Chapter 1

Preliminaries on Coxeter groups

In this chapter we develop much of the basic theory regarding Coxeter groups that will be used throughout the remainder of the text. Although the material contained in this chapter is not comprehensive, enough background is provided to make study of the text largely self-contained. The interested reader may consult either [Bourbaki (1981)] or [Humphreys (1990)] for a more exhaustive treatment of Coxeter groups in general. Additional references are provided as needed in order to direct the reader to more throrough discussions of other topics mentioned in this chapter.

1.1 Historical background and motivation

Coxeter groups have been studied *pro se* for nearly three quarters of a century. One of the first rigorous treatments of these groups is found in the work of H.S.M. Coxeter ([Coxeter (1934)]), although a number of talented mathematicians of the late nineteenth and early twentieth centuries considered such groups at some point in their work (often without explicit indication of the relevant group structure).

Coxeter groups arose as a natural generalization of groups of symmetry and crystallographic groups. For Coxeter, at least initially, they are the groups generated by involutions R_i, in which the angles at which the reflecting "mirrors" corresponding to the involutions intersect are prescribed by means of relations of the form

$$(R_i R_j)^{m_{ij}}$$

for specified integers m_{ij}. (Coxeter goes so far as to invite the reader to create a fundamental domain for the action of a finite Coxeter group by using actual mirrors and a candle flame!)

Throughout his life, Coxeter retained the term he used for these groups in his earliest investigations, calling the objects of his study *groups generated by reflections* or *reflection groups.* (They would not be called *Coxeter groups* until Tits coined this term many years later.) Much of the basic theory of these groups is due to Coxeter himself. For instance, the enumeration of all finite Coxeter groups was first provided by Coxeter in one of the earliest papers written on the subject ([Coxeter (1935)]).

Following the lead established by Coxeter's initial studies, Coxeter groups have frequently been considered from a geometric point of view. Much as was done by Coxeter, Coxeter groups are often defined as groups generated by reflections in a particular vector space which is assigned the appropriate bilinear form (see 1.2.2).

More recent geometric characterizations see Coxeter groups acting isometrically on spaces which exhibit very nice metric properties. For example, we will consider below (in 1.5.3) a complex defined by M. Davis ([Davis (1983)]), building on the work of E. B. Vinberg. This complex, which carries a CAT(0) metric (as proven by Moussong in [Moussong (1996)]), will prove very useful in our studies. It is closely related to another complex, introduced by Tits. The latter object (generally called the Coxeter complex) admits significant generalization, leading to the very rich theory of buildings. In Chapter 6 we consider yet another complex (the chamber complex, also related to buildings) on which the given group acts. Coxeter groups have also been shown to act on trees ([Dranishnikov and Januszkiewicz (1999)]) and on CAT(0) cubical complexes ([Niblo and Reeves (2003)]).

Coxeter groups can also be realized in a straightforward combinatorial fashion, wherein the presentation of the group is of central importance, and analysis of the group is frequently performed with little reference to the group's geometric structure. Methods such as small cancellation theory and van Kampen diagrams can often be applied. The combinatorial viewpoint will facilitate the proofs of many results, and it is the manner in which we first address Coxeter groups in the following sections.

In whatever light they are studied, Coxeter groups arise naturally in crystallography, Lie theory, commutative algebra, representation theory, low-dimensional topology, combinatorics, and geometric group theory. They are useful creatures indeed!

It is clear that the subject of Coxeter groups is a broad one, any study of these groups must by necessity adopt a narrowed focus. We ask, then: what sort of problems will we address in this volume?

Our primary concern will be to understand the ways in which Coxeter groups may be presented. For instance, we may ask to what extent presentations for these groups are "unique". The ultimate goal is to find a solution to the Isomorphism Problem for Coxeter groups. That is, can we define an algorithm which tells us, given two presentations for Coxeter groups, if the presentations define isomorphic groups? This is the third, and most difficult, of the fundamental problems of combinatorial group theory posed by Max Dehn in his seminal 1910 work on surface groups. (The other two, the Word Problem and the Conjugacy Problem, are both known to be solvable for Coxeter groups. See Section 2.4.) Although a solution to the Isomorphism Problem has not yet been obtained, a number of the results proven in this text provide close approximations to a solution in a number of special cases.

The Isomorphism Problem is closely related to the structure of the automorphism group $\mathrm{Aut}(W)$. We will examine $\mathrm{Aut}(W)$ more carefully in the final chapter of this volume.

We now begin our study by defining Coxeter groups, any by putting forth the rudiments of the general theory of these groups that will be useful later on.

1.2 Coxeter systems and Coxeter groups

1.2.1 *A combinatorial definition*

A *Coxeter system* is a pair (W, S) for which $S = \{s_i \mid i \in I\}$ is a distinguished generating set with index set I, and

$$W \cong \langle S \mid R \rangle, \text{ for } R = \{(s_i s_j)^{m_{ij}} \mid m_{ij} \in \{1, 2, ..., \infty\}\}. \tag{1.1}$$

We demand that $m_{ij} = m_{ji}$ and $m_{ij} = 1 \Leftrightarrow i = j$ for all $i, j \in I$. By $m_{ij} = \infty$ we mean that $s_i s_j$ has infinite order. (Such infinite relators may be omitted. We include them in order to remain consistent with the definition frequently adopted, in which the values m_{ij} are related to the entries of the symmetric *Coxeter matrix*, defined in 1.2.2 below.) It can be proven (see [Bourbaki (1981)]) that the product $s_i s_j$ has order m_{ij} in the group W so defined. (This is not true *a priori*. In fact, it is not even clear that the relations R do not identify some distinct generators s_i and s_j.)

The index set I may have any cardinality, though we will often consider only the case in which I is finite. The set S is known as the *fundamental generating set* for the system (W, S). If W is a group for which there exists a presentation as in (1.1), we call W a *Coxeter group*.

There is a convenient way of representing Coxeter groups by means of graphs. We first review the necessary definitions from graph theory, as the notation and terminology are far from standard.

A *graph* X is a pair (V, E), where V is the set of *vertices* of X, and E is the set of *edges* of X. We will be concerned only with undirected graphs for which two vertices are connected by at most one edge, and for which the endpoints of any edge are distinct. Thus for our purposes an edge is completely determined by its two endpoints, and we may identify any element $e \in E$ with a pair $\{v_1, v_2\}$, $v_1, v_2 \in V$. We will denote the edge so defined by $[v_1 v_2]$. We say that $v_1, v_2 \in V$ are *adjacent* if $e = [v_1 v_2]$ is an edge in E; in this case, both v_1 and v_2 are said to be *incident* the edge e. Given $v_0, v_1, ..., v_k \in V$ such that $[v_i v_{i+1}] \in E$ for all i, $0 \leq i \leq n-1$, the sequence $v_0, v_1, ..., v_k$ defines a *path* P (*of length* n) in X. The graph X is said to be *connected* if for any two vertices $v_1, v_2 \in V$, there is a path from v_1 to v_2. An *edge-labeled graph* is a graph which comes equipped with a function lab : $E \to A$ assigning a label lab(e) from the set A to each edge $e \in E$.

Now let (W, S) be a Coxeter system. We define the *Coxeter diagram* $\mathcal{V}$ corresponding to (W, S) to be an edge-labeled graph whose vertex set is in one-to-one correspondence with S, and for which there is an edge $[s_i s_j]$ in $\mathcal{V}$ whenever $m_{ij} < \infty$. For any edge $e = [s_i s_j]$ in $\mathcal{V}$, we define lab$(e) = m_{ij}$. We often suppress the labeling function itself and simply say that an edge e is labeled by a particular number.

It is clear that given $\mathcal{V}$, the presentation $\langle S \mid R \rangle$ is completely determined, and *vice versa*.

Remark 1.1 The Coxeter diagram must be distinguished from the *Coxeter graph*, used in [Bourbaki (1981)] and [Humphreys (1990)], among other writings. In the latter construction, no edge is included between generators s_i and s_j which commute, and an edge is included between generators s_i and s_j for which $s_i s_j$ has infinite order. Furthermore, the label 3 is omitted from any edge $[s_i s_j]$ for which $s_i s_j$ has order 3. This second construction (which facilitates study of direct product decompositions, rather than free product decompositions) is often more convenient when there is a great deal of commutativity between elements of S. In order to maintain consistency throughout the text, we will use only Coxeter diagrams as defined above. Unfortunately, this may require a good deal of mental "translation" between graphs and diagrams for those more familiar with the former!

We will frequently omit the word "Coxeter" when referring to Coxeter groups, systems, and diagrams, as long as there will be no confusion.

There are some classes of Coxeter systems to which we will devote a great deal of attention. Whether or not a given Coxeter group belongs to a certain class can often be decided merely by examining its diagram. Therefore we define some of these classes now in terms of the Coxeter diagram.

Let (W, S) be a Coxeter system with diagram $\mathcal{V}$. If every edge in $\mathcal{V}$ has label 2, we say that (W, S) (or $\mathcal{V}$) is *right-angled.* At the other extreme, we say that (W, S) (or $\mathcal{V}$) is *large-type* if no edge in $\mathcal{V}$ is labeled 2. (Such systems are also called *skew-angled*, as in [Mühlherr and Weidmann (2002)].) Appel and Schupp (in [Appell and Schupp (1983)]) introduce a yet more extreme case, saying a Coxeter system is *extra-large-type* if every edge in the corresponding diagram has label at least 4. If every edge in $\mathcal{V}$ is even, we call (W, S) (or $\mathcal{V}$) *even.* If, given the group W, a system (W, S) exists which is right angled, (extra-)large-type, or even, we call W itself right angled, (extra-)large-type, or even, respectively. (However, we will see that the distinction between systems and groups made here will not always be necessary. This fact is the focus of much of this volume.)

