Chapter 1

Introduction

This final volume handles most of all the construction of a Markov process starting with a pseudo-differential operator with a negative definite symbol. Further we discuss some applications within mathematics.

We start with a chapter on "Essentials from Probability Theory" in which we collect results needed later on. We do not long for a completely self-contained presentation, but we want to make the reader's life, especially when having a more analytic background, a bit more easy. In writing this chapter we used as standard reference the monographs [30] and [31] of H. Bauer which in Germany form still the unchallenged introduction to mathematical probability theory. The topics treated in Chapter 1 are rather straightforward for probabilists: σ-fields and measures, some integration theory, some topological measure theory, independence, conditional expectation, martingales and stopping times, and elementary remarks on stochastic processes. As the corresponding chapter "Essentials from Analysis" in volume I this chapter should be taken for reference purposes.

We discuss mainly three approaches of constructing a stochastic process starting with a pseudo-differential operator: the construction of a Feller process starting with a semigroup, the well-posedness of the martingale problem, and the construction of a Hunt process associated with and L^p-sub-Markovian semigroup.

In Chapter 3 we start with the "usual" Kolmogorov approach of constructing a canonical process. We feel the reader should see details of the construction but of course these are best known results and we follow closely H. Bauer [30]. In Section 3.1 the projective limit and the canonical process construction are discussed, and Section 3.2 treats semigroups of Markovian kernels, their relations to transition functions and canonical processes. Section 3.3 prepares some more advanced topic. We introduce and handle first properties of càdlàg functions since the processes we are interested in will have almost surely càdlàg paths. In particular we introduce the Skorohod topology and the Skorohod space. The first part of Section 3.4 follows once again closely H. Bauer [30] and introduces the notion of a Markov process as well as that of a universal Markov process $\left((X_t)_{t\geq 0}, P^x\right)_{x\in\mathbb{R}^n}$. Then we show that to every Feller semigroup we can associate a universal Markov process. Hence for all pseudo-differential operators we proved (in volume II) that they generate a Feller semigroup, we know by now that they give rise to a Markov process, called a Feller process. Using properties of the underlying Feller semigroup we can show further that every Feller process admits a càdlàg modification, i.e. almost surely every path is a càdlàg function. For many considerations the Markov property is not sufficient, but the strong Markov property is needed. This is the theme of Section 3.5 and we discuss the strong Markov property by using the shift operator, a tool used also later on. As a main result we find that the càdlàg modification of a Feller process leads to a strong Markov process. In Section 3.6 we give a martingale characterization of Feller processes which allows us to introduce the notion of the ($\mathcal{D}_n$-) martingale problem. This establishes a different link between generators and processes. In particular the well-posedness of the martingale problem entails the existence of a Markov process associated with the generator.

Section 3.7 treats Lévy processes. From our point of view Lévy processes are those processes being generated by a pseudo-differential operator the symbol of which is a continuous negative definite function, i.e. the symbol has "constant coefficients". However it is worth to have a more probabilistic approach and to introduce Lévy processes as stochastic processes with independent and stationary incremenents. From this we derive that their transition function originates from a convolution semigroup of probability measures which in turn lead to the characterization of a Lévy process as a Feller process generated by a pseudo-differential operator of the type described above.

Processes with independent and stationary increments have been first investigated by P. Lévy and it was S. Bochner who made Fourier analysis a central tool for studying Lévy processes. Since with the monographs of K. Sato [310] and J. Bertoin [37] we have at least two standard references at hand we do not treat properties of Lévy processes in detail. However we briefly discuss the Poisson process, the compound Poisson process, we mention Brownian motion and symmetric stable processes as well as the Meixner process. Further we discuss now subordination on the level of processes. Finally we give R. Schilling's proof of the continuity of paths of Brownian motion knowing already that the paths are càdlàg.

