Chapter 1

From Geometry to the Quantum

According to one legend, Lucifer was God's favorite angel before stealing light from him and bringing it to mankind. For this, to us a generous act, Lucifer was expelled from heaven and subsequently became the top angel in hell. Most of us are not able to steal possessions from God, but we can at least admire his most marvellous creation — light. Quantum optics is the theory describing our most sophisticated understanding of light.

This book intends to acquaint you with the basic ideas of how physics describes the interaction of light and matter at three different levels: classical, semi-classical and quantum. You will be able to understand basic principles of laser operation leading to the ideas behind non-linear optics and multiphoton physics. You will also become familiar with the ideas of field quantization (not only the electromagnetic field, but also a more general one), nature of photons, and quantum fluctuations in light fields. These ideas will bring you to the forefront of current research. At the end of this book, I not only expect you to understand the basic methods in quantum optics, but also to be able to apply them in new situations — this is the key to true understanding. The notes contain five sets of problems, which are intended for your self-study. Being able to solve problems is definitely crucial for your understanding, and a great number of problems have been chosen from the past exam papers at Imperial College London set by me. I also hope — and this is I believe really very important — that the book will teach you to appreciate the way that science has developed within the last 100 years or so and the importance of the basic ideas in optics in relation to other ideas and concepts in science in general. The book contains a number of topics from thermodynamics, statistical mechanics and information theory that will illustrate that quantum optics is an integral part of a much larger body of scientific knowledge. I hope that at the end of it all, and this is really my main motivation, you will appreciate how quantum description of light forms an important part of our cultural heritage.

Optics itself is an ancient subject. Like any other branch of science, its roots can be found in Ancient Greece, and its development has always been inextricably linked to technological progress. The ancient Greeks had some rudimentary knowledge of geometrical optics, and knew of the laws of reflection and refraction, although they didn't have the appropriate mathematical formalism (trigonometry) to express these laws concisely. Optics was seen as a very useful subject by the Greeks: Archimedes was, for example, hired by the military men of the state to use mirrors and lenses to defend Syracuse (Sicily) by directing the Sun's rays at enemy ships in order to burn their sails. And like most of human activity (apart from some forms of art and mathematics) the Greek knowledge was frozen throughout the Middle Ages only to awaken more than 10 centuries later in the Renaissance. At the beginning of the 15th century, Leonardo da Vinci designed a great number of machines using light and was apparently the first person to record the phenomenon of *interference* — now so fundamental to our understanding not only of light, but matter too (as we will see later in this book). However, the first proper treatment of optics had to wait for the genius of Fermat and Newton (and, slightly later, Huygens) who studied the subject, making full use of mathematical rigor. It was then, in the 16th and 17th centuries, that optics became a mature science and an integral part of physics.

If you could shake a little magnet 428 trillion times per second, it would start making red light. This is not because the magnet would be getting hotter — the magnet could be cold and situated in the vacuum (so that there is no friction). This is because the electromagnetic field would be oscillating back and forth around the magnet which produces red light. If you could wiggle the magnet a bit faster, say 550 trillion times per second, it would glow green, while at around 800 trillion times per second it would produce light that is no longer visible — faster still and it would become ultraviolet. In the same respect, we can think of atoms and molecules as little magnets producing light — and their behavior as they do so is the subject of quantum optics.

From our modern perspective, optics can be divided into three distinct areas which are in order of increasing complexity and accuracy (they also follow the historical development):

- *Geometrical optics* is the kind of optics you would have done in your sixth form and the first year of university, prior to learning that light is an electromagnetic wave. Despite the fact that this is the lowest approximation of treating light, we can still derive some pretty fancy results with it — how lenses work, for instance, or why we see rainbows. I will assume that you are fully familiar with geometrical optics.

- *Physical optics* is based on the fact that light is an electromagnetic wave and, loosely speaking, contains geometrical optics as an approximation when the wavelength of light can be neglected ($\lambda \to 0$). Behavior of light as described by physical optics can be entirely deduced from Maxwell's equations, and it is this level of sophistication that we will investigate at the very beginning of the book.
- *Quantum optics* takes into account the fact that light is quantized in chunks of energy (called photons), and this theory is the most accurate way of treating light known to us today. It contains physical optics (and hence geometrical optics) as an approximation when the Planck constant can be neglected ($\hbar \to 0$). This treatment will be the core of the book.