One more definition for now. It may be that W can be easily decomposed as a direct product in a nontrivial fashion. If $S = S_1 \cup S_2$ such that $S_1 \cap S_2 = \emptyset$ and $s_i \in S_i (i = 1, 2) \Rightarrow s_1 s_2 = s_2 s_1$, then $W = W_1 \times W_2$, where $W_i = \langle S_i \rangle$. If this sort of decomposition is not possible, we call W *irreducible.* Many questions about Coxeter groups can be reduced to questions about irreducible groups.

We will have need to examine still other classes of groups as we continue our study.

1.2.2 *A geometric definition*

Suppose we are given an index set I and a collection of values $m_{ij} \in \{1, 2, ..., \infty\}$ $(i, j \in I)$ satisfying $m_{ij} = m_{ji}$ and $m_{ij} = 1 \Leftrightarrow i = j$. (As before, I can be either finite or infinite, although we will often be concerned only with finite I.) This information is often encoded in a symmetric matrix (called the *Coxeter matrix*) $A = (a_{ij})$ for which $a_{ij} = -\cos \frac{\pi}{m_{ij}}$ when $m_{ij} < \infty$ and $a_{ij} \leq -1$ when $m_{ij} = \infty$. This method of bookkeeping suggests the definition which now follows.

Let V be a vector space of dimension $|I|$ over the real numbers, with basis $\{\alpha_i \mid i \in I\}$ index by I. We define a symmetric bilinear form $(\cdot, \cdot)$ on

V by means of the matrix A. That is, we demand that

$$(\alpha_i, \alpha_j) = a_{ij}, \tag{1.2}$$

Denote by H_i the subspace of V orthogonal to α_i relative to this bilinear form; H_i is then complementary to the line in V containing α_i.

We note that the bilinear form $(\cdot, \cdot)$ motivates the terminology *right-angled* and *skew-angled* introduced in 1.2.1. Indeed, if W is right-angled and α_i and α_j are any two distinct vectors defined as above, either α_i and α_j are orthogonal or $(\alpha_i, \alpha_j) \leq -1$. In case W is skew-angled, no two vectors α_i and α_j are orthogonal to one another.

We now define the *reflection* $r_i : V \to V$ corresponding to α_i by

$$r_i(v) = v - 2(\alpha_i, v)\alpha_i \tag{1.3}$$

for every $v \in V$. It is easy to see that each r_i preserves the bilinear form, that each point of H_i remains fixed by r_i, and that $r_i(\alpha_i) = -\alpha_i$ (see exercises), as shown in Figure 1.1. The mappings r_i (with the operation of composition) generate a subgroup W of the group $\mathrm{GL}(V)$ of linear transformations of V. This group is a Coxeter group as defined in 1.2.1, with system $(W, \{r_i \mid i \in I\})$.

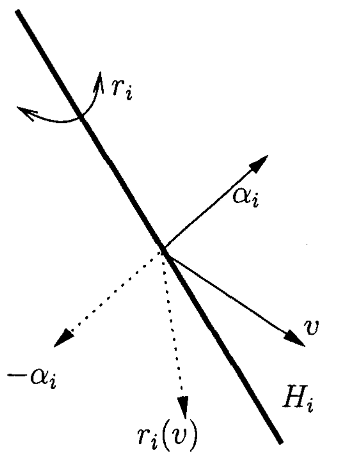

Fig. 1.1 The vector α_i and its corresponding reflection

Remark 1.2 Alternately, we could begin with W as in 1.2.1 and map each element $s_i \in S$ to $r_i \in \mathrm{GL}(V)$ for V a vector space of dimension $|S|$

spanned by α_i, where the bilinear form is defined as in (1.2), and r_i is defined as in (1.3). That this gives a faithful linear representation of W is proven carefully in [Humphreys (1990)]. That is, Coxeter groups are linear groups, and therefore enjoy a number of desirable properties. (For instance, all linear groups have solvable word problem, are residually finite, and are virtually torsion free.)

The vectors α_i are called *simple roots.* Denote by Φ the set $\{w(\alpha_i) \mid w \in W, i \in I\}$. (Here of course $w(\alpha_i)$ is the vector which results from the application of the linear transformation w to α_i.) Then Φ is known as the *root system* corresponding to W. Each element of Φ is called a *root.* The reader is asked to show in Exercise 1 that Φ is finite if and only if W is finite.

Because every α_i is a unit vector, and because every w preserves the bilinear form $(\cdot, \cdot)$, every element of Φ is a unit vector. Because $\{\alpha_i \mid i \in I\}$ is a basis of V, every root α can be expressed as a unique linear combination of the vectors α_i. If all of the coefficients in this sum are positive (resp., negative), we call α a *positive root* (resp. *negative root*). We will denote by Π the set of positive roots (so that, clearly, $-\Pi$ comprises the set of negative roots). Although we will not prove it, Φ is the disjoint union of Π and $-\Pi$.

Given a collection T of roots in some root system Φ, we may consider the subgroup W_T of W generated by the reflections r_α corresponding to $\alpha \in T$. This subgroup itself has a root system (sometimes called a *subsystem* of Φ), which we may denote by Φ_T:

$$\Phi_T = \{\alpha \in \Phi \mid r_\alpha \in W_T\}.$$

1.2.3 *Examples*

We now examine some of the simplest (and most useful) Coxeter groups from both points of view introduced above.

In case W is finite, we can realize W as an orthogonal linear reflection group in some finite dimensional Euclidean space (see [Benson and Grove (1996)], [Bourbaki (1981)], or [Humphreys (1990)] for more details). There are a number of cases of interest.

The simplest examples of Coxeter groups are provided by the dihedral group D_n of order $2n$ and the symmetric group S_n (the group of permutations of $\{1, ..., n\}$) of order $n!$. Indeed, it is easy to see that D_n has presentation $\langle a, b \mid a^2, b^2, (ab)^n \rangle$ and that S_n has presentation

$\langle s_1, ..., s_{n-1} \mid (s_i s_j)^{m_{ij}} \rangle$, where $m_{ij} = 1 \Leftrightarrow i = j$, $m_{ij} = 2 \Leftrightarrow |j - i| \geq 2$, and $m_{ij} = 3 \Leftrightarrow |i - j| = 1$ (with arithmetic modulo $n - 1$). These groups furnish actions upon Euclidean space which are easily understood.

For instance, the group D_n corresponds to simple roots α and β in $\mathbb{E}^2$ which meet at an angle of π/n. As α and β we may choose unit vectors with a common vertex at the origin; the product $ab \in W$ then corresponds to a rotation about the origin through an angle of $2\pi/n$. (We have recaptured the realization of D_n as the group of symmetries of a regular n-gon.)

The action of S_n (as presented above) on $\mathbb{E}^n$ is nearly as simple. Consider the standard orthonormal basis $\{e_1, ..., e_n\}$ of $\mathbb{E}^n$. We let r_i act by permuting the ith and $(i+1)$st elements of this basis ($r_i(e_i) = e_{i+1}$ and $r_i(e_{i+1}) = e_i$), leaving the remaining basis vectors fixed. It is easy to show that α_i is a unit vector pointing in the direction $e_i - e_{i+1}$, and that the line spanned by the vector $e_1 + \cdots + e_n$ is fixed (pointwise) by every r_i, and therefore by every element of W. Therefore S_n also acts on the hyperplane orthogonal to the vector $e_1 + \cdots + e_n$, and the action on this subspace of $\mathbb{E}^n$ has no nonzero fixed points (such an action is called *essential*).

Remark 1.3 In the literature on Coxeter groups and elsewhere, the dihedral group D_n is often denoted by $I_2(n)$, and the symmetric group S_n by A_{n-1}, in accordance with the notation commonly used in Lie theory.

There are a number of other finite Coxeter groups, comprising two other infinite classes of groups, as well as six "exceptional" groups. Some of these arise as the groups of symmetry of polyhedra of appropriate dimensions (see Exercise 4), and others can be realized in more exotic fashions. The reader may consult [Benson and Grove (1996)] or Chapter 2 of [Humphreys (1990)] for more details.

Before leaving the finite Coxeter groups behind, we mention that as simple as dihedral groups are, they will prove to be extremely important throughout our investigation. One reason for this is that many dihedral groups admit more than one Coxeter system. Indeed, if $n = 2k$ for k odd,

$$\langle c, d, g \mid c^2, d^2, g^2, (cg)^2, (dg)^2, (cd)^k \rangle \tag{1.4}$$

gives an alternate presentation for D_n. (The reader is asked to interpret this presentation geometrically as well.)

The simplest infinite Coxeter group is the infinite dihedral group, D_∞, with presentation

$$\langle a, b \mid a^2, b^2 \rangle. \tag{1.5}$$

Because the Coxeter matrix

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{1.6}$$

associated with this presentation is not positive definite, it cannot correspond to an inner product, and therefore we cannot put things in a nice Euclidean setting, as in the finite case. However, it is easy to find a natural "affine" action of D_∞ which generalizes the action of D_n by reflections in linear hyperplanes.

