Chapter 1

Introduction

Representation theory is the study of groups G and algebras A by means of homomorphisms

$$d : G \to \mathsf{GL}_K(V) \quad \text{and} \quad D : A \to \mathsf{End}_K(V),$$

into the group of invertible linear endomorphisms, respectively, the algebra of endomorphisms of a vector space V over a field K. This brings powerful tools of linear algebra to bear on the theory of groups and algebras. The group algebra KG of G over the field K connects the two theories, since, by linearity, any *representation* $d : G \to \mathsf{GL}_K(V)$ of G extends to a *representation* $D : KG \to \mathsf{End}_K(V)$ of KG, which is *unital* in the sense that the image of the identity element of G under D is the identity of $\mathsf{End}_K(V)$. By restriction, any unital representation $D : KG \to \mathsf{End}_K(V)$ defines a homomorphism $d : G \to \mathsf{GL}_K(V)$. The representation theory of groups may thus be viewed as a special case of the representation theory of algebras with identity, with the restriction to unital representations.

The theory of *modules* for groups and algebras works equally well, because any unital representation d of G, or D of A, turns V into a unital G-module, or A-module, by setting

$$vg := v(gd) \qquad \text{or} \qquad va := v(aD),$$

for all $v \in V$ and $g \in G$, $a \in A$. Conversely, any unital G-module, or A-module, provides, in an obvious way, a representation of G, or A. In what follows, all modules are assumed to be unital and finite-dimensional, since the representation theory of finite symmetric groups is the focus of this study.

A linear subspace N of an A-module M is an *A-submodule* of M if N is closed under the action of A, that is, if $na \in N$ for all $n \in N$ and $a \in A$.

The building blocks of the representation theory of an associative algebra A with identity are the *simple* or *irreducible A-modules* $M \neq \{0_M\}$ whose only A-submodules are $N = \{0_M\}$ and $N = M$. An A-module M is *semi-simple* or *completely reducible* if M is the direct sum of irreducible A-modules. Equivalently, any A-submodule N of M has an A-module complement in M, that is, there exists an A-submodule N' of M such that $N \cap N' = \{0_M\}$ and $M = N + N'$.

Assuming right multiplication, the algebra A is an A-module which is called the *regular A-module* and denoted by A^R. The A-submodules of A^R are the right ideals of A. The algebra A is called semi-simple if the regular A-module A^R is semi-simple.

It can be shown that, in this sense, if A is semi-simple, then so is every A-module.

The principal tasks of the representation theory of semi-simple algebras may be summarised as follows:

1. Classify the isomorphism classes of irreducible A-modules and display a representative of each class.
2. Find ways to decompose an arbitrary A-module into irreducible A-submodules.

In the representation theory of finite groups, the following result is crucial.

Maschke's Theorem. *If G is a finite group and K is a field of characteristic not dividing the order of G, then KG is semi-simple.*

Let M be a G-module and let $d : G \to \mathsf{GL}_K(M)$ denote the corresponding representation of G. The mapping

$$\chi_M : G \to K\,, \quad g \mapsto \mathsf{tr}(gd)$$

is the *character of G afforded by M*, where $\mathsf{tr} : \mathsf{End}_K(M) \to K$ denotes the trace function. It is readily seen that χ_M is constant on the conjugacy classes of G. Any such map $G \to K$ is a *class function* of G. The linear space of all class functions of G is denoted by $\mathcal{C}\ell_K(G)$.

Let KG be semi-simple, then the character afforded by M indeed "characterises" M. For, in this situation, two G-modules M and M' are isomorphic if and only if $\chi_M = \chi_{M'}$. We say that χ_M is irreducible if M is irreducible. Then any character of G is a linear combination of irreducible characters with nonnegative integer coefficients, and these coefficients determine the underlying G-module, up to isomorphism. If, in addition, K is a so-called *splitting field* of G, there is the explicit formula

$$\alpha = \sum_{\chi} (\alpha, \chi)_G \, \chi$$

expressing any $\alpha \in \mathcal{C}\ell_K(G)$ as a linear combination of the irreducible characters χ of G, with coefficients given by the scalar product

$$(\alpha, \chi)_G = 1/|G| \sum_{g \in G} \alpha(g^{-1})\chi(g).$$

As a consequence, for any G-module M, there is the isomorphism of G-modules

$$M \cong_G \bigoplus_{\chi} (\chi_M, \chi)_G \, I_\chi \,,$$

where I_χ is an irreducible G-module affording χ, for each irreducible character χ of G. A short and self-contained dispatch of the theory of finite group characters, up to this point, is contained in Appendix A.

