Chapter 1

Single-Degree-of-Freedom Systems

The main goal of this chapter is the introduction of the asymptotic approach based on the complex representation of equations of motion.

At the same time we try to throw new light on some well-explored problems, e.g., secondary resonances at forced oscillation of single-degree-of-freedom (1DOF) systems. We show that such notions as internal resonance, exchange of energy between various modes (resulting in nonstationary oscillations), bifurcational values of energy and *etc.* have sense already for 1DOF systems. As a result of our detailed consideration, the understanding of more complicated behavior of several-degrees-of-freedom systems becomes more comprehensible.

1.1 Free Oscillations in Systems Close to Linear Ones

1.1.1 *Complex equations of motion and solution by the multiple scales method*

Let us consider a nonlinear single-degree-of-freedom system described by the equation

$$\frac{d^2u}{d\tau^2}+u=\varepsilon\ f(u,\frac{du}{d\tau}), \tag{1.1.1}$$

where ε is a small parameter, τ and u are nondimensional time and dependent variable respectively, and f is a piecewise-differentiable function of its variables. Let us write the equation of motion (1.1.1) as a system of two first order equations

$$\frac{d\nu}{d\tau}+u=\varepsilon f(u,\nu), \quad \frac{du}{d\tau}-\nu=0, \tag{1.1.2}$$

and introduce the complex variable

$$\psi=\nu+iu. \tag{1.1.3}$$

Denoting the complex conjugation by sign (*), one has:

$$\psi^*=\nu-iu, \quad \nu=\frac{1}{2}(\psi+\psi^*), \quad u=\frac{1}{2i}(\psi+\psi^*). \tag{1.1.4}$$

Multiplying the second equation (1.1.2) by imaginary unit i and adding it to the first one we obtain

$$\frac{d\psi}{d\tau}-i\psi=\varepsilon f\left[\frac{1}{2i}(\psi-\psi^*),\frac{1}{2}(\psi+\psi^*)\right]. \tag{1.1.5}$$

Thus instead of the second order equation (1.1.1) we deal with the first order complex equation (1.1.5). The solution of this equation will be obtained by the multiple scale method (Cole, 1968a; Nayfeh, 1981) as follows.

Let us introduce times of various scales $\tau_n=\varepsilon^n\tau$ (n=0, 1,...) and consider the required complex function as a function of variables $\tau_o(=\tau), \tau_1,\ldots$. Using the differentiation rule for compound functions one can obtain the operator of differentiation by τ in the form of expansion:

$$\frac{d(\ldots)}{d\tau}=D_0(\ldots)+\varepsilon D_1(\ldots)+\varepsilon^2 D_2(\ldots)+\ldots, \quad D_k(\ldots)\equiv\frac{\partial(\ldots)}{\partial\tau_k}. \tag{1.1.6}$$

The function $\psi(\tau)$ is expanded in the asymptotic series by the small parameter ε:

$$\psi(\tau)=\psi_0(\tau)+\varepsilon\psi_1(\tau)+\varepsilon^2\psi_2(\tau)+\ldots. \tag{1.1.7}$$

In order the functions $\psi_k(\tau)$ to be uniquely defined some additional conditions should be imposed on them, similarly to those for the averaging method in real variables (Bogolyubov, Mitropolsky, 1963). These are the orthogonality conditions:

$$\int_0^{2\pi}\psi_0(\tau)\psi_1^*(\tau)d\tau=0,\quad \int_0^{2\pi}\psi_0(\tau)\psi_2^*(\tau)d\tau=0,\quad \int_0^{2\pi}\psi_1(\tau)\psi_2^*(\tau)d\tau=0. \quad (1.1.8)$$

Besides, initial conditions are imposed on the function $\psi_0(\tau)$; functions $\psi_k(\tau)$ are supposed to satisfy zero initial conditions: $\psi_k(0)=0$ (k=1,2,...).