The real line, subdivided into intervals at integral points, is the "limit" of the sequence of regular n-gons, as n goes to infinity. we may embed this line into the Euclidean plane by identifying it with the line $\{(x, 1) \mid x \in \mathbb{R}\}$. For each $n \in \mathbb{Z}$, the point $(n, 1)$ determines a ray with the origin as its endpoint. Let X denote the union of all the triangular sectors determined by such rays, along with the rays themselves. (That is, X is the open upper half plane, along with the origin.) The action of a on X fixes pointwise the ray through $(0, 1)$ and exchanges sectors (rays) which are "mirror images" with respect to this ray. The action of b on X fixes pointwise the ray through $(1, 1)$ and exchanges sectors (rays) which are "mirror images" with respect to this ray.

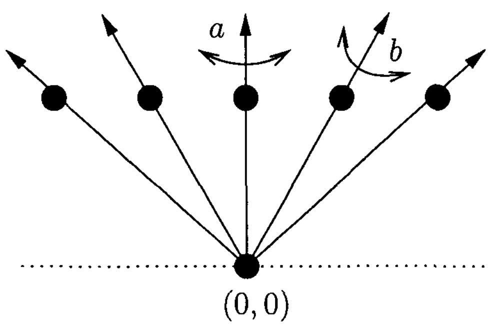

Fig. 1.2 X, on which D_∞ acts by "linear reflections"

Remark 1.4 The space X described in the previous paragraph is known as the *Tits cone*, and is described more carefully in 1.5.2.

If we do not demand that W act by "linear" reflections, but only by affine reflections, we need not embed the real line into $\mathbb{E}^2$. Indeed, D_∞ already acts on the real line by affine reflections. In this way the group

D_∞ can be viewed as a two-dimensional *affine Euclidean reflection group*; it is the only such group in two dimensions. (There are three such groups in three dimensions, corresponding to tessellations of the Euclidean plane by regular hexagons; squares and octagons; and squares, hexagons, and dodecagons. A complete list of all affine Eucliean reflection groups, corresponding to positive semidefinite symmetric bilinear forms, is given in Chapter 4 of [Humphreys (1990)].)

1.3 Basic properties of Coxeter groups

1.3.1 *Reflections*

We return now to the combinatorial definition of the Coxeter group.

Let (W, S) be a Coxeter system as defined in 1.2.1. Any element of the form wsw^{-1} for $s \in S$ and $w \in W$ is known as a *reflection* of the system (W, S). (The elements of the fundamental generating set S are often called *simple reflections.*) The set of all reflections of the system will be denoted by $R(S)$. The set $R(S)$ does indeed depend on S.

For instance, consider the presentations of D_n ($n = 2k$, k odd) given in 1.2.3. An automorphism ϕ of D_n may be defined by

$$\phi(c) = a, \phi(d) = ababa, \phi(g) = (ab)^k. \tag{1.7}$$

In the second system given, the element g is a reflection relative to the generating set $\{c, d, g\}$, whereas g is not a reflection relative to the first system (which is most easily seen from 1.3.2 below).

We will explore the relationship between $R(S)$ and $R(S')$ for two systems (W, S) and (W, S') throughout the text.

Remark 1.5 Clearly, the term *reflection* comes from the geometric notion of the same name. Indeed, performing the construction of W indicated in 1.2.2, we see that the elements r_i of W act by reflection across the codimension-1 subspace H_i. Moreover, if w is any element of W defined as in 1.2.2, it is easy to show that wr_iw^{-1} acts by reflection across the codimension-1 subspace orthogonal to the vector $w\alpha_i$.

1.3.2 *Lengths of words and geodesics*

Let (W, S) be a Coxeter system, and let $w \in W$. It may be possible to express w as a product of simple reflections $s_1 \cdots s_n$ in more than one fashion. Let n be the smallest number for which the equality $w = s_1 \cdots s_n$ holds in the group W. The *length of* w *with respect to* S, denoted $l_S(w)$, is defined to be n. (When the generating set S is clear, we may write merely $l(w)$ for the length of w.) If $w =_W s_1 \cdots s_n$ and $l_S(w) = n$, we say the product $s_1 \cdots s_n$ is *reduced*, or that it is a *geodesic* expression for w.

The following proposition sums up some of the elementary properties of the length function. The verification of these facts is simple, and is left as an exercise.

Proposition 1.1 *Let $l(w)$ denote the length of $w \in W$ with respect to the generating set S, and let $w, w' \in W$.*

1. $l(w) = l(w^{-1})$.
2. $l(w) = 0 \Leftrightarrow w = 1$.
3. $l(ww') \leq l(w) + l(w')$.
4. $l(ww') \geq l(w) - l(w')$.
5. $l(w) - 1 \leq l(ws) \leq l(w) + 1$ *for every* $s \in S$.

There is very nice interplay between the length function $l(w)$ and the action of W upon the vector space V (and its corresponding root system Φ) as defined in 1.2.2. Following [Humphreys (1990)], we denote by $n(w)$ the number of positive roots in Φ sent to negative roots of Φ by the action of w; that is, $n(w) = |(\Pi \cap w^{-1}(-\Pi))|$. We refer the interested reader to Chapter 5 of [Humphreys (1990)] for a proof of the following theorem.

Theorem 1.1 *Let (W, S) be a Coxeter group, and let $w \in W$. Then $l(w) = n(w)$.*

The following is an important special case ($w = s_i$ for some $i \in I$) of the theorem.

Proposition 1.2 *Suppose that α_i is the simple root in Φ corresponding to $s_i \in S$. Then the set $\Pi \setminus \{\alpha_i\}$ is stable under the action of s_i. (That is, $s_i(\Pi \setminus \{\alpha_i\}) = \Pi \setminus \{\alpha_i\}$.)*

Proof. Let λ be a positive root, $\lambda \neq \alpha_i$. Then λ is expressible as a linear combination $\sum_{j \in I} a_j \alpha_j$, where for some $j \neq i$, $a_j > 0$. Applying the reflection s_i to both sides of the equation $\lambda = \sum_{j \in I} a_j \alpha_j$ shows that $s_i(\lambda) = \lambda - 2(\lambda, \alpha_i)\alpha_i$ is a linear combination involving the same coefficients for all α_j, $j \neq i$. Thus (since every root is either negative or positive) $s_i(\lambda)$ is again positive. Now $s_i(\lambda) \neq \alpha_i$, for then $\lambda = s_i s_i(\lambda) = s_i(\alpha_j) = -\alpha_j$, contradicting positivity. Therefore $\lambda \in \Pi \setminus \{\alpha_i\}$, and s_i acts by permuting the elements of $\Pi \setminus \{\alpha_i\}$. ∎

From Proposition 1.2 we obtain the following useful result:

Proposition 1.3 *Suppose that α_i is the simple root in Φ corresponding to $s_i \in S$, and that $w \in W$. Then the following are true regarding the function $n(w)$.*

1. $w(\alpha_i) > 0 \Rightarrow n(wr_i) = n(w) + 1$.
2. $w(\alpha_i) < 0 \Rightarrow n(wr_i) = n(w) - 1$.
3. $w^{-1}(\alpha_i) > 0 \Rightarrow n(r_i w) = n(w) + 1$.
4. $w^{-1}(\alpha_i) < 0 \Rightarrow n(r_i w) = n(w) + 1$.

Proof. If $w(\alpha_i) > 0$, then ws_i negates each of the roots $s_i(\alpha)$, where α is a positive root negated by w. (That all such roots are positive follows from Proposition 1.2.) Also, $ws_i(\alpha_i) \in -\Pi$ and α_i is not negated by w. Thus $n(ws_i) = n(w) + 1$. Similar arguments apply for the remaining cases, as the reader is encouraged to check. □

1.3.3 *Parabolic subgroups*

Let (W, S) be a Coxeter system. For any subset $T \subseteq S$, we define the subgroup $W_T \leq W$ by taking as a generating set T and by applying only those relators from R which involve only letters of T. This subgroup is called the *standard parabolic subgroup* generated by T. Any conjugate wW_Tw^{-1} ($w \in W$) is called a *parabolic subgroup* of W. (When T consists of a single generator t, we often write simply W_t instead of $W_{\{t\}}$.) Whether or not a group is a parabolic subgroup of W clearly depends on the system (W, S) that we choose.

Parabolic subgroups behave nicely with respect to intersection, as the following result demonstrates.

Proposition 1.4 *Let $T_1, ..., T_k \subseteq S$, and let $T = \bigcap_{i=1}^{k} T_i$. Then*

$$W_T = \bigcap_{i=1}^{k} W_{T_i}.$$

Proposition 1.4 follows from the fact that any two geodesic words representing the same group element must contain the same letters of the generating set S (*cf.* Proposition 1.5).

If W_T is a finite group, we call W_T a *spherical subgroup* of W. We may call T spherical as well. Since it is clear (prove!) that the full subgraph of $\mathcal{V}$ induced by the vertices of T is complete, we often say that T is a spherical *simplex* of $\mathcal{V}$. (This justifies our frequent use of letters typically reserved for simplices, such as σ, τ, *etc.*, to denote these collections.) If W_T is spherical and no spherical $T' \subsetneq S$ properly contains T, we say that T is a *maximal* spherical simplex .