If the characteristic of K is positive and divides the order of G, then the group ring KG is not semi-simple. The so-called *modular* representation theory of G arising therefrom is of a totally different nature. Therefore, it is assumed throughout that K *is a field containing the field* $\mathbb{Q}$ *of rational numbers as a subfield.*

* * *

Let $\mathbb{N} := \{1, 2, 3, \ldots\}$ be the set of positive integers. Put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and

$$\underline{n} := \{1, \ldots, n\}$$

for all $n \in \mathbb{N}_0$. The symmetric group $\mathcal{S}_n$ consists of all bijections (or permutations) $\pi : \underline{n} \to \underline{n}$. In examples, we write $\pi \in \mathcal{S}_n$ as a word or as a product of cycles as usual. For instance, the permutation

$$\pi := \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 4\,1\,3\,6\,5\,2\,7\,9\,8 \end{pmatrix} \in \mathcal{S}_9$$

reads as $\pi = (1\,4\,6\,2)\,(3)\,(5)\,(7)\,(8\,9) = (1\,4\,6\,2)\,(8\,9)$ as a product of cycles and as $\pi = 4\,1\,3\,6\,5\,2\,7\,9\,8$ as a word. Products $\pi\sigma$ of permutations are to be read from left to right: first π, then σ. We write id_n for the identity in $\mathcal{S}_n$.

We shall see later on that $\mathbb{Q}$ is a splitting field of $\mathcal{S}_n$. In addressing the representation theory of $\mathcal{S}_n$, the general theory sketched above therefore shows that the following two problems require solutions:

1. Determine the irreducible characters of $\mathcal{S}_n$.
2. Find ways to evaluate scalar products $(\alpha, \chi)_{\mathcal{S}_n}$, where α is an arbitrary character of $\mathcal{S}_n$ and χ is an irreducible character of $\mathcal{S}_n$.

Specht modules and matrix representations of the symmetric group will not be considered here.

The symmetric group $\mathcal{S}_k$ and also the direct product $\mathcal{S}_k \times \mathcal{S}_{n-k}$ occur as subgroups of $\mathcal{S}_n$ in a multitude of ways whenever $1 \leq k \leq n-1$. In this sense, any character of $\mathcal{S}_n$ yields a character of $\mathcal{S}_k$, or of $\mathcal{S}_k \times \mathcal{S}_{n-k}$, by restriction. Conversely, any character of $\mathcal{S}_k$, or of $\mathcal{S}_k \times \mathcal{S}_{n-k}$ yields a character of $\mathcal{S}_n$, by induction. There is the general idea to deduce results for $\mathcal{S}_n$-characters from the character theory of $\mathcal{S}_k$, $\mathcal{S}_{n-k}$, and $\mathcal{S}_k \times \mathcal{S}_{n-k}$, which are assumed to be "well understood", since k and $n-k$ are both $< n$. The major tools of this *inductive method* are induction and restriction of characters together with Frobenius' reciprocity law.

The inductive method may be described elegantly in terms of the *bialgebra of class functions* [Gei77], which is defined as follows. The underlying vector space of this bialgebra is the direct sum

$$\mathcal{C} = \bigoplus_{n \geq 0} \mathcal{C}\ell_K(\mathcal{S}_n).$$

To each class function α of $\mathcal{S}_k$ and each class function β of $\mathcal{S}_{n-k}$ may be associated a class function $\alpha \# \beta$ of $\mathcal{S}_k \times \mathcal{S}_{n-k}$, in a natural way (see 3.2). The product on $\mathcal{C}$ arises from the concept of induction:

$$\alpha \bullet \beta = (\alpha \# \beta)^{\mathcal{S}_n}.$$

The restriction of class functions leads to a *coproduct* on $\mathcal{C}$, that is, a linear mapping $\mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$. It is defined by

$$\alpha \downarrow = \sum_{k=0}^{n} (\alpha|_{\mathcal{S}_k \times \mathcal{S}_{n-k}})\, i_{k,n-k}^{-1}$$

for all class functions α of $\mathcal{S}_n$. Here $i_{k,n-k}$ denotes the natural linear isomorphism $\mathcal{C}\ell_K(\mathcal{S}_k) \otimes \mathcal{C}\ell_K(\mathcal{S}_{n-k}) \to \mathcal{C}\ell_K(\mathcal{S}_k \times \mathcal{S}_{n-k})$. The coproduct $\downarrow$ is an algebra map $(\mathcal{C}, \bullet) \to (\mathcal{C} \otimes \mathcal{C}, \bullet_\otimes)$, where $\bullet_\otimes$ is the product on $\mathcal{C} \otimes \mathcal{C}$ arising from $\bullet$ (see 2.8). In other words, $(\mathcal{C}, \bullet, \downarrow)$ is a *bialgebra*. By orthogonality, the scalar products $(\cdot,\cdot)_{\mathcal{S}_n}$ on the linear spaces $\mathcal{C}\ell_K(\mathcal{S}_n)$ extend to a single bilinear form on $\mathcal{C}$. The bialgebra $\mathcal{C}$ is *self-dual* with respect to this form, that is,