The different normalization of the Fourier transform in probability theory and in analysis leads to one unfortunate problem in our presentation: the symbol of the generator of a Lévy process is the conjugate complex of its characteristic exponent. We keep therefore both notions, but of course they are essentially the same and this makes the theory smooth and satisfactory: probabilistic and analytic concepts (essentially) coincide.

Although we do not prove any results, we summarize in Section 3.8 path properties of Lévy processes including the Lévy-Itô decomposition. The short Section 3.9 in some sense summarizes parts of our philosophy: We know that with every Feller process we can associate a symbol which is the symbol of the generator of the corresponding Feller semigroup. Moreover, this symbol also determines probabilistic properties of the process. Very beautiful results in this direction, especially in relation to path properties are due to R. Schilling. Since he is working on a monograph on this topic we just mention a few of his results without proof.

The martingale problem was introduced by D. W. Stroock and S. R. S. Varadhan to study diffusions and later it was applied to the study of other processes. We investigate in Chapter 4 the martingale problem for pseudo-differential operators with negative definite symbols. When discussing general results we often follow the presentation of St. Ethier and Th. Kurtz [98] but mostly by incorporating the work of W. Hoh [155]. However all results in Chapter 4 related directly to pseudo-differential operators are due to W. Hoh and are taken from [155] and from others of his papers.

First, in Section 4.1 we need to study families of probability measures on the Skorohod space and in particular we need some tightness results. We then discuss the existence of solutions to the $\mathcal{D}_n$-martingale problem, Section 4.2, and in Section 4.3 we give a uniqueness criterion. Section 4.4 explains a localization procedure. We are then prepared for Section 4.5 where we give a proof of W. Hoh's results on the well-posedness of the $\mathcal{D}_n$-martingale problem for a large class of pseudo-differential operators. In some sense this is still the most general result stating that a given pseudo-differential operator generates a Markov process. Under additional assumptions it is possible to prove that the process is indeed a Feller process. This result is once again due to W. Hoh and proved in Section 4.5.

Chapter 5 is devoted to the construction of a Hunt process starting with an L^p-sub-Markovian semigroup. Clearly, in the end we are interested in L^p-sub-Markovian semigroups generated by a pseudo-differential operator. In Section 5.1 we briefly explain why the Feller theory can not cover all cases. The case $p = 2$, i.e. the Hilbert space situation is best known due to the work of M. Fukushima and his investigations make it clear that we have to study now Hunt processes. In Section 5.2 we discuss Hunt processes following the presentations of M. Fukushima, Y. Oshima and M. Takeda [115]. Section 5.3 is the core of Chapter 5. We re-examin Fukushima's proof of the existence of a Hunt process associated with a (symmetric) L^2-sub-Markovian semigroup in order to get an analogous L^p-result. Although a lot of work was done before, in the literature we could not find a throughout proof without a gap covering some general situation. In fact our main result, Theorem 5.3.38, is far from being as satisfactory as the L^2-result is. The central problem is that we have to handle a non-linear theory for $p \neq 2$. In Section 5.4 we indicate a different approach suggested for $p = 2$ by M. Fukushima, namely to start with an L^p-sub-Markovian semigroup having a pointwise defined transition function, in fact we assume to have a pointwise defined density.

In Chapter 3–5 we made already much use of potential theory. As a first application of our theory within mathematics we handle in Chapter 6 more detailed the links of Markov processes and potential theory. Section 6.1 gives some heuristic reasons why one should expect such links and we mention also some pioneers in the field. In Section 6.2 we more systematically develop these links by discussing certain notions from potential theory and their probabilistic counterparts. It turns ou that one should make a distinction between the Feller theory and the L^2- or L^p-theory. In the end we long to reduce results to conditions on symbols of pseudo-differential operators. For this reason we have a closer look to the situation in case of Lévy processes, see Section 6.3, whereas Section 6.4 treats some problems for general pseudo-differential operators. We address problems such as transience, recurrence or conservativity, just to mention a few.