Substitution of (1.1.6) and (1.1.7) in Eq. (1.1.5) gives:

$$D_0(\psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2+...)+\varepsilon D_1(\psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2+...)+$$
$$+\varepsilon^2 D_2(\psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2+...)-i(\psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2+...)=$$
$$=\varepsilon F(\psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2+...,\psi_0^*+\varepsilon\psi_0^*+\varepsilon^2\psi_0^*+...), \quad (1.1.9)$$

where

$$F(\psi,\psi^*)\equiv f\left[\frac{1}{2i}(\psi-\psi^*),\frac{1}{2}(\psi+\psi^*)\right]. \quad (1.1.10)$$

Then we expand function F in series in parameter ε

$$F(\psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2+...,\psi_0^*+\varepsilon\psi_0^*+\varepsilon^2\psi_0^*+...)=$$
$$=F(\psi_0,\psi_0^*)++\varepsilon[F'_\psi(\psi_0,\psi_0^*)\psi_1+F'_{\psi^*}(\psi_0,\psi_0^*)\psi_1^*]+\dots \,. \quad (1.1.11)$$

(F'_ψ denotes the derivative of F with respect to ψ). Equating coefficients at increasing powers of ε to zero one can obtain the following equations:

$$D_0\psi_0-i\psi_0=0, \quad (1.1.12)$$

$$D_0\psi_1-i\psi_1=-D_1\psi_0+F(\psi_0,\psi_0^*), \quad (1.1.13)$$

$$D_0\psi_2-i\psi_2=-D_1\psi_1-D_2\psi_0+F'_\psi(\psi_0,\psi_0^*)\psi_1+F'_{\psi^*}(\psi_0,\psi_0^*)\psi_1^*. \quad (1.1.14)$$

It follows from equation (1.1.12) that

$$\psi_0=Ae^{i\tau}, \quad (1.1.15)$$

where A depends on the "slow" time: A=$A(\tau_1,\tau_2,...)$. Then the orthogonality conditions (1.1.8) give

$$\int_0^{2\pi} e^{-i\tau}\psi_1(\tau)\,d\tau = 0, \quad \int_0^{2\pi} e^{-i\tau}\psi_2(\tau)\,d\tau = 0, \ldots \tag{1.1.16}$$

Let us take now the equation (1.1.13). As solution of (1.1.13) should not include secular terms, the right hand side (r.h.s.) of (1.1.13) should be orthogonal to ψ_0(1.1.15). It means that the expansion of the r.h.s. in the Fourier's complex series at interval (0, 2π) should not contain the first harmonics. Accordingly to (1.1.15) the derivative of ψ_0 in slow time is

$$\frac{\partial\psi_0}{\partial\tau_1} = \frac{\partial A}{\partial\tau_1}e^{i\tau}. \tag{1.1.17}$$

We present the Fourier expansion of function $\Phi(A,A^*,\tau) \equiv F(\psi_0,\psi_0^*)$ in the form

$$\Phi(A,A^*,\tau) = \sum_{-\infty}^{\infty}\Phi_n(A,A^*)e^{in\tau}, \tag{1.1.18}$$

where

$$\Phi_n(A,A^*) = \frac{1}{2\pi}\int_0^{2\pi}\Phi(A,A^*,t)e^{-in\tau}\,d\tau \tag{1.1.19}$$

and ψ_0 is given by (1.1.15).

Then the condition of absence of the first harmonics in the r.h.s. of (1.1.13) can be written in the form

$$\frac{\partial A}{\partial\tau_1} = \Phi_1(A,A^*). \tag{1.1.20}$$

It is clear that $\Phi_1(A,A^*)$ can be usually found immediately in view of $F(\psi_0,\psi_0^*)$ by substituting (1.1.15), without expanding $F(\psi_0,\psi_0^*)$ in Fourier's series (1.1.18), (1.1.19). Function $\Phi_1(A,A^*)$ includes only coefficients at those terms of $F(\psi_0,\psi_0^*)$ which are proportional to $\exp(i\tau)$. Equation (1.1.20) governs the slow change of the amplitude and the phase in the first approximation (1.1.15).