$$(\alpha \bullet \beta, \gamma)_{\mathcal{C}} = (\alpha \otimes \beta, \gamma{\downarrow})_{\mathcal{C}\otimes\mathcal{C}}$$

for all $\alpha, \beta, \gamma \in \mathcal{C}$, where $(\,\cdot\,,\,\cdot\,)_{\mathcal{C}\otimes\mathcal{C}}$ is the bilinear form on $\mathcal{C} \otimes \mathcal{C}$ such that

$$(\alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2)_{\mathcal{C}\otimes\mathcal{C}} = (\alpha_1, \beta_1)_{\mathcal{C}}(\alpha_2, \beta_2)_{\mathcal{C}}$$

for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{C}$. It is this self-duality that mirrors Frobenius' reciprocity law.

The bialgebra of class functions contains the classical representation theory of the symmetric group — in terms of ordinary, that is to say, *commutative character theory.*

For example, if χ is an irreducible character of $\mathcal{S}_k$, and ψ is an irreducible character of $\mathcal{S}_{n-k}$, then $\chi \bullet \psi$ is a character of $\mathcal{S}_n$. It is a classical and fairly intricate problem to decompose this induced character into irreducible characters of $\mathcal{S}_n$. An answer is provided by the Littlewood–Richardson Rule [LR34]. In terms of the bialgebra $\mathcal{C}$, this remarkable result becomes a description of the structure constants with respect to that linear basis which consists of all irreducible characters of all symmetric groups $\mathcal{S}_n$.

Chapter 2 contains parts from the theory of coproducts and bialgebras, to the extent needed, while more details concerning the definition and basic properties of the bialgebra $\mathcal{C}$ are given in Chapter 3.

Each of the bialgebras $\mathcal{A}$ considered here is an inner direct sum

$$\mathcal{A} = \bigoplus_{n \in \mathbb{N}_0} \mathcal{A}_n$$

of linear subspaces $\mathcal{A}_n$ such that

$$\mathcal{A}_n \star \mathcal{A}_m \subseteq \mathcal{A}_{n+m} \quad \text{and} \quad \mathcal{A}_n\delta \subseteq \sum_{k=0}^{n} \mathcal{A}_k \otimes \mathcal{A}_{n-k}\,,$$

for all $n, m \in \mathbb{N}_0$ and $\mathcal{A}_0 \cong K$, where $\star$ denotes the product and δ denotes the coproduct on $\mathcal{A}$. This means that $\mathcal{A}$ is *graded* and *connected.* Any such bialgebra $\mathcal{A}$ is actually a *Hopf algebra.*

The algebra of symmetric functions Λ is isomorphic to $\mathcal{C}$. Andreij Zelevinski's approach [Zel81b] to the representation theory of finite classical groups builds on this particular type of commutative Hopf algebra.

However, knowledge of the theory of symmetric functions, or the theory of Hopf algebras, is not necessary here.

* * *

The origin of noncommutative character theory is Solomon's discovery [Sol76] of a subalgebra of the group algebra of an arbitrary finite Coxeter group W which maps into the algebra of class functions of W. His results in case $W = \mathcal{S}_n$ are briefly revisited below. In our approach, they serve as backdrop and source of motivation, and not as structural building blocks.

Some notations are needed. Many interesting objects in the theory are indexed by *partitions* or *compositions.* It is convenient to represent these indices in terms of words in a free monoid $\mathbb{N}^*$ over the alphabet $\mathbb{N}$. The multiplication in $\mathbb{N}^*$ is *concatenation.* We write $q.r$ for the concatenation product of $q, r \in \mathbb{N}^*$ in order to avoid confusion with the ordinary product in $\mathbb{N}$. Any $q \in \mathbb{N}^*$ may be written uniquely as a product

$$q = q_1.q_2.\cdots.q_k,$$

where $q_1, q_2, \ldots, q_k \in \mathbb{N}$. The identity element $\varnothing$ of $\mathbb{N}^*$ is the empty product. If $q_1 + q_2 + \cdots + q_k = n \in \mathbb{N}_0$, then q is a *composition* of n, denoted by $q \models n$. If, in addition, $q_1 \geq q_2 \geq \cdots \geq q_k$, then q is a *partition* of n and we write $q \vdash n$.

