnc_L}s + c_{Lnc_L}}{((s-pc_L)(s-\overline{pc_L}))^{nc_L}}\end{aligned}$$

where $R(s)$ is a polynomial with real coefficients, a_{ij}, b_{ij}, and c_{ij} are real constants, p_i are real poles[6] of the fraction, pc_i and $\overline{pc_i}$ are complex poles of the fraction, l is the number of distinct real poles of the fraction, L is the number of distinct pairs of complex poles, n_i is the number of times the real pole p_i is repeated, and nc_i is the number of times the complex pole pair $pc_i, \overline{pc_i}$ is repeated. Note that $R(s) = 0$ as long as the order of the denominator is greater than the order of the numerator—the most common case in control theory applications. As $(s-a)(s-\overline{a}) = s^2 - 2\Re(a)s + |a|^2$ is a polynomial with real coefficients, we find that all of the terms in the expansion above are rational function with real coefficients. The expression above is called the *partial fraction expansion* of $\frac{P(s)}{Q(s)}$. (It is often used in calculus to evaluate integrals of rational functions[Tho68].)

1.3.1 *Some Simple Inverse Transforms*

(1) $P(s)/Q(s) = 1/(s+1)^3$. This fraction is already in partial fraction form. We must determine the function whose Laplace transform it "obviously" is. We know that $\mathcal{L}(e^{-t})(s) = 1/(s+1)$. We note that the second derivative of the Laplace transform is $2/(s+1)^3$. Taking a second derivative in the Laplace domain is, according to Theorem 4, the same as multiplying the time function by t^2. Thus, $\mathcal{L}(t^2e^{-t}/2)(s) = 1/(s+1)^3$.

(2) $P(s)/Q(s) = 1/(s^2-1)$. We can factor the denominator into $(s-1)(s+$

[6]The poles of a rational function are those points at which the function becomes unbounded.

1). We find that:

$$\frac{1}{s^2-1} = \frac{a}{s-1} + \frac{b}{s+1}.$$

A standard way of finding the coefficients a and b is to multiply both sides of the equation by $s^2 - 1$. One finds that:

$$1 = a(s+1) + b(s-1) \Leftrightarrow 1 = (a+b)s + (a-b).$$

Equating coefficients of like powers of s, we find that $a + b = 0$, and $a - b = 1$. Adding the two equations we find that $a = 1/2$. Clearly $b = -1/2$. We find that:

$$\frac{1}{s^2-1} = \frac{1/2}{s-1} - \frac{1/2}{s+1}.$$

Recognizing that:

$$\mathcal{L}((1/2)e^{-t})(s) = \frac{1/2}{s-1}$$

and that:

$$\mathcal{L}((1/2)e^{t})(s) = \frac{1/2}{s+1},$$

we find that:

$$\mathcal{L}((1/2)(e^{-t} - e^{t}))(s) = \frac{1}{s^2-1}.$$

(3) $P(s)/Q(s) = 1/(s+1)^2$. This fraction is also in partial fraction form. We once again must determine its "obvious" inverse transform. One way to proceed is to note that $1/s^2$ is the Laplace transform of t and a shift by -1 in the s-plane is equivalent to multiplication by e^{-t} in time. The "obvious" inverse Laplace transform is te^{-t}. Another way to get to the same answer is to note that $\mathcal{L}(e^{-t})(s) = 1/(s+1)$ and that $-d/ds\{1/(s+1)\} = 1/(s+1)^2$. As the differentiation of the Laplace transform is equivalent to multiplication of the original function by $-t$, we find that the original function must have been te^{-t}.

(4)

$$P(s)/Q(s) = \frac{s+1}{s^3+4s^2+4s}.$$

We can factor the denominator as follows:

$$s^3 + 4s^2 + 4s = s(s^2+4s+4) = s(s+2)^2.$$

We see that:

$$\frac{s+1}{s^3+4s^2+4s} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2}.$$