Spherical subgroups (and in particular, the maximal ones) play an important role in the theory of Coxeter groups. It is important that we be able to recognize the spherical subgroups of a given group W. (The enumeration of all spherical Coxeter groups was first done by Coxeter himself, in [Coxeter (1935)].) Clearly, any vertex generates a spherical subgroup, as does any edge. To detect three-generated spherical subgroups, we will make frequent use of the following lemma, often without mention.

Lemma 1.1 *Let (W, S) be a Coxeter system with diagram $\mathcal{V}$. Let the distinct vertices $T = \{s_1, s_2, s_3\}$ be form a triangle in $\mathcal{V}$, and let $\{k, l, m\}$ be the multiset of edge labels for this triangle. Then W_T is spherical if and only if*

$$\{k, l, m\} \in \Big\{\{2,3,4\}, \{2,3,5\}, \{2,4,5\}\Big\} \cup \Big\{\{2,2,n\} \mid n \in \{2,3,...\}\Big\}.$$

The following lemma (proven below in 1.5.4) will also be used frequently.

Lemma 1.2 *Let (W, S) be a Coxeter system, and let G be an arbitrary finite subgroup of W. Then there exists a spherical subgroup $W_T \leq W$ and a group element $w \in W$ such that $G \leq wW_Tw^{-1}$.*

In particular, the conjugates of maximal spherical subgroups of W are the maximal finite subgroups of W.

As an application of this lemma, let us prove the following fact, in which W is realized (as in 1.2.2) as a group of orthogonal transformations in the vector space V, with associated root system Φ:

Lemma 1.3 *Let $T \subset \Phi$ be a finite set of roots closed under the action of W. Then the subgroup of W generated by the reflections in the roots $T' = \Phi \cap \operatorname{span}(T)$ is a finite parabolic subgroup of W.*

This will be used in Chapter 7 when we examine the automorphism group of certain Coxeter groups.

Proof. Because T is finite, span(T) is a Euclidean vector subspace of V, and T' is also finite. At this point, an application of Lemma 1.2 and an unraveling of the definitions gives the desired result. □

Given a system (W, S), we denote by $\mathcal{S}(S)$ (or simply by $\mathcal{S}$ when no confusion will arise) the collection of subsets T of S for which W_T is a spherical subgroup with respect to the generating set S. $\mathcal{S}(S)$ is a partially ordered set (poset) under inclusion. We denote by $W\mathcal{S}(S)$ (or by $W\mathcal{S}$) the set of *spherical cosets* $\{W/W_T \mid T \in \mathcal{S}(S)\}$. This set too is a poset under inclusion (see Exercise 8). We will have use for these collections in Section 1.5.3.

A well-known result from the general theory of Coxeter groups asserts that (W_T, T) is itself a Coxeter system, with the presentation described above. This fact highlights a possible ambiguity concerning the length function. To be precise, if $w \in W_T$ for some T properly contained in S, it is conceivable that $l_S(w) < l_T(w)$. However, this turns out to not be the case:

Lemma 1.4 *Let W_T be a standard parabolic subgroup of W, $W_T \neq W$. Then for any $w \in W_T$, $l_T(w) = l_S(w)$.*

The proof of this lemma is left as an easy exercise. Because of this result, it makes no difference whether we measure length in the subgroup W_T or in the ambient group W.

Given a parabolic subgroup W_T of W, it is often desirable to determine a set of *minimal coset representatives* for W_T. Let $T \subsetneq S$ and define W^T by

$$W^T = \{w \in W \mid l(ws) > l(w) \text{ for all } s \in T\}. \tag{1.8}$$

That is, W^T consists of those elements of W for which reduced expressions cannot end in any letter of T. W^T is in some sense "complementary" to W_T, as demonstrated by the following fact.

Lemma 1.5 *Suppose W_T is a parabolic subgroup and W^T is defined as above. Given $w \in W$ there are unique elements $u \in W^T$ and $v \in W_T$ such that $w = uv$. Moreover, $l(w) = l(u) + l(v)$ holds, and u is the unique element of shortest length in the coset wW_T.*

The proof of this lemma too is left as an exercise. One means of proving it requires the Deletion Condition, a fundamental fact concerning Coxeter groups to which we now turn.

1.3.4 *The Deletion Condition*

The *Deletion Condition* not only provides a great deal of information concerning the reduced expressions for a given element $w \in W$; it also serves to characterize Coxeter groups in a very well-defined sense (Theorem 1.3).

The Deletion Condition is logically equivalent to the proposition known as the *Exchange Condition.* We will not prove the equivalence of these conditions, directing the reader to [Bourbaki (1981)] or [Humphreys (1990)] for further information. However, a simple proof of the Deletion Condition will be possible at the end of the next section.

Theorem 1.2 *Let (W, S) be a Coxeter system, and let $w \in W$.*

1. **[Deletion Condition]** *Suppose that $w = s_1 \cdots s_n$ holds in W, and let $l(w) < n$. Then there exist indices i and j satisfying $1 \leq i < j \leq n$ such that $w = s_1 \cdots s_{i-1}s_{i+1} \cdots s_{j-1}s_{j+1} \cdots s_n$ holds in w.*
2. **[Exchange Condition]** *Suppose that $w = s_1 \cdots s_n$ holds in W, where this expression is not necessarily reduced. Suppose that $l(ws) < l(w)$ for some $s \in S$. Then for some index i, $ws = s_1 \cdots s_{i-1}s_{i+1} \cdots s_n$. Moreover, if the expression is reduced, this index is unique.*

We note without proof here the following fact, which underscores the importance of the Deletion Condition to the theory of Coxeter groups.

Theorem 1.3 *Let G be a group generated by a set S of involutions (that is, S consists of elements of order 2). Then (G, S) is a Coxeter system if and only if G satisfies the Deletion Condition with respect to the set S. (That is, if $g = s_1 \cdots s_n$ in G and $l(g) < n$, then g can also be expressed as a product obtained from $s_1 \cdots s_n$ by omission of exactly two letters s_i, s_j.)*

More general groups satisfying the Deletion Condition are considered in Exercise 17.

In order to prove the Deletion Condition, we introduce a tool which will be useful later as well.

1.4 Van Kampen diagrams

In this section we present a brief account of the first powerful method from combinatorial group theory which we will have occasion to use. After defining van Kampen diagrams and stating a few fundamentral results concerning them, we will use them immediately to give an easy proof of the Deletion Condition. (This proof is originally due to A. Yu. Ol'Shanksii.) We will make regular use of van Kampen diagrams in the following chapters. For further applications of these diagrams to the study of Coxeter groups and related groups, the reader is encouraged to read [Appell and Schupp

(1983)], [Kapovich and Schupp (preprint)], and [Schupp (preprint)]. For the general theory of van Kampen diagrams, consult [Lyndon and Schupp (1977)] or [Ol'Shanskii (1991)].

Suppose that we are given a group presentation $G = \langle S \mid R \rangle$ where R is *symmetrized.* By this we mean that R is closed under taking inverses and cyclic permutations, and all elements of R are cyclically reduced.

A *map* Δ is a finite, connected, planar graph. Unlike the underlying graph of a Coxeter diagram, we regard Δ as a *directed graph*; that is, each edge comes equipped with an orientation, and its endpoints are referred to as its *initial* and *terminal vertices.* We will further regard a map as a planar 2-complex, considering also the 2-dimensional faces whose boundaries are formed by the edges of the graph Δ. Note that there is one unbounded face; namely, the one whose boundary is the perimeter of the graph Δ itself.

We define a *van Kampen diagram* Δ over the symmetrized presentation $\langle S \mid R \rangle$ (often simply called an R-diagram) to be a map with the following properties:

1. If e is an edge in Δ, then e is labeled by an element $\text{lab}(e) = s$ in $S^{\pm 1}$, and $\text{lab}(e^{-1}) = s^{-1}$.
2. If D is a face in Δ with boundary given by the edges $e_1, e_2, ..., e_k$, then the word $\text{lab}(e_1)\text{lab}(e_2) \cdots \text{lab}(e_k)$ is (letter-for-letter; that is, as a word in the free group on S) a word in R.

Figure 1.3 illustrates this definition.

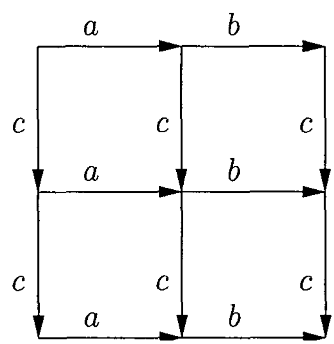

Fig. 1.3 A van Kampen diagram over the presentation $\langle a, b, c \mid aca^{-1}c^{-1}, bcb^{-1}c^{-1} \rangle$

Remark 1.6 In more general applications, lab(e) is merely assumed to be a word in the letters of $S^{\pm 1}$. Subdivision of edges of Δ (with concomitant subdivision of labeling words) then yields an R-diagram Δ as we have defined it.