Let $n \in \mathbb{N}_0$ and $q = q_1.\cdots.q_k \models n$. Denote by $P^q = (P^q_1, \ldots, P^q_k)$ the set partition of $\underline{n}$ consisting of the successive blocks of order q_i in $\underline{n}$, for all $i \in \underline{k}$. For example, $P^{2.1.2} = (\{1,2\}, \{3\}, \{4,5\})$. The *Young subgroup of type* q in $\mathcal{S}_n$ is

$$\mathcal{S}_q := \{\, \pi \in \mathcal{S}_n \mid P^q_i \pi = P^q_i \text{ for all } i \in \underline{k} \,\}.$$

If $\pi, \sigma \in \mathcal{S}_n$, then clearly $\mathcal{S}_q\pi = \mathcal{S}_q\sigma$ if and only if $P^q_i\pi = P^q_i\sigma$ for all $i \in \underline{k}$. Therefore, the set

$$\mathcal{S}^q := \{\, \nu \in \mathcal{S}_n \mid \nu|_{P^q_i} \text{ is increasing for all } i \in \underline{k} \,\}$$

is a transversal of the right cosets of $\mathcal{S}_q$ in $\mathcal{S}_n$. The symmetric group $\mathcal{S}_n$ acts on these right cosets, by right multiplication. The corresponding character $\xi^q = (1_{\mathcal{S}_q})^{\mathcal{S}_n}$ is the *Young character of type* q. The set of Young characters $\{\, \xi^p \mid p \vdash n \,\}$ is a linear basis of $\mathcal{C}\ell_K(\mathcal{S}_n)$ (see 12.3). Observe that the linear space $\mathcal{C}\ell_K(\mathcal{S}_n)$ is a ring with multiplication defined by $(\alpha\beta)(\pi) = \alpha(\pi)\beta(\pi)$ for all $\alpha, \beta \in \mathcal{C}\ell_K(\mathcal{S}_n)$ and $\pi \in \mathcal{S}_n$.

The Mackey formula [CR62, (44.3)] yields a multiplication rule

$$\xi^r \xi^q = \sum_{s \models n} m_q^r(s)\, \xi^s$$

and a combinatorial description of the coefficients $m_q^r(s) \in \mathbb{N}_0$. Solomon's far-reaching discovery was a noncommutative refinement of this rule for the elements $\Xi^q := \sum_{\nu \in \mathcal{S}^q} \nu$, $q \models n$, of the group algebra $K\mathcal{S}_n$:

$$\Xi^r \Xi^q = \sum_{s \models n} m_q^r(s)\, \Xi^s, \tag{1.1}$$

with the same coefficients as above. As a consequence, there is the following result.

1.1 Theorem. (Solomon, 1976) *Let $n \in \mathbb{N}$. The linear span $\mathcal{D}_n$ of the elements Ξ^q, $q \models n$, is a subalgebra of the group algebra $K\mathcal{S}_n$.*

Furthermore, the linear map $c_n : \mathcal{D}_n \to \mathcal{C}\ell_K(\mathcal{S}_n)$, defined by $\Xi^q \mapsto \xi^q$ for all $q \models n$, is an epimorphism of algebras.

Several different proofs of this result have since been given (see, for example, Tits' appendix to Solomon's original paper [Sol76], or [BBHT92; BL93; Bro00; GR89; Ges84; Reu93; vW98]). A short and transparent proof of Solomon's theorem may also be found in Appendix B.

The elements Ξ^q, $q \models n$, actually form a linear basis of $\mathcal{D}_n$. To see this, observe that for any $\pi \in \mathcal{S}_n$, we have $\pi \in \mathcal{S}^q$ if and only if the *descent set*

$$\mathsf{Des}(\pi) = \{\, i \in \underline{n-1} \mid i\pi > (i+1)\pi \,\}$$

of π is contained in the set $\{q_1, q_1 + q_2, \ldots, q_1 + \ldots + q_{l-1}\}$ of partial sums of q. The elements

$$\Delta^D := \sum_{\mathsf{Des}(\pi) = D} \pi \qquad (D \subseteq \underline{n-1})$$

of $K\mathcal{S}_n$ are clearly linearly independent, and

$$\Xi^q = \sum_{D \subseteq \{q_1, q_1 + q_2, \ldots\}} \Delta^D$$

for all $q \models n$. An inclusion/exclusion argument implies that both sets $\{\, \Xi^q \mid q \models n \,\}$ and $\{\, \Delta^D \mid D \subseteq \underline{n-1} \,\}$ are linear bases of $\mathcal{D}_n$. Accordingly, the algebra $\mathcal{D}_n$ is referred to as the *Solomon descent algebra* of $\mathcal{S}_n$.

The direct sum

$$\mathcal{D} = \bigoplus_{n \in \mathbb{N}_0} \mathcal{D}_n ,$$

has K-basis $\{ \Xi^q \mid q \in \mathbb{N}^* \}$. We define the structure of a bialgebra on $\mathcal{D}$, as follows. The multiplication is given by

$$\Xi^r \star \Xi^q := \Xi^{r.q}$$

for all $q, r \in \mathbb{N}^*$, and linearity, so that $(\mathcal{D}, \star)$ is a free associative algebra over the set of (noncommuting) variables $\{ \Xi^n \mid n \in \mathbb{N} \}$. As a consequence, there is a unique coproduct $\downarrow$ on $\mathcal{D}$ such that $(\mathcal{D}, \star, \downarrow)$ is a bialgebra and

$$\Xi^n \downarrow = \sum_{k=0}^{n} \Xi^k \otimes \Xi^{n-k}$$

for all $n \in \mathbb{N}$. Here, by definition, Ξ^0 is the identity $\Xi^\varnothing$ of $(\mathcal{D}, \star)$.