Multiplying through by s^3+4s^2+4s we find that:

$$s+1 = A(s^2+4s+4) + B(s^2+2s) + Cs.$$

Equating coefficients of like powers of s we find that:

$$\begin{aligned} A + B &= 0 \\ 4A + 2B + C &= 1 \\ 4A &= 1. \end{aligned}$$

We find that $A = 1/4$, $B = -1/4$, and $C = 1/2$. Following the logic of the previous example, we see that:

$$\mathcal{L}(te^{-2t})(s) = \frac{1}{(s+2)^2}.$$

Thus, the original function must have been:

$$\left(\frac{1}{4} - \frac{1}{4}e^{-2t} + \frac{1}{2}te^{-2t}\right) u(t).$$

1.3.2 *The Quadratic Denominator*

Let us consider a fraction of the form:

$$\frac{\alpha s + \beta}{as^2 + bs + c}$$

where all of the constants are real. The poles of the function are the zeros of the denominator. Using the quadratic formula we find that the poles are:

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If:

$$b^2 - 4ac \geq 0,$$

then there are two real poles and we can find the inverse transform quite easily using the techniques demonstrated in the previous example. Suppose, however, that:

$$b^2 - 4ac < 0.$$

Then there are two complex conjugate poles:

$$s = \frac{-b \pm j\sqrt{4ac - b^2}}{2a}.$$

In this case, the simplest way to proceed is to complete the squares in the denominator and then manipulate the fractions into fractions whose inverse transforms we know. We find that:

$$\begin{aligned}
\frac{\alpha s + \beta}{as^2 + bs + c} &= \frac{\alpha s + \beta}{a(s^2 + (b/a)s + (c/a))} \\
&= \frac{\alpha s + \beta}{a\left(\left(s + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right)} \\
&= \frac{1}{a}\frac{\alpha\left(s + \frac{b}{2a}\right) - \frac{\alpha b - 2a\beta}{2a}}{\left(s + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}} \\
&= \frac{1}{a}\left(\frac{\alpha\left(s + \frac{b}{2a}\right)}{\left(s + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}} - \frac{\frac{\alpha b - 2a\beta}{2a}}{\left(s + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}}\right) \\
&= \frac{1}{a}\left(\frac{\alpha\left(s + \frac{b}{2a}\right)}{\left(s + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}} - \frac{\alpha b - 2a\beta}{2a\sqrt{\frac{4ac-b^2}{4a^2}}}\frac{\sqrt{\frac{4ac-b^2}{4a^2}}}{\left(s + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}}\right) \\
&= \frac{1}{a}\left(\frac{\alpha\left(s + \frac{b}{2a}\right)}{\left(s + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}} - \frac{\alpha b - 2a\beta}{\sqrt{4ac - b^2}}\frac{\sqrt{\frac{4ac-b^2}{4a^2}}}{\left(s + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}}\right)
\end{aligned}$$

After these simple (albeit tedious) operations, we find that "by inspection" the inverse transform is:

$$\frac{e^{-\frac{b}{2a}t}}{a}\left(\alpha \cos\left(\sqrt{\frac{4ac - ba^2}{4a^2}}t\right) - \frac{\alpha b - 2a\beta}{\sqrt{4ac - b^2}} \sin\left(\sqrt{\frac{4ac - ba^2}{4a^2}}t\right)\right) u(t).$$

Comparing the inverse Laplace transform and the poles of the fraction, we find that the rate of growth of the inverse Laplace transform is a function of the real part of the pole—$-b/(2a)$—and the frequency of the oscillations is controlled by the imaginary part of the pole:

$$\pm j\sqrt{\frac{4ac - ba^2}{4a^2}} = \pm j\frac{\sqrt{4ac - ba^2}}{2a}.$$

1.4 Integro-Differential Equations

An integro-differential equation is an equation in which integrals and derivatives appear. Many systems can be modeled using integro-differential equations. From Theorem 2 and Corollary 3, it is clear that both linear constant coefficient differential and integro-differential equation in $y(t)$ lead to expressions for $Y(s)$ that are rational functions of s. Using the techniques of the previous section, we should be able to find the inverse Laplace transform of such expressions. We consider a few examples.