Equation (1.1.13) with account of (1.1. 17), (1.1.20) takes the form

$$\frac{\partial\psi_1}{\partial\tau} - i\psi_1 = F(\psi_0,\psi_0^*) - \Phi_1(A,A^*)e^{i\tau}. \tag{1.1.21}$$

It allows to determine the correction $\psi_1(\tau_1,...)$ for the second approximation.

Then we can analyze Eq. (1.1.14) to obtain the $A-\tau_2$ dependence (if the next approximation is necessary). Similar considerations yield to the following equation for $\partial A/\partial\tau_2$:

$$\frac{\partial A}{\partial \tau_2}=\frac{1}{2\pi}\int_0^{2\pi}\left(F'_\psi(\psi_0,\psi_0^*)\psi_1+F'_{\psi^*}(\psi_0,\psi_0^*)\psi_1^*-\frac{\partial\psi_1}{\partial\tau_1}\right)e^{-i\tau}\,d\tau\,. \quad (1.1.22)$$

Equation (1.1.14) with account of (1.1.22) and orthogonality conditions (1.1.16) determines $\psi_2(\tau)$, and so on.

The obtained expansion (1.1.7) is a uniformly suitable first order solution, if we retain only the first term in the form (1.1.15) with A determined by Eq. (1.1.20). The above mentioned expansion turns out to be the second order uniformly suitable solution in case we add also the second term (1.1.21), and use a more exact expression for A obtained with account of (1.1.22). This can be shown similarly to the averaging procedure (Bogolyubov, Mitropolsky, 1963) or to the multiple scales procedure applied to equations in real variables (Nayfeh, 1981). Note that the above procedure has certain advantages of simplicity.

1.1.2 *Applications*

1.1.2.1 *Nonlinear oscillator with cubic nonlinearity (Duffing equation with damping)*

As the first example we take nonlinear oscillatory system described by the equation

$$m\ddot{U}+2n\dot{U}+c_1U+c_3U^3=0, \quad (1.1.23)$$

with initial conditions: $U=U_0$, $\dot{U}=V_0$ at $t=0$, where ($^\bullet$) means the derivative with respect to time t.

After introducing dimensionless variables $\tau=\omega_0 t$, $u=U/U^*$, Eq. (1.1.23) and initial conditions take the form

$$\frac{d^2u}{d\tau^2}+2\varepsilon\,\mu\frac{du}{d\tau}+u+8\varepsilon\alpha\,u^3=0\,, \tag{1.1.24}$$

$$\tau=0:\quad u=u_0\equiv\frac{U_0}{U^*};\quad \frac{du}{d\tau}=\nu_0\equiv\frac{V_0}{\omega_0 U^*}\,,$$

where $\omega_0=\sqrt{c_1/m}$, U^* is a characteristic scale of the problem (e.g., $U^*=U_0$ or $U^*=\sqrt{U_0^2+V_0^2}$), and

$$\varepsilon\,\mu=\frac{n}{\sqrt{c_1 m}}\,,\quad 8\alpha\,\varepsilon=\frac{c_3U^{*2}}{c_1}\,. \tag{1.1.25}$$

After transferring the terms with ε to the r.h.s. Eq. (1.1.23) takes the form (1.1.1), where

$$f(u,\frac{du}{d\tau})=-2\,\mu\frac{du}{d\tau}-8\alpha\,u^3\,. \tag{1.1.26}$$

Then according to (1.1.10)

$$F(\psi,\psi^*)=-\mu\,(\psi+\psi^*)-i\alpha\,(\psi-\psi^*)^3\,, \tag{1.1.27}$$

so with account of (1.1.15)

$$\begin{aligned}\Phi(A,A^*,\tau)\equiv F(\psi_0,\psi_0^*)&=-\mu\,(Ae^{i\tau}+A^*e^{-i\tau})-\\ -i\alpha\,(Ae^{i\tau}-A^*e^{-i\tau})^3&=-\mu\,(Ae^{i\tau}+A^*e^{-i\tau})-\\ -i\alpha\,(A^3e^{3i\tau}-3|A|^2Ae^{i\tau}&+3|A|^2A^*e^{-i\tau}-A^{*3}e^{-3i\tau})\,.\end{aligned} \tag{1.1.28}$$