The bialgebra $(\mathcal{D}, \star, \downarrow)$ is isomorphic to the *algebra of noncommutative symmetric functions.* The latter was introduced by Gelfand et al. in [GKL⁺95] and has been further studied extensively in so far five subsequent papers [KLT97; DKK97; KT97; KT99; DHT02].

Defining a bilinear form on $\mathcal{D}$ by

$$(\Xi^r, \Xi^q)_{\mathcal{D}} := |\mathcal{S}^r \cap (\mathcal{S}^q)^{-1}|$$

for all $q, r \in \mathbb{N}^*$, we are in a position to state and proof the following result.

1.2 Theorem. *The linear map $\mathcal{D} \to \mathcal{C}$, defined by $\Xi^q \mapsto \xi^q$ for all $q \in \mathbb{N}^*$, is an isometric epimorphism of bialgebras with respect to $(\cdot,\cdot)_{\mathcal{D}}$ and $(\cdot,\cdot)_{\mathcal{C}}$.*

The homomorphism rules for the products and the coproducts on $\mathcal{D}$ and $\mathcal{C}$ follow from the definition of $(\mathcal{D}, \star, \downarrow)$ and the corresponding rather immediate identities in $\mathcal{C}$:

$$\xi^r \bullet \xi^q = \xi^{r.q} \quad \text{and} \quad \xi^n \downarrow = \sum_{k=0}^{n} \xi^k \otimes \xi^{n-k}$$

for all $r, q \in \mathbb{N}^*$ and $n \in \mathbb{N}$ (see 3.2 and 3.9).

The interesting fact that the simultaneous linear extension of Solomon's epimorphisms c_n is an isometry, follows from Solomon's theorem: observe first that the identity $\mathcal{S}^n = \{\mathrm{id}_n\}$ and Frobenius' reciprocity law imply

$$(\Xi^q, \Xi^n)_{\mathcal{D}} = 1 = (1_{\mathcal{S}_q}, 1_{\mathcal{S}_q})_{\mathcal{S}_q} = ((1_{\mathcal{S}_q})^{\mathcal{S}_n}, \xi^n)_{\mathcal{S}_n} = (\xi^q, \xi^n)_{\mathcal{C}}$$

for all $q \models n$. Now, for arbitrary $q, r \models n$, comparing the coefficient of id_n on both sides of (1.1), gives

$$\begin{aligned}(\Xi^r, \Xi^q)_{\mathcal{D}} &= \sum_{s \models n} m_q^r(s)(\Xi^s, \Xi^n)_{\mathcal{D}} \\ &= \sum_{s \models n} m_q^r(s)(\xi^s, \xi^n)_{\mathcal{C}} \\ &= (\xi^r \xi^q, \xi^n)_{\mathcal{C}} \\ &= (\xi^r, \xi^q)_{\mathcal{C}} \,.\end{aligned}$$

As a consequence of Theorem 1.2, there is a noncommutative character theory which is satisfying at least to some extent. Every $\mathcal{S}_n$-character has a counterpart in $\mathcal{D}_n$ and the epimorphism $\mathcal{D} \to \mathcal{C}$ allows one to transfer every problem involving restriction, induction and scalar products of characters. However, this noncommutative setup builds on the formal procedure of passing from a polynomial ring in the set $\{ \xi^n \mid n \in \mathbb{N} \}$ of commuting variables to the free associative algebra over the set $\{ \Xi^n \mid n \in \mathbb{N} \}$ of noncommuting variables. This turns out to be too restrictive, even in view of the classical results on irreducible characters of $\mathcal{S}_n$. For, unfortunately, suitable noncommutative counterparts in $\mathcal{D}$ of irreducible characters (which should allow significantly simplified arguments in the noncommutative superstructure) cannot be found.