The Laplace Transform of $\sinh(\omega t)$—An Example

If $y(t) = \sinh(\omega t)$, then $y'(t) = \omega\cosh(\omega t)$, and $y''(t) = \omega^2\sinh(\omega t)$. Clearly $y(0) = 0$, and $y'(0) = \omega$. Let us find the Laplace transform of the solution of the equation $y''(t) = \omega^2 y(t)$, $y(0) = 0, y'(0) = \omega$ in order to find the Laplace transform of $\sinh(\omega t)$.

$$\begin{aligned} y''(t) &= \omega^2 y(t), \quad y(0) = 0, y'(0) = \omega \Leftrightarrow \\ s^2\mathcal{L}(y(t))(s) - y'(0) - sy(0) &= \omega^2\mathcal{L}(y(t))(s) \Leftrightarrow \\ \left(s^2 - \omega^2\right)\mathcal{L}(y(t))(s) &= \omega \Leftrightarrow \\ \mathcal{L}(y(t))(s) &= \frac{\omega}{s^2 - \omega^2}. \end{aligned}$$

At this point we have found the Laplace transform of $\sinh(\omega t)$. We note that the Laplace transform is not in partial fraction form. As $s^2 - \omega^2 = (s - \omega)(s + \omega)$, find that:

$$\frac{\omega}{s^2 - \omega^2} = \frac{A}{s - \omega} + \frac{B}{s + \omega}$$

It is easy to see that $A = 1/2$ and $B = -1/2$. The inverse transform of $\mathcal{L}(y(t))$ is thus:

$$\sinh(\omega t) = \frac{e^{\omega t} - e^{-\omega t}}{2}$$

which is just as it should be.

A Simple Circuit—An Example

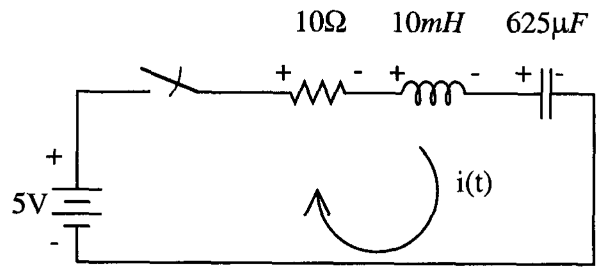

Fig. 1.1 A Simple R-L-C Circuit

If one has a series circuit composed of a switch, a 5V battery, a 10Ω resistor[7], a 10 mH inductor[8] and a $625\,\mu F$ capacitor[9] in series (see Figure 1.1) and there is initially no current flowing, the charge on the capacitor is initially zero, and one closes the switch when $t = 0s$, then from Kirchoff's voltage law[10] (KVL) one finds that the resulting current flow, $i(t)$, is described by the equation:

$$\overbrace{10i}^{resistor} + \overbrace{.01\frac{di}{dt}}^{inductor} + \overbrace{1600\int_0^t i(z)\,dz}^{capacitor} = \overbrace{5u(t)}^{battery}$$

where t it the time measured in seconds. Taking Laplace trans-

[7]Ohm's law states that the voltage across a resistor, $V_R(t)$, is equal to the current flowing through the resistor, $i_R(t)$, times the resistance, R, of the resistor. I.e. $V_R(t) = i_R(t)R$.

[8]Recall that the voltage across an inductor, $V_L(t)$, is equal to the inductance, L, of the inductor times the time derivative of the current flowing through the inductor, $di_L(t)/dt$. That is, $V_L(t) = L di_L(t)/dt$.

[9]The charge stored by a capacitor, $Q_C(t)$, is equal to the capacitor's capacitance, C, times the voltage across the capacitor, $V_C(t)$. I.e. $Q_C(t) = CV_C(t)$. Recall further that charge, $Q_C(t)$, is the integral of current, $i_C(t)$. Thus, $Q_C(t) = \int_0^t i_C(t)\,dt + Q_C(0)$.

[10]Kirchoff's voltage law states that the sum of the voltage drops around any closed loop is equal to zero. In our example KVL says that:

$$-5u(t) + V_R(t) + V_L(t) + V_C(t) = 0.$$

forms, one finds that:

$$(10 + .01s + 1600/s)\, I(s) = 5/s.$$

Rearranging terms, we find that:

$$I(s) = \frac{500}{(s^2 + 1000s + 160000)}.$$

The denominator has two real roots:

$$p_{1,2} = -200, -800.$$

Thus:

$$I(s) = \frac{500}{(s+200)(s+800)} = \frac{A}{s+200} + \frac{B}{s+800}.$$

Multiplying the fractions by $(s + 200)(s + 800)$, we find that:

$$500 = A(s + 800) + B(s + 200) \Rightarrow A = 5/6, B = -5/6.$$

We find that the solution of the equation is:

$$i(t) = \frac{5}{6}\left(e^{-200t} - e^{-800t}\right) u(t)$$

where the current is measured in amperes.