We call the 2-dimensional faces of a van Kampen diagram *cells.* Much as products in the free group may often be reduced by cancellation of adjacent mutually inverse letters, a van Kampen diagram Δ may be put in a simpler form by cancellation of adjacent mutually inverse cells. That is, suppose that Δ contains two cells Π_1 and Π_2 for which at least one edge e appears in the boundary of both cells. Furthermore, suppose that the labels on the edges of Π_1, reading clockwise starting with lab(e), give the same word in the free group on S as the labels on the edges of Π_2, reading counterclockwise starting with lab(e). We may perform a *reduction* on Δ by removing the two cells Π_i, thus creating a "hole", and then "sewing up" the hole by identifying the edges on the boundaries of Π_1 and Π_2 which were not held in common (in the obvious fashion). This action does not effect the remaining cells of the diagram Δ, and we may perform many reductions, in sequence, on the same diagram. If no such reduction is possible, we say that Δ is *reduced.*

Combined with the method of small cancellation theory (see [Lyndon and Schupp (1977)]), van Kampen diagrams provide a powerful tool for proving theorems about groups with certain presentations. The most fundamental fact is the following theorem, often known as van Kampen's Lemma. For a proof, try Exercise 14, or see [Lyndon and Schupp (1977)].

Lemma 1.6 [**van Kampen's Lemma**] *Given a symmetrized group presentation $G \cong \langle S|R\rangle$ and a word w in the letters S, the element $\bar{w}$ represented by w is trivial in G if and only if there is a reduced R-diagram Δ whose boundary is labeled (letter-for-letter) by the word w.*

We now use van Kampen diagrams to analyze a Coxeter system (W, S). Since every generator $a \in S$ is its own inverse, we may consider an R-diagram over the symmetrized version of the presentation $\langle S \mid R\rangle$ as an undirected graph in which each edge is labeled by a single generator. For the remainder of this section, let Δ be such a van Kampen diagram.

Since each relator in the set R is of even length, we may speak unambiguously of the edge *opposite* a given edge on a face whose boundary label is given by a relator $r \in R$. Consider any edge on the boundary of Δ described above. This edge lies on the boundary of exactly one bounded face of Δ, and we may determine the edge that is opposite the original edge on this face. By planarity this edge either lies on the perimeter of Δ

or in exactly one other bounded face of Δ. If the latter condition holds, we may once more select the edge opposite this edge in this next bounded face. Continuing this process until we have reached another edge on the perimeter of Δ, we obtain the *band* beginning at the first edge chosen.

As a warm-up to the proof of the Deletion Condition, we prove the following fact:

Lemma 1.7 *Consider Δ as above, and construct a band as described above. Then the band does not cross itself.*

The forbidden situation is indicated in Figure 1.4.

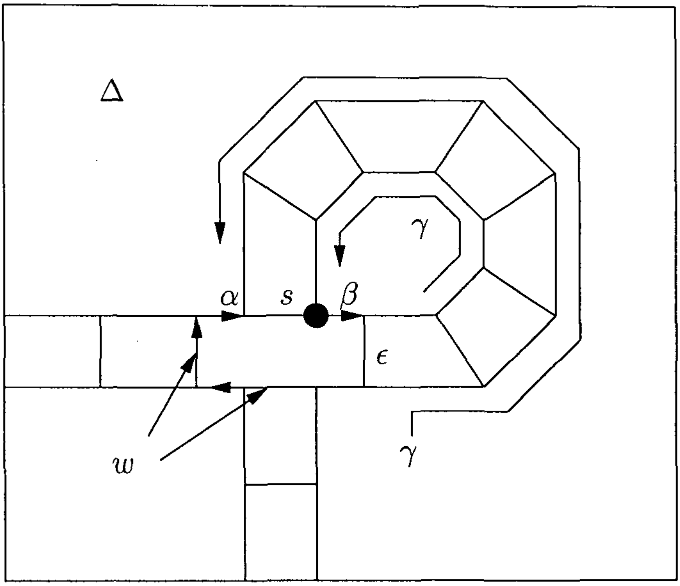

Fig. 1.4 A self-crossing band

Proof. Suppose by way of contradiction that there is a band which self-crosses. We use the notation indicated in Figure 1.4. In particular, w is the word in the letters s and s' which makes up one half of the relator at which the band self-intersects, and therefore $\bar{w}\bar{\alpha}s\bar{\beta}\epsilon = 1$, where $\epsilon \in \{s, s'\}$ is the letter occurring at the end of the relator face farther from the boundary of the diagram Δ.

We claim that this presents a contradiction to the order of ss'. Indeed, note that the word γ can be read both on the "inside" of the arc formed by the band, as shown, and on the "outside". Therefore, reading from the basepoint indicated by a dot in Figure 1.4, we see that

$$\bar{\beta}\bar{\gamma} = s\bar{\alpha}^{-1}\bar{w}\bar{\gamma}s = 1.$$

Thus $\bar{\alpha}\bar{\beta} = \bar{w}$. But this is nonsense, as w is geodesic and $\alpha\beta$ is strictly shorter than w. ☐

We now give the promised proof of the Deletion Condition.

Proof. Suppose that w is a freely reduced word representing $\bar{w}$ so that w is not geodesic. Let u be a geodesic word representing this same group element, and consider a van Kampen diagram Δ, as shown in Figure 1.5, with w labeling the "top", and u labeling the "bottom" of Δ. Such a diagram exists, because of van Kampen's Lemma. (We assume here that both wu^{-1} and $u^{-1}w$ are freely reduced. There are further subtleties regarding the exact appearance of Δ; these are left to the reader to resolve in Exercise 10.)

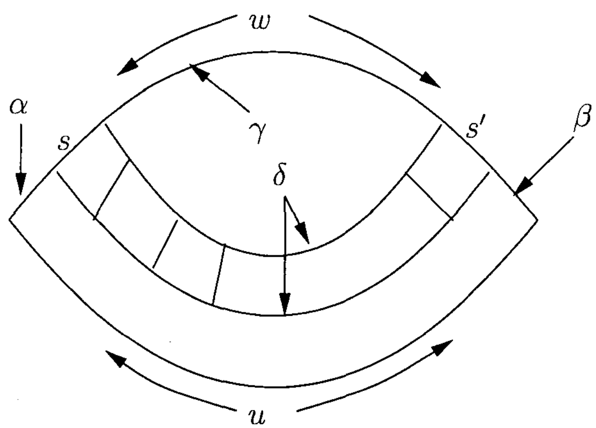

Fig. 1.5 The proof of the Deletion Condition

For any edge lying in the "top", labeled by w, we can construct a band as described above, which does not intersect itself (by Lemma 1.7) and which terminates somewhere on the boundary of Δ. We claim that for some edge lying in w, the corresponding band must terminate at another edge lying in w. Indeed, because u is strictly shorter than w, it cannot be that every band beginning in w terminates inside u.

Therefore, as shown in Figure 1.5, there is some band which begins and ends at different edges in w. Adopting the notation used in that figure, $\bar{\gamma} = \bar{\delta}$, and therefore

$$\bar{w} = \bar{\alpha} s \bar{\gamma} s' \bar{\beta} = \bar{\alpha} s \bar{\delta} s' \bar{\beta} = \bar{\alpha} \bar{\delta} \bar{\beta} = \bar{\alpha} \bar{\gamma} \bar{\beta},$$

and $\alpha\gamma\beta$ is obtained from w by the removal of exactly two letters (s and s'). □

As another application of van Kampen diagrams, let us prove the following fact.

Proposition 1.5 *Let (W, S) be an even Coxeter system, and let $w \in W$. If the length of w is k and $s_1 s_2 \cdots s_k$ and $s_1' s_2' \cdots s_k'$ are both words in the letters of S which represent w, then every letter $s \in S$ appears the same number of times in the first word as in the second.*

Proof. Consider a van Kampen diagram Δ whose "top" edge is labeled $s_1 \cdots s_k$ and whose "bottom" edge is labeled $s_1' \cdots s_k'$. Let $s_i = s$. Because (W, S) is even, the band beginning at s_i must end at another occurrence of s. If this band ends at some edge in the top of Δ, we obtain a contradiction to the length of w. Thus the band ends at an occurrence of s in $s_1' \cdots s_k'$. Since s_i was arbitrary, the proof is complete. □

Proposition 1.5 is a special case of the more general fact that any two geodesic words for the same group element must contain the same generators.

Remark 1.7 A large number of the fundamental results contained in [Bourbaki (1981)] and [Humphreys (1990)] are very easily proven using van Kampen diagrams. Further application of these diagrams will be found in this chapter's exercises, and throughout the remainder of the text.

1.5 Other geometric viewpoints: the Coxeter complex and the Davis complex

We conclude this chapter by considering a pair of geometric objects associated to a given Coxeter system (W, S), on which W acts in a natural fashion. (A third object, known as a *chamber system*, will be introduced in Chapter 6.)

The first construction is more "classical". The objects considered are called *Coxeter complexes*. The study of Coxeter complexes leads to the theory of buildings, developed primarily by J. Tits and his colleagues in the 1960s and 1970s, and detailed for instance, in [Brown (1989)] and [Ronan (1989)]. (Buildings have proven useful in algebraic geometry, CAT(0) geometry, and combinatorics, among other fields.)

The second construction is more recent, and was developed by M. Davis ([Davis (1983)]), building upon the work of E. B. Vinberg in [Vinberg (1971)]. (The more general spaces defined in [Vinberg (1971)] will be examined in 4.3.4.) The result of this construction will be called the *Davis complex* corresponding to a given system (W, S).

Before describing either of these objects, we provide a brief review of (abstract) simplicial complexes.