Section 6.5 is more involved. It discusses the analogue to the classical Dirichlet problem for the Laplacian. The non-local character of the operators being involved, or equivalently the fact that the paths of the processes under consideration have jumps, causes significant changes which was first noted by M. Riesz when treating fractional powers of the Laplacian. In the Feller case we may embed our problem into the theory of balayage spaces introduced by J. Bliedtner and W. Hansen. The corresponding Dirichlet problem we call "balayage Dirichlet problem". Following joint work with W. Hoh we prove that many pseudo-differential operators generating a Feller semigroup indeed give rise to a balayage space allowing us to solve the corresponding balayage Dirichlet problem. A further approach uses (modified) Hilbert space methods which leads to a generalized Dirichlet problem admitting variational solutions. Following the joint work with W. Hoh further on we give existence results and we identify solutions of these two problems using a probabilistic representation formular.

We continue to discuss applications in Chapter 7. First, in Section 7.1, we treat problems involving fractional derivatives. Here we can rely on work of A. Krägeloh and V. Knopova. In particular we obtain first existence results for semigroups — hence processes — with state space being the half-space. Section 7.2 points out the general problem when handling pseudo-differential operators with negative definite symbols in bounded domains: in the generic case the transmission condition does not hold. This section contains a type of case study when subordinating the Dirichlet and the Neumann operator associated with certain second order elliptic differential operators (more precisely we consider fractional powers only). The results are obtained jointly with R. Schilling and were extended to non-smooth domains with W. Farkas. We work within the frame of (symmetric) Dirichlet forms and determine domains of generators as well as of forms in terms of classical Sobolev spaces of fractional order. In these spaces the boundary behavior is incorporated. In some cases we give also pathwise representation of the subordinate processes, i.e. the analogue to the Skorohod decomposition for reflected diffusions.

Section 7.3 introduces a nice idea due to O. Barndorff-Nielsen and S. Levendorskiĭ. They pointed out that one comes naturally to pseudo-differential operators generating Markov processes when making parameters of characteristic exponents of Lévy processes state space dependent. We explain this in more detail by discussing the Meixner process following the work of B. Böttcher.

The classical Feynman-Kac formula is a very powerful tool in the spectral theory of the Laplacian. M. Demuth and J. van Casteren developed a theory called stochastic spectral analysis by substituting the Laplacian (or Brownian motion) by a general symmetric Feller generator (or Feller process). In Section 7.4 we briefly describe the frame of their theory.

The final Section 7.5 discusses two topcis on function spaces associated with continuous negative definite functions. One result is joint work with R. Schilling and embeds a result of V. Maz'ya and J. Nagel into the theory of Dirichlet spaces. The second result is due to W. Farkas and H.-G. Leopold and deals with function spaces of generalized smoothness.

In Section 7.3–7.5 we give no proofs but discuss results only. We added five appendices. Appendix D compiled by B. Böttcher gives further examples of continuous negative definite functions and Appendix E compiled by R. Schilling gives more examples of (complete) Bernstein functions.

Appendix A-C indicate recent developements in the analytic part of the theory – in some sense they belong to the discussion of volume II. In Appendix A we discuss the construction of a parametrix for the fundamental solution of the evolution equation associated with an (elliptic) pseudo-differential operator with a symbol in Hoh's class, a result due to B. Böttcher. In Appendix B we summarize results jointly obtained with A. Tokarev to get a parameter-dependent Hoh-calculus. Appendix C is devoted to a discussion of Roth's method to construct Feller semigroups. These results are due to A. Potrykus.

Two technical remarks: 1. Since some readers might be interested to study the L^p-theory independent of the Feller theory, Chapter 3 and Chapter 5 show some redundancy. 2. Sometimes we write in an abuse of notation $q(x, \xi) \in S^m$ etc. when we mean that $(x, \xi) \mapsto q(x, \xi)$ is a symbol in the class S^m. We feel that this if often helpful.