A Look at Resonance—An Example

Consider the equation satisfied by a spring-mass system with an m Kg mass and a spring whose spring constant is k N/M that is being excited by a sinusoidal force of amplitude A newtons and angular frequency Ω rad/sec. (See Figure 1.2.) Let $y(t)$ by the variation of the mass from it equilibrium position (measured in meters). From Newton's second law[11] we see that:

$$\overbrace{my''(t)}^{ma} = \overbrace{-ky(t) + A\sin(\Omega t)}^{\sum F}.$$

Assume that the mass starts from its equilibrium position, $y(0) = 0$, and that the mass starts from rest, $y'(0) = 0$. Taking the Laplace

[11] Which states that:

$$\text{sum of forces} = \sum F = ma.$$

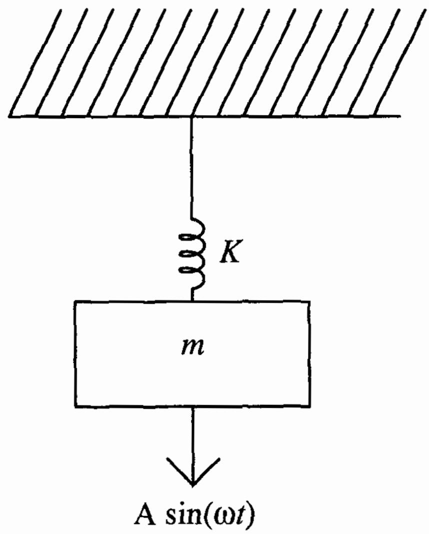

Fig. 1.2 The Free Body Diagram of the Spring-Mass System

transform of the equation, we find that:

$$ms^2Y(s) = -k\mathcal{L}(y(t))(s) + \frac{A\Omega}{s^2+\Omega^2}.$$

That is:

$$Y(s) = \frac{A\Omega}{s^2+\Omega^2}\frac{1/m}{s^2+k/m}.$$

Assume that $k/m \neq \Omega^2$. Then the above fraction is not in partial fraction form. The partial fraction expansion of the fraction is:

$$\frac{A\Omega/m}{(s^2+\Omega^2)(s^2+k/m)} = \frac{As+B}{s^2+\Omega^2} + \frac{Cs+D}{s^2+k/m}.$$

Multiplying both sides of the equation by $(s^2+\Omega^2)(s^2+k/m)$ leaves us with the equation:

$$(As+B)(s^2+k/m) + (Cs+D)(s^2+\Omega^2) = A\Omega/m \Leftrightarrow$$
$$(A+C)s^3 + (B+D)s^2 + \left(\frac{Ak}{m} + C\Omega^2\right)s + \frac{Bk}{m} + D\Omega^2 = \frac{A\Omega}{m}.$$

Equating coefficients of powers of s in the last equation gives us:

$$A + C = 0 \tag{1.2}$$
$$B + D = 0 \tag{1.3}$$
$$C\Omega^2 + \frac{Ak}{m} = 0 \tag{1.4}$$
$$\frac{Bk}{m} + D\Omega^2 = \frac{A\Omega}{m}. \tag{1.5}$$

Combining (1.2) and (1.4), we find that $A = C = 0$. Combining (1.3) and (1.5), we find that $B = A\Omega/(k - m\Omega^2)$ and $D = -B$. Thus, we find that:

$$y(t) = \frac{A}{k - m\Omega^2}\left(\sin(\Omega t) - (\Omega/\sqrt{k/m})\sin(\sqrt{k/m}t)\right)$$

where $y(t)$ is measured in meters.