Geometrical optics can be summarized in a small number of fundamental principles. For those of you interested in the colorful history of optics, I mention Huygens' *Treatise on Optics* as a good place to read about the early understanding of light. Here are the three basic principles that completely characterize all the phenomena in geometrical optics:

Geometrical Optics Principles

(1) In a homogeneous and uniform medium, light travels in a straight line.
(2) The angle of incidence is the same as that of reflection.
(3) The law of refraction is governed by the law of sines — to be detailed below (see Figure 1.1).

Are these laws independent of each other or can they be derived from a more fundamental principle? It turns out that they can be summarized in a very beautiful statement due to Fermat.

Fermat's Principle

Fermat's principle of least time. Light travels such that the time of travel is extremized (i.e. minimized or maximized).

All the above three laws can be derived from Fermat's principle. We will now briefly demonstrate this. The fact that in a homogeneous and uniform medium light travels in a straight line is simple, as the speed of light is the same everywhere in such a medium (by definition of the medium), and therefore a straight line, being the shortest path between two points, also leads to the shortest time of travel. The same reasoning applies for the incidence and reflection angles. The law of sines is a bit more complicated to derive, but I will now show you how to do so in a few lines. Suppose that light is going from a medium of refractive index 1 to a medium of refractive index n as shown in Figure 1.1.

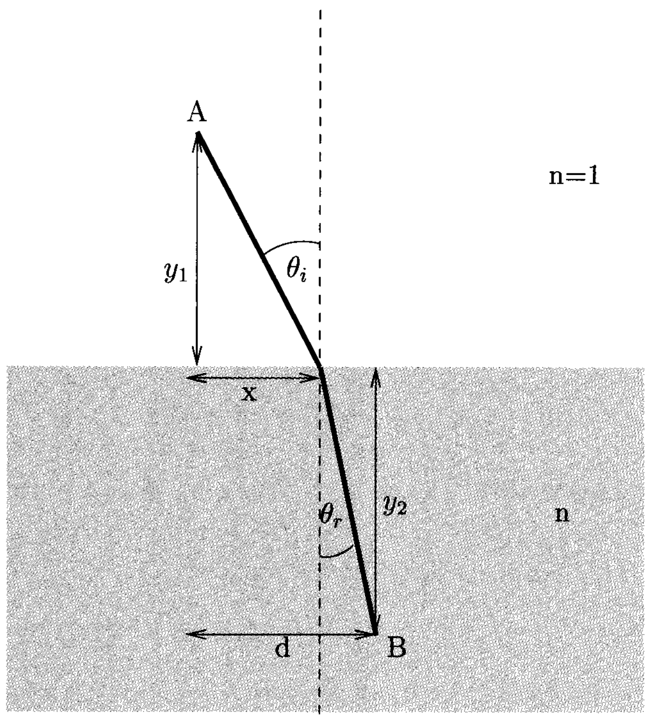

Fig. 1.1 The law of sines can be derived from Fermat's principle of least time. The full derivation is in the notes.

The total time taken from the point A to the point B is

$$t \propto \sqrt{x^2 + y_1^2} + n\sqrt{y_2^2 + (d-x)^2} \tag{1.1}$$

Note that the second term is multiplied by n, as the speed of light is smaller in the medium of refractive index n, being equal to c/n where c is the speed of light in vacuum. Now, Fermat's principle requires that the time taken is extremized, leading to

$$\frac{dt}{dx} \propto \frac{x}{\sqrt{x^2 + y_1^2}} - \frac{(d-x)}{n\sqrt{y_2^2 + (d-x)^2}} = 0 \tag{1.2}$$

which, after a short restructuring, gives

$$\sin\theta_i = n\sin\theta_r \tag{1.3}$$

since $\sin\theta_i = x/\sqrt{x^2+y_1^2}$ and $\sin\theta_r = (d-x)/n\sqrt{y_2^2+(d-x)^2}$. Therefore, all three basic laws of geometrical optics can be derived from Fermat's least time principle. We can, of course, also ask "Why Fermat's principle?". But the reason for this cannot be found in geometrical optics. We need a more sophisticated theory to explain this.