The function $\Phi_1(A,A^*)$ is obtained from the terms in the r.h.s. proportional to $\exp(i\tau)$:

$$\Phi_1(A,A^*)=-\mu\,A+3i\alpha\,|A|^2A\,. \tag{1.1.29}$$

So Eq. (1.1.20) has the form

$$\frac{dA}{d\tau_1}+\mu\,A-3i\alpha\,|A|^2A=0\,. \tag{1.1.30}$$

First let us consider the conservative case μ=0. Then (1.30) is reduced to the equation

$$\frac{dA}{d\tau_1} - 3i\alpha\,|A|^2 A = 0. \tag{1.1.31}$$

It follows from (1.1.31) that vector $dA/d\tau_1$ is orthogonal to vector A, so $|A|=a=$const. Therefore (1.1.31) is a linear equation, and its solution has the form

$$A = a_0\, e^{3i\alpha a^2 \tau_1}, \tag{1.1.32}$$

where a_0 is a complex constant.

Substitution of (1.1.32) in (1.1.15) leads to

$$\psi = a_0\, e^{i(1+3\varepsilon\alpha a^2)\tau}. \tag{1.1.33}$$

From the initial conditions we have $a_0 = v_0 + i u_0$, so the first approximation solution (1.1.33) in real variables is

$$u = u_0 \cos\widetilde{\omega}\tau + v_0 \sin\widetilde{\omega}\tau, \tag{1.1.34}$$

where

$$\widetilde{\omega} = 1 + 3\varepsilon\alpha a^2.$$

Using (1.1.24), (1.1.25) one can easily write the solution in the dimensional variables:

$$U = U_0 \cos\omega_* t + \frac{V_0}{\omega_0}\sin\omega_* t, \tag{1.1.35}$$

where

$$\omega_* = \omega_0 \left(1 + \frac{3}{8}\frac{c_3}{c_1} a_*^2\right) \tag{1.1.36}$$

($a_* = U^* a$ is the amplitude of oscillation).

Let us return now to the general case $\mu \neq 0$. To obtain the solution of Eq. (1.1.30) in the explicit form we make the following change of the dependent variable

$$A = e^{-\mu\tau_1}\vartheta. \tag{1.1.37}$$

Substitution of (1.1.37) in (1.1.30) yields

$$\frac{d\vartheta}{d\tau_1} - 3i\alpha\, e^{-2\mu\tau_1}|\vartheta|^2\vartheta = 0. \qquad (1.1.38)$$

We see that in this case vector $d\vartheta/d\tau_1$ is orthogonal to vector ϑ, so $|\vartheta| = a_0$ ==const. Therefore Eq. (1.1.38) is also a linear equation, and its solution has the form

$$\vartheta = C_0\, e^{\frac{3i\alpha}{2\mu}a_0^2\left(1-e^{-2\mu\tau_1}\right)}. \qquad (1.1.39)$$

Then functions A, ψ_0 are as follows:

$$A = C_0\, e^{-\mu\tau_1} e^{\frac{3i\alpha}{2\mu}a_0^2\left(1-e^{-2\mu\tau_1}\right)},$$

$$\psi_0 = C_0\, e^{-\mu\varepsilon\tau} e^{i\left[\tau+\frac{3\alpha}{2\mu}a_0^2(1-e^{-2\mu\varepsilon\tau})\right]}. \qquad (1.1.40)$$

After separation of the real and imaginary parts expression (1.1.40) coincides (for the particular case of the pendulum) with the first approximation solution on the real equation basis (Bogolyubov, Mitropolsky, 1963).