1.5.1 *Simplicial complexes*

Recall that an *abstract simplical complex* C with *vertex set* V is a collection Σ of finite subsets of C (each of which is called a *simplex* in C) satisfying the following conditions:

1. every singleton set $\{v\}$ $(v \in V)$ lies in Σ, and
2. if $\sigma \in \Sigma$ and $\tau \subseteq \sigma$, then $\tau \in \Sigma$. (In this case, τ is called a *face* of σ.)

The *rank* of a simplex σ is defined to be its cardinality as a set; the *dimension* of $\sigma \in \Sigma$ is defined to be one less than its rank. A complex C is called *finite dimensional* if there is a finite upper bound on the dimension of its simplices. Clearly any simplicial complex Σ is a poset, where order is defined by the face relation: $\tau \leq \sigma \Leftrightarrow \tau \subseteq \sigma$.

The *geometric realization* $|C|$ of a an abstract simplicial complex C is constructed by taking the union over all simplices $\sigma \in \Sigma$ of the *open geometric simplices* $|\sigma|$, where $|\sigma|$ is the interior of a standard Euclidean simplex of the same dimension as σ. More precisely, we may begin with a vector space with basis V. We then have (for each $\sigma \in \Sigma$) an open simplex $|\sigma|$ consisting of the convex combinations

$$\sum_{x \in \sigma} \lambda_x x,$$

where $\sum_{x\in\sigma} \lambda_x = 1$ and $\lambda_x > 0$ for all $x \in \sigma$. C (under the face relation) is isomorphic to the poset of *geometric* simplices of the complex $|C|$ so constructed (under set inclusion).

If the data we are given to begin with describe a poset rather than an abstract simplicial complex, we may define a related simplicial complex in the following fashion. Given the poset P, let $\mathrm{Ord}(P)$ be the collection of all finite chains (totally ordered subsets) of P. Considering each such chain as a simplex, $\mathrm{Ord}(P)$ is an abstract simplicial complex, isomorphic to the poset of simplices of its geometric realization, which we will denote by $\mathrm{Geom}(P)$. Note that, for instance, each element of P corresponds to a vertex of $\mathrm{Geom}(P)$, and a set of vertices of $\mathrm{Geom}(P)$ spans a simplex if and only if the corresponding elements of P form a totally ordered subset of P.

Given any lower set $\downarrow_P x = \{y \in P \mid y \leq x\}$ or upper set $\uparrow_P x = \{y \in P \mid y \geq x\}$ in P, we may define subcomplexes of $\mathrm{Geom}(P)$ by $\mathrm{Geom}(\downarrow_P x)$ and $\mathrm{Geom}(\uparrow_P x)$, which are called, respectively, *faces* and *dual faces* of $\mathrm{Geom}(P)$.

Recall that an action of a group G on a simplicial complex C is called *simplicial* if for every $g \in G$ and for every simplex $\sigma \in C$, $g \cdot \sigma$ is again a simplex of C.

1.5.2 *The Coxeter complex*

Given (W, S), we define a *special coset* to be any coset wW_T, where $w \in W$ and $T \subseteq S$. We define a poset P whose elements are the special cosets. For $w_1W_{T_1}, w_2W_{T_2} \in P$, we say that $w_1W_{T_1} \leq w_2W_{T_2}$ if and only if $w_2W_{T_2} \subseteq w_1W_{T_1}$. The Coxeter complex $X = X(W, S)$ is now defined to be the geometric realization $\mathrm{Geom}(P)$ of this poset.

X possesses a great deal of structure beyond its simpliciality. The maximal dimensional simplices of X are called its *chambers*, and the codimension-1 faces of a chamber are called its *mirrors*. The support of a given mirror (that is, the affine subspace determined by the mirror) is called its *wall*. The walls of X are thus codimension-1 (affine) hyperplanes in the space in which X is realized. The complement of each wall consists of two disjoint open sets, each called a *half-space* of X. It can be shown that every mirror is a face of exactly 2 chambers, and that any two chambers can be connected by a sequence of "adjacent" chambers known as a *gallery*. These properties make X into what is known as a *chamber complex*.

The group W acts on X in a natural way: $w \cdot w_1W_T = (ww_1)W_T$. This action turns out to be simply transitive on the set of chambers (that is, the action is transitive and the only element w taking any given chamber to itself is the trivial element). We can identify some chamber C as the *fundamental chamber*, and all other chambers are translates of C under W's action. Each $s \in S$ then acts as a "reflection" in some wall H_s supporting a codimension-1 face of C. A single vertex of C is moved by this reflection, so we can unambiguously label each vertex of C with that generator. Moreover, each vertex v of X can be labeled by an element of S by translating v back to a (unique!) vertex of C and giving v the label of that vertex. This yields a canonical *labeling* of X which makes X into a *labeled* chamber complex, in which the given labeling respects the action of W on X in a very natural way.

For the precise definitions of the terms and concepts introduced above, the reader is encouraged to consult [Brown (1989)]. We will examine X in more detail as it becomes necessary to do so (in Chapter 7). We content ourselves now with a few examples.

If W is a finite Coxeter group, then W can be realized as a subgroup of the group $GL(V)$ of invertible linear transformations on some finite-dimensional Euclidean vector space V. In this case, it turns out that the simplices of $X(W, S)$ are in one-to-one correspondence with the simplices of V defined by the intersection of the hyperplanes in which the elements of W act by reflections. (That is, this cell structure on V is isomorphic, as an abstract simplicial complex, to X. If, for example, $W = D_n$ with the usual presentation, $V = \mathbb{E}^2$ is decomposed into $2n$ triangular sectors, each having an angle $\frac{\pi}{n}$ at the origin, as shown in Figure 1.6. *Cf.* 1.2.3.)

In case W is infinite, X is not quite so simple. Although in general V can still be represented as a subgroup of the linear group GL(V) for some finite-dimensional vector space V, V itself may no longer be the carrier for the complex X. However, we may identify X with a certain subset of the dual space V^*, properly subdivided into simplices. This subspace, known as the *Tits cone*, is defined most easily in terms of a fundamental domain for W's action.

Precisely, let C be defined by

$$C = \{f \in V^* \mid f(v) > 0 \text{ for all } v \in V\}. \tag{1.9}$$

(It is known that this set is not empty.) Then, given $w \in W$ and $f \in V^*$, we define $w \cdot f$ by $(w \cdot f)(v) = f(w^{-1}v)$ for all $v \in V$. Then the Tits cone is defined to be the union

$$\bigcup_{w \in W} w\bar{C}$$

where $\bar{C}$ is the closure of C in V^*.

As an example, consider the case of D_∞, examined in 1.2.3. Here C is the sector determined by the rays through the points $(0, 1)$ and $(1, 1)$,

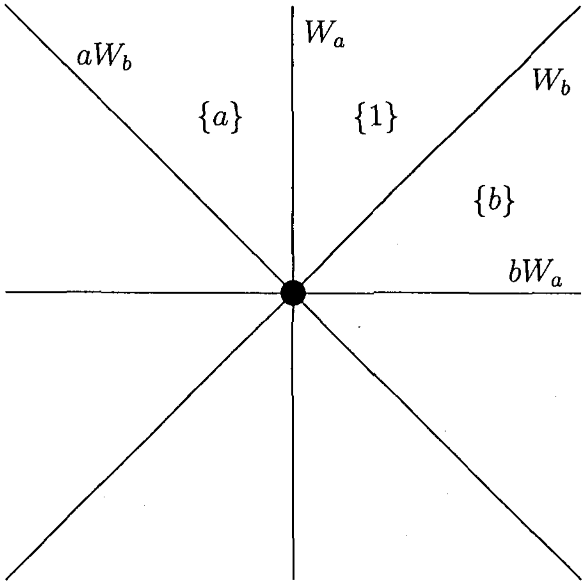

Fig. 1.6 The Coxeter complex $X(D_4, \{a, b\})$

and the Tits cone is the open upper half-space of $\mathbb{E}^2$, plus the origin. (The origin is the vertex of X corresponding to the group $W = D_\infty$ itself.)

More information on the Tits cone can be found in Section 5.13 of [Humphreys (1990)] and in [Howlett, *et al.* (1997)], for instance. It will play a major role in our work only in Section 7.2.

1.5.3 *The Davis complex*

We now define a different simplicial complex $\Sigma = \Sigma(W, S)$ on which a given Coxeter group W acts in a manner similar to the action on $X(W, S)$ above. The construction of Σ is due to M. Davis (in [Davis (1983)]), following a more general construction of E. B. Vinberg ([Vinberg (1971)]) which will be discussed briefly in Section 4.3, where it will be required in order to complete the proof of Theorem 4.4.

The complex Σ has the advantage that it can be given a metric that yields a very desirable geometry.

Theorem 1.4 *Let (W, S) be a Coxeter system. Then there is a locally finite simplicial complex $\Sigma = \Sigma(W, S)$ on which W acts simplicially, cocom-*

pactly, and properly by isometries. Moreover, Σ admits a natural piecewise Euclidean CAT(0) *metric.*

Recall that a group G is said to act *cocompactly* on the metric space X if there exists some compact subset K of X such that $G \cdot K = X$. The action is said to be *proper* if for every point $x \in X$, there is a number $r > 0$ such that the set

$$\{g \in G \mid g \cdot B(x,r) \cap B(x,r) \neq \emptyset\}$$

is finite. ($B(x,r)$ is the open ball of radius r around x.) Note that the action of W on the Coxeter complex $X(W,S)$ is proper, although not cocompact.