* * *

Due to Jöllenbeck ([Jöl98], see also [Jöl99]), there is the crucial idea to construct a proper extension of the bialgebra $\mathcal{D}$ (and Theorem 1.2), in order to fill this gap. The *bialgebra of permutations*, introduced by Malvenuto and Reutenauer in [MR95], provides the general framework for this purpose. The underlying vector space of this bialgebra is the direct sum

$$\mathcal{P} = \bigoplus_{n \in \mathbb{N}_0} K\mathcal{S}_n$$

and has the set of all permutations as a linear basis. The product in $\mathcal{P}$ may be described by

$$\pi \star \sigma := \sum_{\gamma} \gamma$$

for all permutations $\pi \in \mathcal{S}_n$, $\sigma \in \mathcal{S}_m$, where the sum is taken over all permutations $\gamma \in \mathcal{S}_{n+m}$ such that $i\gamma < j\gamma$ if and only if $i\pi < j\pi$ for all $i, j \in \underline{n}$, and $i\gamma < j\gamma$ if and only if $(i-n)\sigma < (j-n)\sigma$ for all $i, j \in \underline{n+m} \backslash \underline{n}$. For example,

$$12 \star 21 = 1243 + 1342 + 1432 + 2341 + 2431 + 3421.$$

The unique element $\emptyset$ of $\mathcal{S}_0$, the empty permutation, is the identity of $(\mathcal{P}, \star)$.

A combinatorial description of the coproduct on $\mathcal{P}$ is

$$\pi{\downarrow} := \sum_{k=0}^{n} \alpha_k \otimes \beta_k$$

for all $\pi \in \mathcal{S}_n$, where, for each $k \in \underline{n} \cup \{0\}$, $\alpha_k \in \mathcal{S}_k$ and $\beta_k \in \mathcal{S}_{n-k}$ are determined by the conditions that $i\alpha_k^{-1} < j\alpha_k^{-1}$ if and only if $i\pi^{-1} < j\pi^{-1}$, for all $i, j \in \underline{k}$, and $(i-k)\beta_k^{-1} < (j-k)\beta_k^{-1}$ if and only if $i\pi^{-1} < j\pi^{-1}$, for all $i, j \in \underline{n} \backslash \underline{k}$. For example,

$$4132{\downarrow} = \emptyset \otimes 4132 + 1 \otimes 321 + 12 \otimes 21 + 132 \otimes 1 + 4132 \otimes \emptyset.$$

A bilinear form on $\mathcal{P}$ is defined by

$$(\pi, \sigma)_{\mathcal{P}} := \begin{cases} 1 & \text{if } \pi = \sigma^{-1}, \\ 0 & \text{otherwise,} \end{cases}$$

for all permutations π and σ. Malvenuto and Reutenauer [MR95] showed that $\mathcal{P}$ is a self-dual bialgebra, and that $(\mathcal{D}, \star, {\downarrow})$ as defined above is a subbialgebra of $\mathcal{P}$. The bialgebra of permutations is introduced in Chapter 5. In view of applications, it is suitable to use an approach which builds on Stanley's theory of P-partitions [Sta72], which is briefly revisited in Chapter 4 for that reason.

The algebra $\mathcal{P}$ maps onto $\mathcal{C}$, as follows. The *cycle type* of a permutation π in $\mathcal{S}_n$ is the partition p of n obtained by concatenating the lengths of the cycles occurring in the cycle decomposition of π, in a non-increasing fashion. It is well-known that two permutations π, σ in $\mathcal{S}_n$ are conjugate in $\mathcal{S}_n$ if and only if they have the same cycle type.

Let $\Pi_n \in K\mathcal{S}_n$ be primitive for all $n \in \mathbb{N}$, that is, $\Pi_n{\downarrow} = \Pi_n \otimes \emptyset + \emptyset \otimes \Pi_n$. Assign to $\varphi \in K\mathcal{S}_n$ the class function $c_\Pi(\varphi)$ of $\mathcal{S}_n$ which maps each π in $\mathcal{S}_n$ of cycle type $p = p_1. \dots .p_l \vdash n$ to

$$c_\Pi(\varphi)(\pi) := (\varphi, \Pi_{p_1} \star \cdots \star \Pi_{p_l})_{\mathcal{P}}\,.$$

This defines a linear map

$$c_{\Pi} : \mathcal{P} \rightarrow \mathcal{C}.$$

It is an important observation that c_{Π} is in fact a homomorphism of algebras from $(\mathcal{P}, \star)$ into $(\mathcal{C}, \bullet)$; see Chapter 7. Assuming a mild condition on the primitive elements Π_n (namely that the coefficient $(\Pi_n, \mathsf{id}_n)_{\mathcal{P}}$ of the identity in Π_n is 1), c is onto and coincides with Solomon's epimorphism c_n from $\mathcal{D}_n$ onto $\mathcal{C}\ell_K(\mathcal{S}_n)$ when restricted to $\mathcal{D}_n$, for all $n \in \mathbb{N}_0$.

The algebra map $c_{\Pi} : \mathcal{P} \rightarrow \mathcal{C}$ is not a bialgebra map nor an isometry. However, for properly chosen primitive elements Π_n, c_{Π} has these properties when restricted to the *coplactic algebra* $\mathcal{Q} \subseteq \mathcal{P}$, which contains $\mathcal{D}$ as well as suitable noncommutative irreducible characters. This bialgebra was discovered by Poirier and Reutenauer [PR95]. Its definition relies on the famous Robinson–Schensted correspondence [Rob38; Sch61]. In his thesis [Jöl98], Jöllenbeck considered a smaller extension $\mathcal{F}$ of $\mathcal{D}$ which is revisited in Chapter 6.