Now let us assume that $\Omega^2 = k/m$. Then the Laplace transform is already in partial fraction form. In this case, we must find the inverse transform of:

$$\frac{A\Omega/m}{(s^2 + \Omega^2)^2}.$$

Note that:

$$\mathcal{L}(t\sin(\Omega t))(s) = \frac{2s\Omega}{(s^2 + \Omega^2)^2}.$$

We are interested in $A\mathcal{L}(t\sin(\Omega t))(s)/(2ms)$. From Corollary 3 we know that this is:

$$\frac{A}{2m}\int_0^t z\sin(\Omega z)\,dz = \frac{A}{2m}\left(-\frac{t\cos(\Omega t)}{\Omega} + \frac{\sin(\Omega t)}{\Omega^2}\right).$$

We find that so long as the resonant frequency of the spring, $\sqrt{k/m}$, is not the same as the forcing frequency, Ω, the output of the system is bounded. If the resonant frequency is the same as the forcing frequency, then the solution grows "essentially linearly" in time.

1.5 An Introduction to Stability

It is important for us to know that a given system behaves in a "reasonable fashion." In this section we make the kind of reasonableness that we want precise, and we give a criterion that a system must satisfy in order to behave reasonably. First, however, we put our generic system in a form that makes reasonableness easy to check.

1.5.1 *Some Preliminary Manipulations*

Consider an arbitrary linear, constant-coefficient, inhomogeneous integro-differential equation in $y(t)$. After differentiating the equation a sufficient number of times to eliminate all of the integrals, one can convert the equation into a linear, constant-coefficient, differential equation.

A Simple R-C Circuit—An Example

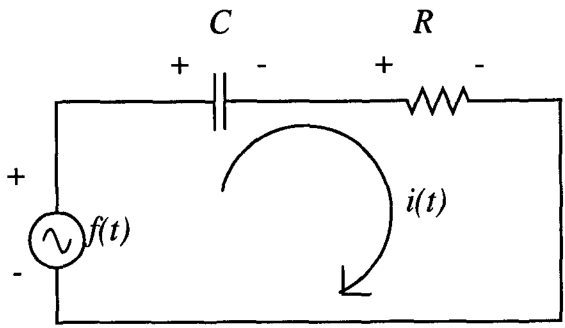

Fig. 1.3 A Simple R-C Circuit

Consider a circuit in which a capacitor of capacitance C farads, and a resistor of resistance R ohms are in series with a voltage source that sources the voltage $f(t)$ volts. Let $i(t)$ be the current (measured in amperes) traversing the circuit at time t (measured in seconds) and let $Q(0)$ be the charge on the capacitor at time $t = 0$. (See Figure 1.3.) The equation that describes the current in the circuit at time t is:

$$Ri(t) + \left(\int_0^t i(y)\,dy + Q(0)\right)/C = f(t) \tag{1.6}$$

We would like to manipulate this into an ordinary differential equation; we differentiate the equation once. We find that:

$$Ri'(t) + i(t)/C = f'(t).$$

We seem to be missing an initial condition—we must find $i(0)$. Looking at (1.6), we find that when $t = 0$, the equation reads $Ri(0) + Q(0)/C = f(0)$. This equation gives us the initial value of the current—$i(0) = -Q(0)/C + f(0)$. This is the initial condition we need.

Let us assume that we have differentiated the integro-differential equation that describes our system sufficiently many times that we are left with an ODE of the form:

$$a_n y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_0 y(t) = b_m f^{(m)}(t) + \cdots + b_0 f(t)$$
$$y^{(n-1)}(0) = c_{n-1}, \ldots, y(0) = c_0$$

where $f(t)$ is the input to the system and $y(t)$ is the output of the system.
Taking the Laplace transform of our equation, we find that:

$$\left(a_n s^n + a_{n-1}s^{n-1} + \cdots a_0\right) Y(s) = P(s) + \left(b_m s^m + b_{m-1}s^{m-1} + \cdots b_0\right) F(s)$$

where $P(s)$ is related to the initial value of $y(t)$ and its derivatives and of the initial value of $f(t)$ and its derivatives. Denoting $a_n s^n + a_{n-1}s^{n-1} + \cdots a_0$ by $Q(s)$, and denoting $b_m s^m + b_{m-1}s^{m-1} + \cdots b_0$ by $R(s)$, we find that:

$$Y(s) = P(s)/Q(s) + F(s)R(s)/Q(s)$$

where $P(s)$ is determined by our initial conditions, and $Q(s)$ and $R(s)$ are determined by the differential equation.