Newton believed that light is made up of particles. Contrary to him, Huygens, who was his contemporary, believed that light is a wave. He reasoned as follows. If light is made up of particles then when we cross two different light beams, we would expect these particles to collide and produce some interesting effects. However, nothing like this really happens; in reality, the two beams just pass through each other and behave completely independently. The key property that in the end won the argument for Huygens against Newton was interference. That light exhibited interference was beautifully demonstrated by Young in his famous "double slit" experiment. Young basically observed a sinusoidal pattern of dark and light patterns (fringes) on a screen placed behind slits which were illuminated. The only way that this could have been explained was by assuming that light is a wave. However, the scientific community in England was not very favorable towards his findings and did not accept them for some time. Theoretically, the argument was clinched by Maxwell some 60 years after Young's experiment. He first came up with four equations fully describing the behavior of the electromagnetic field. These are the celebrated Maxwell's equations (I write their form in vacuum as this will be the relevant form for us here)

Maxwell's Equations

$$\nabla \mathbf{E} = 0 \tag{1.4}$$

$$\nabla \wedge \mathbf{E} = -\frac{d\mathbf{B}}{dt} \tag{1.5}$$

$$\nabla \mathbf{B} = 0 \tag{1.6}$$

$$\nabla \wedge \mathbf{B} = \mu_0\epsilon_0\frac{d\mathbf{E}}{dt} \tag{1.7}$$

where μ_0 is the permeability of free space and ϵ_0 is permittivity of free space. Maxwell was then very surprised to discover that he could derive a wave equation for the E and B fields propagating at the speed of light. This is very easy to obtain from the above equations (and you can find it in any textbook on electromagnetism):

we need to take a curl of the second equation and substitute the value of $\nabla \wedge \mathbf{B}$ from the last equation. We have

$$\nabla \wedge \nabla \wedge \mathbf{E} = -\nabla \wedge \frac{d\mathbf{B}}{dt} \tag{1.8}$$

$$\nabla \wedge \mathbf{B} = \mu_0 \epsilon_0 \frac{d\mathbf{E}}{dt} \tag{1.9}$$

which leads to the wave equation by using the fact that $\nabla \wedge \nabla \wedge =$ grad div $- \nabla^2$,

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E} \tag{1.10}$$

where $c = 1/\sqrt{\mu_0 \epsilon_0}$ is the speed of light. The same wave equation can be derived for the magnetic field by manipulating the same two equations and reversing our steps (i.e. taking the curl of B first and then using the second equation). That this is so should be immediately clear from the symmetrical form of Maxwell's equations with respect to interchanging B and E. So, Maxwell concluded that light is an electromagnetic wave! Therefore, it displays all the wave properties: interference, in particular.

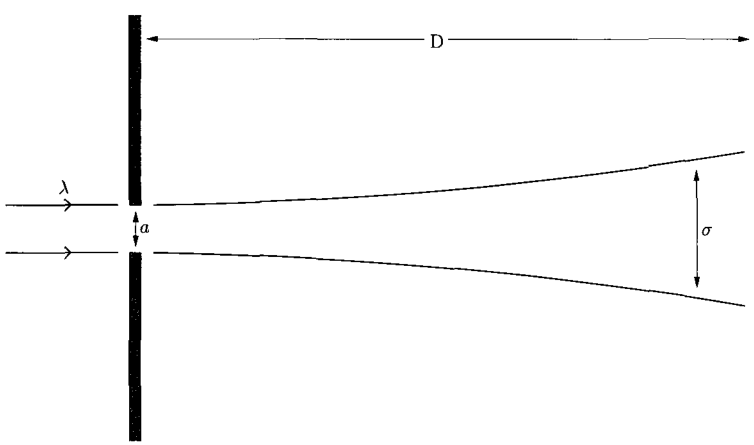

Fig. 1.2 Simple visualization of light diffraction. We observe in the laboratory that a light which passes through a small slit will spread in its width as it propagates. The distance beyond which the spread becomes significant (defined in the text) is called the Fraunhofer limit.