One may expand the exponent in brackets (1.1.40) in Taylor series and retain only two (or three) main terms. Then we obtain

$$\psi_0 = C_0\, e^{-\mu\varepsilon\tau} e^{i\tau\left[1+3\varepsilon\alpha\, a_0^2(1-\mu\varepsilon\tau)\right]}. \qquad (1.1.41)$$

If retaining only the terms of order ε in the exponent, the frequency of oscillations will not depend on the damping. So in the first approximation (with accuracy $O(\varepsilon)$) the damping influences on the oscillation amplitude only. The dependence of the frequency on the damping can be correctly presented in the second approximation.

Note that Eq. (1.1.38) and the conjugated one can be presented at $\mu = 0$ in the hamiltonian form:

$$\frac{d\vartheta}{d\tau_1} = \frac{dH}{d\vartheta^*}; \quad \frac{d\vartheta^*}{d\tau_1} = -\frac{dH}{d\vartheta},$$

where $H = (3/2)\, i\alpha \left|\vartheta\right|^4$, and, hence, the first integral $|\vartheta| = a_0$ could be written as $H = const$.

1.1.2.2 Oscillator with self-excitation due to nonlinear damping (Van der Pol equation).

As the second example we take a nonlinear oscillatory system described by the Van der Pol equation

$$\ddot{u} - \varepsilon\,(1-u^2)\dot{u} + u = 0\,, \tag{1.1.42}$$

with initial conditions $u(0) = u_0$, $\dot{u}(0) = 0$. Here the linear damping is negative, the nonlinear damping being positive. In this case $f(u,\dot{u}) = (1-u^2)\dot{u}$, and function $F(\psi,\psi^*)$ (1.1.10) is

$$F(\psi,\psi^*) = \frac{1}{2}(\psi+\psi^*)\left(1+\frac{1}{4}(\psi-\psi^*)^2\right). \tag{1.1.43}$$

Function $\Phi(A,A^*,\tau) \equiv F(\psi_0,\psi_0^*)$ is as follows

$$\Phi(A,A^*,\tau) = \frac{1}{2}(Ae^{i\tau} + A^* e^{-i\tau})\left(1+\frac{1}{4}(Ae^{i\tau} - A^* e^{-i\tau})^2\right) =$$

$$= \frac{1}{2}\left(Ae^{i\tau} + A^* e^{-i\tau} + \frac{1}{4}(A^3 e^{3i\tau} - |A|^2 Ae^{i\tau} - |A|^2 A^* e^{-i\tau} + A^{*3} e^{-3i\tau})\right). \tag{1.1.44}$$

Function $\Phi_1(A,A^*)$ consists of $e^{i\tau}$ prefactors in the r.h.s. of (1.1.44):

$$\Phi_1(A,A^*) = \frac{1}{2}A - \frac{1}{8}|A|^2 A\,. \tag{1.1.45}$$

So Eq. (1.1.20) has the form

$$\frac{dA}{d\tau_1} = \frac{1}{2}A(1-\frac{1}{4}|A|^2)\,. \tag{1.1.46}$$

If to present A in polar coordinates $A = a\exp(i\theta(\tau_1))$, one easily obtains that $\theta = \theta_0$=const, and Eq. (1.1.46) is written as

$$\frac{da}{d\tau_1} = \frac{1}{2}a(1-\frac{1}{4}a^2)\,. \tag{1.1.47}$$

Eq. (1.1.47) is an equation in separable variables, and its solution is

$$a^2 = \frac{4}{1+C_0 e^{-\tau_1}} . \qquad (1.1.48)$$

From the initial conditions $a(o) = |u_0|$, $\theta_0 = \pi/2$ and

$$C_0 = \frac{4}{u_0^2} - 1 , \qquad a = 2\left(1+\left(\frac{4}{u_0^2}-1\right)e^{-\tau_1}\right)^{-1/2} ,$$

$$\psi = 2i\left(1+\left(\frac{4}{u_0^2}-1\right)e^{-\varepsilon\tau}\right)^{-1/2} e^{i\tau} .$$