1.5.2 *Stability*

We would like to find a condition which guarantees that "if the input to the system is reasonable, then so is the output of the system." The first question one must ask is, how does one define reasonable? We take a simple definition—we say that *a system is stable if* for any *(reasonable) bounded input and* for any *initial conditions the output of the system is bounded.* This is often referred to as BIBO—bounded input bounded output—stability. We claim that a necessary and sufficient condition for a system described by a linear, constant coefficient, time invariant, integro-differential equation to be stable is that *all of the poles of $P(s)/Q(s)$—all of the zeros of $Q(s)$—must lie in the left half-plane.* When we speak of the left half-plane we will always mean the region LHP $\equiv \{s|\Re(s) < 0\}$. The right half-plane will always mean the region RHP $\equiv \{s|\Re(s) \geq 0\}$ unless we specify the open right half-plane—the region $\{s|\Re(s) > 0\}$.

We show that our condition is necessary. First, suppose that $Q(s)$ has a real zero of order n, z, in the open right half-plane and assume that $f(t) \equiv 0$. We find that $Y(s) = P(s)/Q(s)$, where the coefficients of $P(s)$ are related to the initial conditions on $y(t)$. The partial fraction expansion of $Y(s)$ is:

$$Y(s) = \frac{a}{(s-z)^n} + \cdots$$

where $a \neq 0$. Under this condition, $y(t) = a(t^{n-1}/(n-1)!)e^{zt}$ which is a function whose magnitude is unbounded.

Now suppose that $Q(s)$ has two complex conjugate zeros in the right half-plane. Then the partial fraction expansion of $Y(s) = P(s)/Q(s)$ is:

$$\frac{\alpha s + \beta}{as^2 + bs + c} + \cdots$$

where $b^2 - 4ac < 0$. As we found on p. 18, the inverse Laplace transform of such a term grows like $e^{\mathcal{R}(\text{pole})t}$. As the real part of the pole is positive in our case, we find that the magnitude of inverse Laplace transform grows without bound. (The case of multiple roots can be handled similarly.)

Suppose that the $Q(s)$ has a single zero, z, on the imaginary axis. Suppose, for example, that $Q(s)$ has a zero at 0—that $Q(s) = s\tilde{Q}(s)$. Let the initial conditions on the ODE be identically zero. Let the function $f(t) \equiv 1$. Then:

$$Y(s) = \frac{F(s)}{Q(s)} = \frac{1/s}{s\tilde{Q}(s)} \overset{\text{partial fractions}}{=} \frac{a}{s^2} + \cdots$$

Clearly then:

$$y(t) = at + \cdots$$

This function is not bounded as $t \to \infty$ even though the input is. Similarly if $Q(s)$ has an imaginary zero at $z = j\omega$, then using $f(t) = \sin(\omega t)$ will lead one to a solution whose maximum approaches infinity linearly. (See Problem 8.) Because the instability exhibited by systems with poles of multiplicity one on the imaginary axis does not generally lead the system's output to increase without bound, such systems are said to be *marginally stable*. Note that marginally stable systems are actually unstable.

The proof that our condition is sufficient to guarantee stability is more complicated, and we do not present a proof of the sufficiency of the condition.

1.5.3 *Why We Obsess about Stability*

We have seen that if a system is not stable, then for some bounded input the output of the system becomes unbounded. One might think, however, that as long as one avoids the particular inputs that cause the system to behave badly, then one can make use of the unstable system. Let us consider a simple example to see why this is not a practical solution.