Let's describe a very simple interference behavior of a light beam of wave length λ, passing through a single slit of width a. A distance D after the slit we will obtain a bright spot of diameter σ. This spot will in general be larger than the size of the slit, which is the indication that light "bends around corners", i.e. it interferes.[1] There is a very simple relationship between the four quantities just mentioned which can be derived from a more rigorous wave optics treatment (see e.g. *Wave Optics* by Hecht):

$$\lambda D = \sigma a \tag{1.11}$$

(Just think of an equation involving four numbers — dimensionally we have to multiply two numbers and equate them to the product of the other two. A logical way of doing so is to multiply the largest and the smallest number, D and λ respectively and equate them to the other two middle sized numbers — hence the above equation!) The Fraunhofer limit is the distance after which the light starts to spread, i.e. when $a = \sigma$. We therefore deduce

Fraunhofer Diffraction

$$D = \frac{a^2}{\lambda} \tag{1.12}$$

This is a very useful formula to remember as it tells us under what conditions to expect light to start to behave like a wave (rather than travel in a straight line). Suppose that the slit is $1mm$ wide, and that $\lambda = 500nm$. Then for distances larger than $D = 4m$, light would behave like a wave. For distances below $4m$ light would for all practical purposes travel in a straight line — which is why geometrical optics is such a good approximation in the first place! (In a laboratory one would, of course, perform an interference experiment on a much smaller scale, and this would be achieved by putting a lens immediately after the slit to focus the light.)

What happens if light propagates not in the vacuum but in the air? Then there are atoms around which light can interact with. Imagine the following situation: a beam of light encounters two atoms as shown in Figure 1.3. The initial wave vector of light, which also determines the direction of propagation, is $\mathbf{k}$. Suppose that the light changes its wave vector (and hence possibly the propagation direction as well) to $\mathbf{k}'$ after scattering. Now I have to put you in the right frame of mind for calculating what we need from the wave formalism in order to show that light travels in straight lines. When we talk about waves, "amplitudes" become important. We have to add all the amplitudes for various possible ways that contribute to the process to obtain the total amplitude.[2] The final

[1] This behavior is strictly speaking called "diffraction", however, the fundamental process through which it arises is called interference, which is why I prefer to use this term. In fact, all the phenomena of light are just different consequences of the interference property.

[2] The fact that we have to add all the amplitudes is a consequence of the

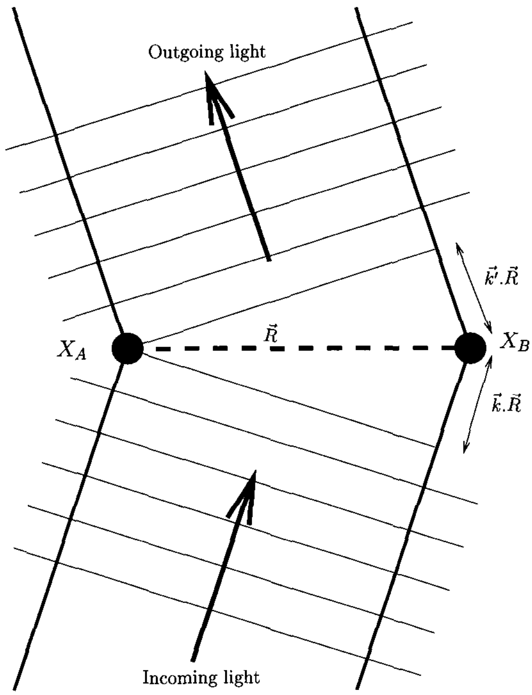

Fig. 1.3 Propagation of light in the air. We can derive the straight line trajectory from the wave theory of light.

total amplitude then has to be squared, leading to the intensity which is then the observable quantity. (Intensity is basically the number of photons falling onto a certain area per unit of time, but I don't really want to mention photons yet as we are not supposed to know quantum optics at this stage!) So, what is the final amplitude for this process? It is given by (strictly speaking, proportional to)

$$e^{\Delta \mathbf{k} \mathbf{x_A}} + e^{\Delta \mathbf{k} \mathbf{x_B}} = e^{\Delta \mathbf{k} \mathbf{x_B}}(1 + e^{\Delta \mathbf{k} \mathbf{R}}) \tag{1.13}$$

where $\mathbf{x_A}$ and $\mathbf{x_B}$ are the position vectors of the two atoms (and $\mathbf{k}$ and $\mathbf{k}'$ are the initial and final light wave vectors respectively). So the intensity in the $\mathbf{k}'$-direction is given by the mod square of the amplitude

$$|1 + e^{\Delta \mathbf{k} \mathbf{R}}|^2 = 2(1 + \cos(\Delta \mathbf{k} \mathbf{R})) \tag{1.14}$$

where $\Delta \mathbf{k} = \mathbf{k} - \mathbf{k}'$. Thus we see that if $\mathbf{k} = \mathbf{k}'$, then the intensity

linearity of wave equation; namely if two waves are solutions of this equation then so is their sum. We will talk about this in more detail later on.