Recall the definitions of the posets $\mathcal{S} = \mathcal{S}(S)$ and $W\mathcal{S} = W\mathcal{S}(S)$ given in 1.3.3. Let $\Sigma = \text{Geom}(W\mathcal{S})$, $K = \text{Geom}(\mathcal{S})$, and $K_T = \text{Geom}(\uparrow_{\mathcal{S}} T)$. There is a natural inclusion of K into Σ induced by $T \mapsto 1 \cdot W_T$. The image of K in Σ under this identification is called the *fundamental chamber* of Σ. Since W acts in a natural way on $W\mathcal{S}$ (by $w \cdot w'W_T = (ww')W_T$, just as in the case of the Coxeter complex), we can extend this to an action of W on Σ. As in the Coxeter complex, the translates of K under this action are called the *chambers* of the action; the sets $K_s = K_{\{s\}}$ are called the *mirrors* of K. It is across the hyperplane in $\Sigma(W,S)$ which supports the mirror K_s that s acts as a reflection. Any translate of such a hyperplane will be called a *wall* of the complex $\Sigma(W,S)$. Any wall of $\Sigma(W,S)$ is exactly the fixed point set of some reflection in (W,S).

We call $\Sigma = \Sigma(W,S)$ the *Davis complex* associated to (W,S).

Remark 1.8 There are two fundamental differences between the Davis complex and the Coxeter complex. First, the Coxeter complex contains a vertex corresponding to *any* coset of a standard parabolic subgroup, whereas the vertices of the Davis complex correspond only to cosets of spherical subgroups. Second, the order in the poset of all special cosets is defined by reverse inclusion, which order is dual to that used in the definition of the Davis complex.

In case W is finite, it is often easy to visualize K and Σ, and this construction yields unsurprising results. For instance, suppose that W is the dihedral group D_n of order $2n$ with the usual presentation: $\langle a, b \mid a^2, b^2, (ab)^n \rangle$. Then Σ can be identified topologically with a $2n$-gon, along with its interior. Combinatorially, we subdivide the $2n$-gon into $4n$ 2-simplices, where K consists of the union of two of these simplices, and of which $2n$ translates fill out Σ. The barycenter of the subdivided $2n$-gon corresponds to the group W itself.

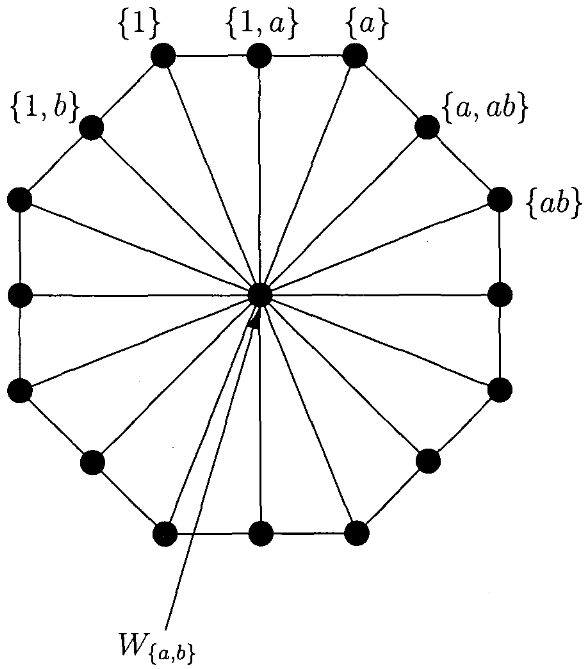

Fig. 1.7 The Davis complex $\Sigma(D_4, \{a, b\})$

The reader is invited to construct the Davis complex for a few other simple groups in the exercises.

The complex Σ has a number of very nice properties, some of which we will investigate in the course of our studies.

For instance, given a simplex σ in Σ, the subgroup Stab(σ) of W of elements which pointwise fix σ is a conjugate of a spherical subgroup of W. Indeed, a given simplex corresponds to a chain of spherical cosets $w_1 W_{T_1} < w_2 W_{T_2} < \cdots < w_n W_{T_n}$. It is easy to show that Stab(σ) is given by $w_1 W_{T_1} w_1^{-1}$. Using this and the fact that Σ carries a CAT(0) metric as defined in the next subsection, we will be able to prove Lemma 1.2, one of the most important intermediate results in this text.

We will now discuss the construction of the CAT(0) metric in very general terms, referring the reader to [Davis and Moussong (1999)] for a detailed study.

1.5.4 *The metric on* Σ

We first provide the definition of the CAT(0) condition, which requires a number of additional geometric notions.

For any real number κ there is (up to isometry) a unique complete, simply connected, Riemannian manifold M^n_κ of dimension n and constant sectional curvature κ. For $\kappa = 0$, M^n_κ is simply n-dimensional Euclidean space. For $\kappa > 0$, M^n_κ is spherical, and for $\kappa < 0$ (the most interesting case for many group theoretic purposes), M^n_κ is hyperbolic. These spaces are frequently called *model spaces* of a given curvature κ because they provide models for the behavior of spaces which exhibit similar curvature, either locally or globally.

Now suppose that X is a metric space with metric d. Let x, y be two points in X, and let α be a continuous function from the interval $[0, l]$ to X which satisfies $\alpha(0) = x$, $\alpha(l) = y$, and $d(\alpha(t), \alpha(t')) = |t - t'|$ for every $t, t' \in [0, l]$. In this case we call α a *geodesic* from x to y, and the image $\alpha([0, l])$ is called a *geodesic segment* from x to y, and may be denoted by $[x, y]$. These definitions makes precise the notion of the "shortest path" between two points in the space X.

Fix a nonpositive real number κ. (The following definitions can be modified to deal with the case in which X is positively curved, but since we are most interested in the case when $\kappa \leq 0$, we will not address the case in which $\kappa > 0$.)

Consider any three points x, y, z in X and construct a triangle Δ whose sides consist of geodesic segments between the respective vertices x, y, z. We can find a (geodesic) triangle Δ' in the model space $M = M^2_\kappa$, with vertices x', y', z', such that $d(x, y) = d(x', y')$, $d(y, z) = d(y', z')$, and $d(z, x) = d(z', x')$ (with obvious abuse of notation). Let a and b be any points on the triangle Δ, and let a' and b' be the corresponding points in Δ'. That is, for instance, if a lies on the geodesic segment $[x, y]$ and $d(a, x) = r$, then a' is the point lying on the geodesic segment $[x', y']$ such that $d(x', a') = r$. If for any choice of a and b

$$d_X(a, b) \leq d_M(a', b') \tag{1.10}$$

holds, we say that Δ satisfies the CAT(κ) inequality. If $\kappa \leq 0$ and (1.10) is satisfied for *every* geodesic triangle in X, X is said to be CAT(κ).

Remark 1.9 The term CAT(κ) was introduced by M. Gromov, and comes from the names E. Cartan, A.D. Alexandrov, and V.A. Toponogov, three mathematicians who considered problems very closely related to the

CAT(κ) condition.

How does one impose a CAT(0) metric upon the Davis complex Σ? Consider a Coxeter system (W, S). Roughly speaking, we define the "shape" of the portions of Σ corresponding to spherical subgroups W_T in such a manner that when these portions intersect nontrivially, the shapes can be glued together isometrically. The geometry of the entirety of Σ is then induced by the fact that K is a fundmental domain for the action of W on Σ. The resulting metric is piecewise Euclidean, and the action of W on Σ is by isometries.

More precisely, suppose first that we are given a system (W, S) such that W is finite, and select any point x in the fundamental domain for the action of W as a reflection group in Euclidean space $\mathbb{E}^n$ (x may be given, for example, by specifying the distance from x to each of the hyperplanes corresponding to the generators of S). We define the *Coxeter cell* C to be the convex hull in $\mathbb{E}^n$ of the orbit Wx of x under the action of W. (For different choices of x, the exact shape of C will vary.) For example, suppose that $W = D_n$ is presented as $\langle a, b \mid a^2, b^2, (ab)^n \rangle$ and x is chosen to be equidistant the two hyperplanes which bound the fundamental domain consisting of a sector of $\mathbb{E}^2$ with angle π/n. Then C will be a regular n-gon.

The following lemma is straightforward (see [Davis and Moussong (1999)]).

Lemma 1.8 *If W is finite and C is a Coxeter cell corresponding to W, then the poset WS is isomorphic to the poset* Faces(C) *of faces of C by the correspondence $w \mapsto wx$.*

For example, a Coxeter cell corresponding to the dihedral group D_n is a $2n$-gon, each of whose edges have one of two lengths (in general), which alternate.

Now consider (W, S) for an arbitrary Coxeter group W. We fix once and for all a map f from S to the positive real numbers which will be used to determine the shape of the Coxeter cells C introduced below. (As above, we can imagine that this map prescribes a point x lying a predetermined distance from each generator's hyperplane.) For each spherical subgroup W_T of W, we define a piecewise Euclidean metric structure on Σ by identifying the face (as defined in 1.5.1) Geom($\downarrow_{WS} W_T$) with the Coxeter cell $C(W_T)$ corresponding to W_T which is defined by restricting the map f to T. Any translate of a face of Geom(W_T) also has shape determined by $C(W_T)$.

For example, consider the group

$$W = \langle a, b, c \mid a^2, b^2, c^2, (ab)^3, (bc)^3, (ac)^3 \rangle,$$

which we will hereafter refer to as $(3,3,3)$. The Davis complex for W is (combinatorially) a triangulation of the Euclidean plane by triangles. The Coxeter cells (i.e., the faces) corresponding to the 2-generated subgroups of W are hexagons, each of which falls into one of at most three isometry classes.