Standard Young tableaux of shape p are realised as permutations in what follows. The link to the usual notion of a tableau (see, for example, [Ful97]) is given by the concept of juxtaposing the rows of a Young diagram. For example, consider the usual picture

5	7	
2	6	8
1	3	4

of an array of numbers increasing in rows and decreasing in columns. Juxtaposing the rows from top to bottom, yields the standard Young tableau

$$5\,7\,2\,6\,8\,1\,3\,4 \in \mathcal{S}_8$$

of shape 3.3.2. The set of all standard Young tableaux of shape $p \vdash n$ is denoted by $\mathsf{SYT}^p \subseteq \mathcal{S}_n$.

The Robinson–Schensted correspondence yields a bijection

$$\mathcal{S}_n \longrightarrow \bigcup_{p \vdash n} \mathsf{SYT}^p \times \mathsf{SYT}^p, \quad \pi \longmapsto (P(\pi), Q(\pi)). \tag{1.2}$$

Following Schensted, its first component $P(\pi)$ is called the *P-symbol* of π, while its second component $Q(\pi)$ is called the *Q-symbol* of π. Collecting together all permutations $\pi \in \mathcal{S}_n$ with a given Q-symbol σ, we obtain a *coplactic class*

$$A_\sigma = \{\, \pi \in \mathcal{S}_n \mid Q(\pi) = \sigma \,\}$$

in $\mathcal{S}_n$, for all $\sigma \in \bigcup_{p \vdash n} \mathsf{SYT}^p$. We write ΣA for the sum of A in $K\mathcal{S}_n$, for all subsets $A \subseteq \mathcal{S}_n$. The coplactic algebra $\mathcal{Q}$ is defined as the linear span of the sums ΣA_σ of all coplactic classes. As already mentioned, it is a sub-bialgebra of the bialgebra $\mathcal{P}$ of permutations and contains $\mathcal{D}$; see Chapter 8.

A theorem of Schützenberger [Sch63] states that $P(\pi) = Q(\pi^{-1})$ for all $\pi \in \mathcal{S}_n$, which implies that

$$(\Sigma A_\sigma, \Sigma A_\nu)_{\mathcal{P}} = \#\{\, \pi \in \mathcal{S}_n \mid Q(\pi) = \sigma,\ P(\pi) = \nu \,\} = \begin{cases} 1 & \text{if } p = q, \\ 0 & \text{otherwise,} \end{cases}$$

for all $\sigma \in \mathsf{SYT}^p$ and $\nu \in \mathsf{SYT}^q$, since (1.2) is a bijection. Furthermore, for each $p \vdash n$, there exists a standard Young tableau σ of shape p such that $\mathsf{SYT}^p = A_\sigma$; hence, denoting by Z^p the sum of SYT^p in $K\mathcal{S}_n$, there are the *noncommutative orthogonality relations*

$$(\mathrm{Z}^p, \mathrm{Z}^q)_{\mathcal{P}} = \begin{cases} 1 & \text{if } p = q, \\ 0 & \text{otherwise,} \end{cases} \tag{1.3}$$

for all partitions p and q of n. Note that, by definition, $(\mathrm{Z}^p, \mathrm{Z}^q)_{\mathcal{P}}$ is equal to the number of standard Young tableaux π of shape p such that π^{-1} is a standard Young tableau of shape q. The reader is encouraged to verify (1.3) for small values of n. Appendix C contains a new proof of the Robinson–Schensted correspondence and related results of Knuth, Schensted and Schützenberger, which builds on [BJ99].

When choosing suitable primitive elements Π_n in $\mathcal{Q}_n := \mathcal{Q} \cap K\mathcal{S}_n$ for all $n \in \mathbb{N}$, we get the following concluding result of Part II, in Chapter 9.

1.3 Main Theorem. $c_\Pi|_{\mathcal{Q}} : (\mathcal{Q}, \star, \downarrow) \to (\mathcal{C}, \bullet, \downarrow)$ *is a graded and isometric epimorphism of bialgebras, that is,* $c_\Pi(\mathcal{Q}_n) = \mathcal{C}\ell_K(\mathcal{S}_n)$ *and*

$$\begin{aligned} (\alpha, \beta)_{\mathcal{P}} &= (c_\Pi(\alpha), c_\Pi(\beta))_{\mathcal{C}}, \\ c_\Pi(\alpha \star \beta) &= c_\Pi(\alpha) \bullet c_\Pi(\beta), \\ (c_\Pi \otimes c_\Pi)(\alpha\downarrow) &= c_\Pi(\alpha)\downarrow \end{aligned}$$

for all $n \in \mathbb{N}_0$ *and* $\alpha, \beta \in \mathcal{Q}$.