An Unstable System—An Example

Consider a system that satisfies the equation:

$$y''(t) = y(t).$$

The general solution of this equation (found using Laplace transforms, for example) is:

$$y(t) = \frac{y(0) + y'(0)}{2}e^{t} + \frac{y(0) - y'(0)}{2}e^{-t}.$$

Suppose that one believes that one has set the initial conditions to $y(0) = 1, y'(0) = -1$. For these initial conditions, the solution is $y(t) = e^{-t}$—a perfectly nice solution. The difficulty with this system is that *practically* it is well nigh impossible to be sure that $y(0) = 1$ and $y'(0) = -1$. Suppose that $y(0) = 1 + \epsilon$ and $y'(0) = -1 + \epsilon'$ where both ϵ and ϵ' are extremely small. The solution of the equation with these initial conditions is

$$y(t) = \frac{\epsilon + \epsilon'}{2}e^{t} + \left(1 + \frac{\epsilon - \epsilon'}{2}\right)e^{-t}.$$

The effect of this infinitesimal change in the initial conditions hardly affects the coefficient of the e^{-t}. However, now we find that the growing exponential e^t has a nonzero coefficient. As t gets large we find that the part of the solution that we wanted, e^{-t}, is swamped by the part of the solution that came from a tiny imprecision in our initial conditions.

Because of this tendency of even very small mistakes to have very large consequences in unstable systems, such systems *cannot generally be used.* Another problem with such systems is that in many cases the possibility of unbounded output is itself a very bad sign. (One does not want one's car's speed to increase without bound!) For these reasons we do not generally use or want unstable systems.

1.5.4 *The Tacoma Narrows Bridge—a Brief Case History*

On November 7, 1940 the four month old Tacoma Narrows bridge collapsed. In the aftermath many explanations were proposed for the collapse[Kou96]. As a bridge is a very complicated structure, nobody has ever been able to say with certainty what caused the collapse of the Tacoma Narrow Bridge. It seems that what caused the bridge to tear itself apart was the wind exciting a *nonlinearly* resonant mode in the bridge[BS91]. (This is somewhat akin to the *linearly* resonant mode in the spring-mass system of Page 22.)

The Tacoma Narrows bridge serves to remind engineers of the importance of treating unstable systems—and even marginally stable systems—with the respect and caution that they deserve.

1.6 MATLAB

Throughout this book we will use MATLAB as a very fancy calculator. We now give a very brief overview of how MATLAB is used. (We make use of the commands found in the fourth edition of the student edition of MATLAB.)

1.6.1 *Assignments*

In MATLAB, one makes assignments by writing `variable = object` where object may be a number, an array, or various other objects about which we will hear more later. Note that neither variables nor arrays need to be declared in MATLAB's language. Here are a number of examples of legal assignments.

(1) `A = 3`. This assigns the value 3 to the variable `A`. It also causes MATLAB to print:

```
A =
      3
```

(2) Generally, if one wishes to suppress printing one ends the assignment with a semi-colon. The command `A = 3;` also assigns three to the variable `A`, but it does not cause anything to be printed.

(3) MATLAB prints the value of a variable if one types the variable's name. If one types `A`, MATLAB responds with:

```
A =
      3
```

(4) `B = [3, 4 ,5];` assigns the array `[3, 4, 5]` to `B`. To refer to the individual elements of `B` one refers to `B(1)` to `B(2)`, and to `B(3)`. Arrays in MATLAB always start from element number `1`. It is worth noting that the commas in the assignment statement are optional. If one leaves a space between two numbers, MATLAB assume that the two numbers are distinct elements of the array.

(5) `C = [3, 4; 5, 6]` assigns the two dimension matrix to the variable `C`. Additionally, because there was no semi-colon after the assignment, MATLAB prints:

```
C =
      3     4
      5     6
```

One can refer to the elements of this array as one would refer to elements of a matrix. To refer to the second element in the first row one refers to `B(1,2)`. Typing `B(1,2)` causes MATLAB to respond with:

```
ans =
      4
```

MATLAB prints `ans` because it is not responding with the value of an entire named object—4 is not the value of `B`; it is the value of a component of `B`.

1.6.2 *Commands*

As we have already seen, we are often interested in finding the roots of a polynomial—for example, the roots of the denominator of the Laplace transform of some function. MATLAB has a command for this purpose—`roots`[12]. Suppose one's polynomial is:

$$c_1x^n + c_2x^{n-1} + \cdots + c_nx + c_{n+1}.$$

To find the roots of this polynomial, one defines the array

$$\texttt{C = } [c_1, c_2, \ldots, c_n, c_{n+1}],$$

and then one makes the assignment/function call `R = roots(C)`. The result is that the roots of the polynomial are now stored in the array `R`.