The action of W upon Σ is then clearly by isometries. It is, however, a non-trivial matter to show that the metric which is induced by this structure is CAT(0). The interested reader may consult [Davis and Moussong (1999)] for details.

Remark 1.10 In [Moussong (1996)], Moussong also characterizes word hyperbolic Coxeter groups. A group G is said to be *word hyperbolic* with respect to a particular generating set S provided the Cayley graph $\Gamma(G,S)$ is a hyperbolic metric space (relative to the usual path metric). This means, roughly, that $\Gamma(G,S)$ satisfies a certain negative-curvature condition. Although we know it to be true for Coxeter groups, it is an open question whether every word hyperbolic group acts properly and cocompactly by isometries on some CAT(0) metric space.

Because W acts so nicely on a CAT(0) space, W enjoys a number of desirable properties. For instance, the Conjugacy Problem for Coxeter groups is solvable (see Section 2.4). For now we content ourselves with a proof of Lemma 1.2.

Proof. We apply the following result (see [Bridson and Haefliger 1999], II.2) to the action of $G \leq W$ on the Davis complex $\Sigma(W,S)$:

Proposition 1.6 *Let G be a finite group of isometries of a complete* CAT(0) *space X. Then the set X^G of points of X fixed by G is a non-empty convex subspace of X.*

Our finite $G \leq W$ is a finite group of isometries of the CAT(0) space Σ, and the above proposition shows that its set of fixed points is non-empty. Given a point v in Σ fixed by G, G fixes the smallest simplex containing v. By the construction of Σ, the isotropy group of a simplex is a finite parabolic subgroup wW_Tw^{-1} (for some $T \subseteq S$). Thus $G \leq wW_Tw^{-1}$ for this T. □

We will apply Lemma 1.2 almost immediately as soon as we begin to prove theorems concerning rigidity.

1.6 Exercises

1. Let W be a finitely generated Coxeter group. Prove that the set Φ of roots associated to a Coxeter group W as in 1.2.2 is finite if and only if W is finite.

2. Define Φ and Π as in 1.2.2. Prove that Π and $-\Pi$ exhaust Φ in case W is finite.

3. Verify all of the statements regarding the mapping r_i made in the paragraph containing (1.3).

4. Let $\{e_1, ..., e_n\}$ be the standard orthonormal basis of $\mathbb{E}^n$. The group W of order $2^n n!$ generated by exchanges $e_i \leftrightarrow e_j$ $(1 \leq i < j \leq n)$ and by exchanges $e_i \leftrightarrow -e_i$ $(1 \leq i \leq n)$ is a Coxeter group. Find a generating set of n reflections for this group, and compute the corresponding Coxeter matrix and diagram. (These groups, typically denoted B_n, are known as the *hyperoctahedral groups*, as they are the groups of symmetry of the regular octahedra of a given dimension.)

5. Prove Proposition 1.1.

6. Prove Lemma 1.4. (*Hint*: use the function $n(w)$ from 1.3.2.)

7. Prove that if (W, S) is a Coxeter system with finite group W, then there is a unique element Δ_W of greatest length in W. Moreover, show that Δ_W has order 2, and that for any $s \in S$, $\Delta_W s \Delta_W^{-1} = s'$ for some $s' \in S$. (The element Δ_W is called the *Garside element* or *Coxeter element* of W. The former term comes from the work of F.A. Garside, who produced pioneering work on certain Artin groups. Artin groups are discussed in Chapter 3 and elsewhere in the sequel.)

8. Prove that $W\mathcal{S}(S)$ is a poset under inclusion, and show that $w_1 W_{T_1} \leq w_2 W_{T_2}$ if and only if $T_1 \subseteq T_2$ and $w_2^{-1} w_1 \in W_{T_2}$.

9. Prove Lemma 1.5. Also, state and prove the corresponding result for right cosets.

10. What subtleties are overlooked in our application of van Kampen diagrams to proving the Deletion Condition? Explain how to get around any such pesky details.

11. Describe (and draw, as well as you can!) the Davis complex for each of the groups Σ_n, $(\mathbb{Z}_2)^n$, and $(3, 3, 3)$, indicating in each case the fundamental

domain K and the walls corresponding to various reflections.

12. (For the definition of the *Cayley graph* of a group G relative to a generating set S, consult any text on group theory.) Let (W, S) be a Coxeter system. Prove that the Cayley graph embeds into the cell complex Σ' (defined by Coxeter cells as in 1.5.4) dual to the Davis complex $\Sigma(W, S)$. (This identification turns out to be a *quasi-isometric* embedding; that is, distances between points on the Cayley graph are distorted in a controllable fashion upon embedding into the Davis complex.)

13. Let (W, S) be a Coxeter system, with diagram $\mathcal{V}$. The *nerve* $N = N(W, S)$ of the system (W, S) is the simplicial complex which results from $\mathcal{V}$ by attaching a k-dimensional simplex according to the vertex set σ ($|\sigma| = k + 1$) whenever W_σ is spherical. Let $\Sigma(W, S)$ be given the cell structure determined by identifying each face of Σ with the Coxeter cell of the corresponding type (as in 1.5.4). For every vertex in Σ so cellulated, prove that $\mathrm{Lk}(v, \Sigma)$ is homeomorphic to N. (Recall that the *link* $\mathrm{Lk}(v, C)$ of a vertex v in the simplicial complex C is the simplicial complex consisting of a simplex σ' of dimension $k - 1$ for each k-simplex σ of C containing v in its closure.)

14. Prove van Kampen's Lemma (Lemma 1.6). (*Hint*: In one direction, you can induct on the number of cells in the given R-diagram. For the other implication, begin by expressing the word w as a product, in the free group on S, of conjugates of relators from R.)

15. Prove: if T is a spherical subset of S, the vertices of T induce a complete subgraph of $\mathcal{V}$.

16. Prove: for any two points x and y in a CAT(0) metric space, if a there exists a geodesic from x to y, then this geodesic is unique. (More is true, in fact, for it turns out that CAT(0) metric spaces are *geodesic*: there exists a geodesic between any two points. See [Bridson and Haefliger 1999] for details.)

17. A *right-angled Artin group* G is any group with generating set $A = \{a_i\}_{i \in I}$ and presentation

$$\langle A \mid R \rangle,$$

where $R = \{a_i a_j = a_j a_i\}$ for some (though not necessarily all) pairs $i \neq j$ in I. Note that every generator has infinite order, so that (G, A) is not a Coxeter system. Use van Kampen diagrams to show that G *does* satisfy the Deletion Condition with respect to A, so that Theorem 1.3 is not true if the generating set S given there does not consist of involutions. (The most general groups satisfying the Deletion Condition are known as weakly partially commutative Artin-Coxeter groups, and are investigated by S.A. Basarab in [Basarab (2002)].)

18. Describe all subgroups of a given dihedral group, D_{2n}, noting any differences between the cases n odd and n even.

19. Let W be realized as the group of orthogonal transformations generated by the reflections $\{r_i\}_{i \in I}$ in some vector space V. Let Φ be the associated root system. Prove that if $v \in V$ is a unit vector (with respect to the bilinear form defined in 1.2.2) such that $r_v \in W$, then $v \in \Phi$. (*Hint:* induct on the length of r_v and use results from 1.3.2.)

20. With the same set-up as in the previous exercise, assume that Φ is irreducible. Prove that for any $v_1, v_2 \in \Phi$, there is $v \in \Phi$ such that $(v_i, v) \neq 0$, $i = 1, 2$.

21. Let W, V, and Φ be as in the previous two exercises. In [Brink and Howlett (1993)], B. Brink and R. Howlett describe a partial order on the set $\Pi \subseteq \Phi$, the set of positive roots; this order was later extended (in [Howlett, *et al.* (1997)]) to all of Φ as follows. Let $\alpha, \beta \in \Phi$. We say that α *dominates* β, and write $\alpha \succeq \beta$, if

$$\{w \in W \mid w\alpha \in -\Pi\} \subseteq \{w \in W \mid w\beta \in -\Pi\};$$

that is, if every element of W negating α also negates β. The corresponding strict order is denoted by $\succ$. In [Brink and Howlett (1993)] it is shown that for all $\alpha, \beta \in \Phi$, $(\alpha, \beta) \geq 1 \Leftrightarrow \alpha \succeq \beta$ or $\beta \succeq \alpha$.

Suppose that $S = \{a, b, c\}$, with corresponding simple roots $\Delta = \{\alpha, \beta, \gamma\}$. Suppose moreover that each of m_{ab}, m_{ac}, and m_{bc} are at least 3. Show that there exist roots $\{\alpha_1, \beta_1, \gamma_1\} \subseteq \Phi$ such that $\alpha_1 \succ \alpha$, $\beta_1 \succ \beta$, and $\gamma_1 \succ \gamma$. (*Hint*: new roots as desired can be obtained from α, β, and γ under the action of the appropriate words of length 2.)

This result can be generalized to all 3-generated infinite irreducible Coxeter systems. Much more is known about the dominance order $\succeq$. For instance, it can be shown that the set of positive roots which are minimal with respect to the order (among all positive roots) is finite. In Chapter 7 we will say more about $\succeq$.