This is a slight extension of a result of Jöllenbeck, who considered a particular series $(\omega_n)_{n\in\mathbb{N}}$ of primitive elements (see 9.4). The epimorphism $\mathcal{P} \twoheadrightarrow \mathcal{C}$ associated with this series will simply be denoted by c. Any inverse image under c in $\mathcal{P}$ of a character χ afforded by the $\mathcal{S}_n$-module M is called a *noncommutative character* corresponding to χ, or M.

* * *

By recourse to the noncommutative theory, the ordinary character theory of the symmetric group can be deduced by means of simple *noncommutative computations* in the coplactic algebra. Many classical results serve as examples in the third part of this book.

To start with, we shall consider the class functions $\zeta^p := c(\mathrm{Z}^p)$ of $\mathcal{S}_n$ indexed by partitions p of n. The Main Theorem together with the noncommutative orthogonality relations (1.3) implies that $\{\,\zeta^p \,|\, p \vdash n\,\}$ is an orthonormal basis of $\mathcal{C}\ell_K(\mathcal{S}_n)$. In Chapter 10, it will be shown that this is in fact the set of all irreducible characters of $\mathcal{S}_n$ and, in particular, that $\mathbb{Q}$ is a splitting field of $\mathcal{S}_n$. Accordingly, the sum Z^p of all standard Young tableaux of shape p in $K\mathcal{S}_n$ is a *noncommutative irreducible character* of $\mathcal{S}_n$, for each $p \vdash n$.

Let δ be a character of $\mathcal{S}_n$ with noncommutative counterpart Δ in $\mathcal{Q}$. If Δ is a sum of permutations, say, $\Delta = \sum_{\pi\in D} \pi$, then

$$(\delta, \zeta^p)_{\mathcal{C}} = (c(\Delta), c(\mathrm{Z}^p))_{\mathcal{C}} = (\Delta, \mathrm{Z}^p)_{\mathcal{P}} = |D^{-1} \cap \mathsf{SYT}^p|.$$

The scalar product on the left hand side yields the multiplicity of the irreducible character ζ^p in δ. Its noncommutative counterpart on the right hand side gives, by definition, a combinatorial description of this multiplicity, namely the number of standard Young tableaux $\pi \in \mathsf{SYT}^p$ such that $\pi^{-1} \in D$. This is a leading point for applications of noncommutative character theory.

For example, the sum $\Sigma\mathcal{S}_n = \sum_{\pi\in\mathcal{S}_n} \pi$ is an inverse image of the regular character $\chi_{K\mathcal{S}_n}$ of $\mathcal{S}_n$ under c and contained in $\mathcal{Q}$ — a noncommutative regular character. It follows without difficulty that

$$(\chi_{K\mathcal{S}_n}, \zeta^p)_{\mathcal{C}} = (\Sigma\mathcal{S}_n, \mathrm{Z}^p)_{\mathcal{P}} = \mathsf{syt}^p := |\mathsf{SYT}^p|.$$

In other words, the multiplicity of ζ^p in the regular $\mathcal{S}_n$-character is equal to the number of standard Young tableaux of shape p. But the same scalar product yields the degree $\deg \zeta^p$ of the irreducible character ζ^p. Hence, denoting by M_p an irreducible module affording the character ζ^p for all partitions p of n, we may conclude that $\dim M_p = \deg \zeta^p = \mathsf{syt}^p$ and

$$KS_n \cong \bigoplus_{p \vdash n} \mathsf{syt}^p\, M_p\,. \tag{1.4}$$

This fundamental identity has as combinatorial refinement the decomposition

$$\Sigma S_n = \sum_{p \vdash n} \sum_{\sigma \in \mathsf{SYT}^p} \Sigma A_\sigma$$

of S_n into coplactic classes. Indeed, $c(A_\sigma) = \zeta^p$ actually for any $\sigma \in \mathsf{SYT}^p$. When applying c, the latter equality thus turns into (1.4), expressed in terms of the corresponding characters.

In Chapters 11–13, few-line proofs are given in the same fashion of some classical results including the Littlewood–Richardson Theorem, the Branching Rule, Young's Rule and a combinatorial description of the Kostka numbers. Also without difficulty, the recursive formula for the irreducible character values known as the Murnaghan–Nakayama Rule can be stated and proved more generally as a formula for so-called skew characters, which reduces to the classical rule (due to Murnaghan and Nakayama) as a special case.

Further applications concern the descent characters $\delta^D = c(\Delta^D)$ indexed by subsets $D \subseteq \underline{n-1}$ in Chapter 14. To conclude, results of Kraskiewicz–Weyman [KW01] and Leclerc–Scharf–Thibon [LST96] are recovered in Chapter 15 on the cyclic characters of S_n, which are induced from the cyclic subgroup generated by a long cycle.