Another command that is useful is the `residue` command. The `residue` command gives the partial fraction expansion of a rational function. One enters two array `B` and `A` which are the coefficients of the numerator polynomial and the denominator polynomial respectively. One uses this command by writing `[R P K] = residue(B,A)`. The vector `P` contains the poles of the various fractions that appear in the partial fraction expansion. Note that if a pole appears more than once then the fraction connected to the n^{th} occurrence of the pole is $1/(s-p)^n$. The vector `R` is the vector of coefficients—the n^{th} coefficient is the numerator of the fraction associated with the n^{th} item in the vector `P`. The vector `K` is the vector of coefficients of the polynomial than one gets if one starts with a fraction whose numerator has degree[13] greater than or equal to the degree of the denominator. We note that `residue` gives all the poles separately—it separates fractions of the form $(As + B)/((s-a)(s-\overline{a})$ into fractions of the form $C/(s-a) + D/(s-\overline{a})$.

How to Use `residue`—An Example

Suppose that one would like to find the inverse transform of the function $\mathcal{L}(f(t))(s) = 1/(s^2-1)$. One assigns `B = 1` and `A = [1, 0, -1]`. Then one performs the operation `[R P K] = residue(B,A)`. MATLAB responds with:

```
R =
```

[12]This command does not work very well for high order polynomials because the roots of an equation are inherently difficult to calculate.

[13]The degree of a polynomial is the highest power that occurs in the polynomial. Thus, the degree of $s+1$ is one, while the degree of s^3+1 is three. Polynomials of degree n are also sometimes said to be of *order* n.

```
      -0.5000
       0.5000

P  =
      -1.0000
       1.0000

K =
         []
```

That means that the array K is empty and that the fraction is equal to:

$$\frac{1}{s^2-1} = \frac{-0.5000}{s-(-1.0000)} + \frac{0.5000}{s-1.0000}.$$

Clearly the inverse Laplace transform of the function is $(e^t - e^{-t})/2$. There are many useful MATLAB commands, and we will see some of them in the course of this book.

1.7 Exercises

(1) Find the Laplace transform of:

(a)

$$\cos(\omega t)u(t), \omega > 0$$

Note that $u(\omega t) = u(t)$ if $\omega > 0$.

(b)

$$f(t) = te^{-t}\cos(t)u(t)$$

(c)

$$f(t) = t^2 e^{-t} u(t)$$

(d)

$$f(t) = e^{-t}\sin(2t)u(t)$$

(2) Find the function whose Laplace transform is:

(a)

$$F(s) = \frac{s+1}{s^2+1}$$

(b)

$$F(s) = \frac{1}{s+1}$$

(c)

$$F(s) = \frac{s}{s^2+1}$$

(d)

$$F(s) = \frac{1}{(s+1)^2}$$

(e)

$$F(s) = \frac{s+1}{s^2+2s+2}$$

Do each problem both "by hand" and using MATLAB's `residue` command.

(3) Use the Laplace transform to solve the differential equation:

$$y''(t) + 5y'(t) + 4y(t) = 1$$

subject to the initial conditions:

$$y(0) = y'(0) = 0$$

(4) Use the Laplace transform to solve the integral equation:

$$\int_0^t y(r)\,dr = -y(t) + u(t)$$

where $u(t)$ is the unit step function.

(5) Use the Laplace transform to solve the integral equation:

$$\int_0^t y(r)\,dr = -y(t) + \sin(\omega t)u(t)$$

where $u(t)$ is the unit step function.

(6) Use the Laplace transform to solve the integro-differential equation:

$$3\int_0^t y(r)\,dr + 4y(t) + y'(t) = 0$$

subject to the initial condition $y(0) = 1$.

(7) Show the Laplace transform of the function $f(t) = u(t)/\sqrt{t}$ is $\sqrt{\pi/s}$ for all $s > 0$. Use the definition of the Laplace transform, the substitution $y = \sqrt{st}$, and the fact that $\int_0^\infty e^{-y^2}\,dy = \sqrt{\pi}/2$.

(8) Suppose that the relation between the input to a system, $f(t)$, and the output of the system, $y(t)$, satisfies:

$$Y(s) = \frac{1}{s^2+1}F(s)$$

By considering $f(t) = \sin(t)$, show that the system is not BIBO stable.