is maximal, so according to Fermat's least time principle the light travels in a straight line. Of course, there will be other directions where we have maxima, given by $\Delta\mathbf{k} = 2n\pi$. So it looks as though light could take other paths than the straight line. However, imagine that there are more than two atoms, randomly distributed (like in the air, for instance, and *unlike* in periodic crystals as in a typical solid-state problem). Then any other direction will be unlikely as contributions from different **R**s will average to zero unless the beam of light travels in a straight line. If it worries you that atoms are not moving in our treatment, just remember that the speed of light is typically 10^6 times larger. Thus, the first postulate of geometrical optics can be derived from the wave theory. With a little more effort it can be seen that the whole of geometrical optics can be derived as an approximation from Maxwell's equations! This reasoning is slightly simplified as light can also propagate in vacuum without any atoms around. The most general way of dealing with this is to take all the possible paths that light can take and add up all the corresponding amplitudes. The resulting amplitude should then be mod squared to yield the total intensity.

What changes in quantum optics? Well, light is again composed of particles (photons), but these particles behave like waves — they interfere (so both Newton and Huygens were somehow right after all). The proof for the existence of photons has built up over the year since Planck made his "quantum hypothesis" (which we will talk about in great detail shortly). I will mention a number of experiments throughout the book which demonstrate that light is composed of particles — photons. Now, however, I want to present a simple experiment to demonstrate the basic properties of quantum behavior of light. This is meant to motivate the rest of the subject without going into too much quantum mechanical detail at this stage.

The apparatus in Figure 1.4 is called the Mach–Zehnder interferometer. It consists of two beam splitters (half-silvered mirrors, which pass light with probability one half and reflect it with the same probability), and two 100 percent reflecting mirrors. Let us now calculate what happens in this set up to a single photon that enters the interferometer. For this we need to know the action of a beam splitter. The action of a beam splitter on the state a is given by the simple rule

Beam Splitter Transformation

$$|a\rangle \rightarrow |b\rangle + i|c\rangle \tag{1.15}$$

which means that the state a goes into an equal superposition of

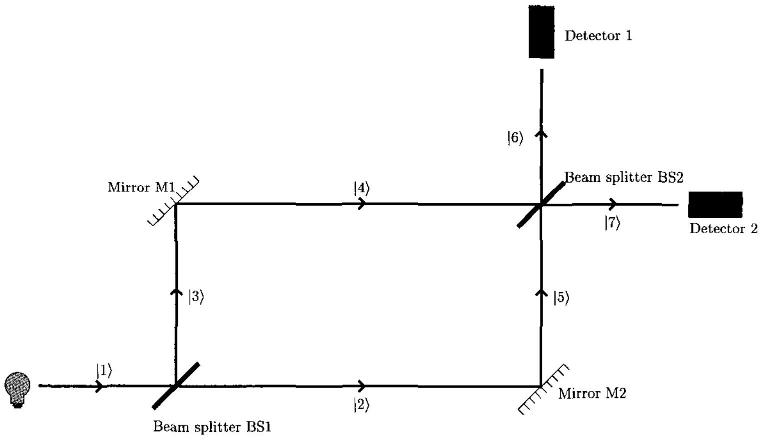

Fig. 1.4 Mach–Zehnder Interferometer. This is one of the most frequently used interferometers in the spectral study of light. In this book we will use it mainly to illustrate the unusual behavior of light in quantum mechanics.

states b and c.[3] The imaginary phase in front of c signifies that when light is reflected from a mirror at 90° it picks up a phase of $e^{i\pi/2} = i$ (the origin of this phase is purely classical, that is, derivable from Maxwell's equations). Now the Mach–Zehnder interferometer works as follows:

Quantum Interference

$$|1\rangle \overset{BS1}{\rightarrow} |2\rangle + i|3\rangle \overset{M1,M2}{\rightarrow} (i|5\rangle) + i(i|4\rangle) \tag{1.16}$$
$$= i|5\rangle - |4\rangle \overset{BS2}{\rightarrow} i(|6\rangle + i|7\rangle) - (i|6\rangle + |7\rangle) \tag{1.17}$$
$$= i|6\rangle - |7\rangle - i|6\rangle - |7\rangle = -2|7\rangle \tag{1.18}$$

Therefore, if everything is set up properly, and if both of the arms of the interferometer have the same length, then the light will come out and be detected by detector 2 only.[4] This is called interference and is a well-known property of waves, as we saw (it's just that quantumly every photon behaves in this way). What would happen if we detected light after the first beam splitter and wanted to know which route it took? Then, half of the photons would be detected in arm 2 and the rest of them would be detected in arm 3. So, it seems that photons randomly choose to move left or right at a beam

[3]Note that this state is not normalized. We need a prefactor of $1/\sqrt{2}$, but since the normalization is the same for both states b and c we will omit it throughout.

[4]Because we did not normalize the initial state and the states throughout the interferometer, there is an extra factor of "2" in the final result which should be ignored. The extra minus sign is just an overall phase and cannot be detected by any experiment.

splitter. And they are particles; we never detect half the photon in one arm and the other half in the other arm — they come in chunks. Thus it seems that this is the same as tossing a coin and registering heads or tails. Well, not quite. In fact, not at all. The interferometer shows why. Suppose that at the first beam splitter the photons goes either left or right, but it definitely goes either left or right (as our experience seems to suggest). Then, at the second beam splitter the photon would again face the same choice, i.e. it would definitely move either left or right. So, according to this reasoning we should expect detectors 1 and 2 to click with an equal frequency. But this is not what we saw! In reality, only detector 2 clicks. The amplitude, and therefore the probability for detector 1 to click is zero. This means that the operation of a beam splitter and the behavior of the photon is not just like coin tossing. The state after the beam splitter is more than just a statistical (random) mixture of the two probabilities. It is, of course, a *superposition*, and the photon takes both of the possible routes (in spite of being a particle). This is the true meaning behind writing its state as a mathematical sum of two vectors, say $|1\rangle + |2\rangle$.[5] This is basically why we use vectors to express states of physical systems.

But we won't be using this most sophisticated description of light immediately from the beginning of the book. Why? There are several reasons for this. Firstly, the mathematics used for the full quantum theory of light is quite advanced. Secondly, there are many important features of light that can be correctly described using the less sophisticated (so called) semi-classical theory. So we don't need to bother with the more complicated manipulations of equations and can postpone this until later. Thirdly, starting with simple things and going towards more complicated stuff has a great pedagogical value. It shows us how our understanding improves and teaches us never to be dogmatic about our understanding since it is very likely that it will be superseded by some better theory. I'd say that this is the most important part of our scientific culture. And finally, if we started with the complicated theory we would miss out on all the beautiful progress that took place at the beginning of the last century, and it was precisely this progress that made it the Century of Physics. Therefore, in the first part of this book (up to Chapter 7) we will have to do a lot of "hand waving" in order to describe the interaction of light with

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[5]So photons (and all other physical systems as it turns out) behave according to Yogi Berra's saying: "When you come to a fork in the road, take it!"

matter, which will only be justified by a rigorous quantum treatment in the second part. However, we are in good company, as this is exactly what Planck and Einstein had to do about 100 years ago!

Final Thought: What have we learnt from the above story? Optics is an old science and the story of light has evolved over many centuries. The first "modern" treatment of light (Newton's *Optics* in 1660) described light as composed of small particles — corpuscules. This was compatible with the fact that light travels in straight lines, but there were phenomena difficult to explain, such as interference. With the discovery of Maxwell's equations, it was firmly established that light was an electromagnetic wave; therefore it interferes and diffracts. However, this theory was also found not to be completely compatible with some experimental evidence, in particular the Compton scattering as we will see later on in the book. Finally, quantum mechanics united the previous two and combined them into a new picture where light is composed of particles which interfere at the individual level. Science thus produces better and better approximations of nature to account for the more thorough experimental evidence that we gather through more developing technology. With every new scientific theory our understanding and picture of the world change dramatically and usually result in a different philosophy. It will not be surprising at all if all the results presented in this book are superseded by a higher level generalization of which they become an approximation in the same way that today classical optics approximates quantum optics. (This, of course, doesn't mean that what we will learn in this book would become useless; on the contrary, it will become crucial in testing quantum mechanics and exploring its domain of validity and applicability.)