Returning to the real variable gives

$$u = 2\left(1+\left(\frac{4}{u_0^2}-1\right)e^{-\varepsilon\tau}\right)^{-1/2} \cos\tau . \qquad (1.1.49)$$

We obtained the well-known solution. The stable stationary regime $\psi = 2i\exp(i\tau)$, or $u = 2\cos\tau$, is an attractor for nonstationary (transient) oscillations at any $u_0 \neq 0$ (a limit cycle). Note that these oscillations are isochronic (in the first approximation).

1.1.3 *Change of the dependent variable*

The above procedure may be somewhat simplified. We return to Eq. (1.1.5):

$$\frac{d\psi}{d\tau} - i\psi = \varepsilon f\left[\frac{1}{2i}(\psi - \psi^*), \frac{1}{2}(\psi + \psi^*)\right].$$

The change of the dependent variable

$$\psi(\tau) = e^{i\tau}\varphi(\tau) \qquad (1.1.50)$$

leads to the following equation for $\varphi(\tau)$:

$$\frac{d\varphi}{d\tau} = \varepsilon \tilde{F}(\varphi, \varphi^*, \tau) , \qquad (1.1.51)$$

$$\widetilde{F}(\varphi,\varphi^*,\tau)=e^{-i\tau}f\left[\frac{1}{2i}(\varphi e^{i\tau}-\varphi^* e^{-i\tau}),\frac{1}{2}(\varphi e^{i\tau}+\varphi^* e^{-i\tau})\right]. \quad (1.1.52)$$

Expanding $\varphi(\tau)$ in the asymptotic series

$$\varphi(\tau)=\varphi_0(\tau)+\varepsilon\varphi_1(\tau)+\varepsilon^2\varphi_2(\tau)+\ldots, \quad (1.1.53)$$

from Eq. (1.1.51) we have

$$D_0\varphi_0=0, \quad (1.1.54)$$

$$D_0\varphi_1=-D_1\varphi_0+\widetilde{F}(\varphi_0,\varphi_0^*), \quad (1.1.55)$$

$$D_0\varphi_2=-D_1\varphi_1-D_2\varphi_0+\widetilde{F}'_\varphi(\varphi_0,\varphi_0^*)\varphi_1+\widetilde{F}'_{\varphi^*}(\varphi_0,\varphi_0^*)\varphi_1^*. \quad (1.1.56)$$

From Eq. (1.1.54) we conclude that φ_0 depends on the "slow" time $A=A(\tau_1,\tau_2,\ldots)$, i.e. in the first approximation φ_0 is constant (its value is determined by the initial conditions). Then the orthogonality conditions (1.1.8) take the form

$$\int_0^{2\pi}\psi_s(\tau)d\tau=0,\quad \int_0^{2\pi}\psi_s(\tau)\psi_l^*(\tau)d\tau=0\,(s,\ l=1,2,\ldots,\ s\neq l). \quad (1.1.57)$$

Because φ_0 is constant in the first approximation, function $\widetilde{F}(\varphi_0,\varphi_0^*,\tau)$ (1.1.52) is a periodical function of τ with quickly varying exponential terms (with period 2π). Its average value $\widetilde{\Phi}_0(\varphi_0,\varphi_0^*)$ is:

$$\widetilde{\Phi}_0(\varphi_0,\varphi_0^*)=\frac{1}{2\pi}\int_0^{2\pi}F(\varphi_0,\varphi_0^*,\tau)d\tau. \quad (1.1.58)$$

Condition of absence of secular terms in the solution of Eq. (1.1.55) now means that the average value of r.h.s. in (1.1.55) (and therefore the average value of derivative $\partial\varphi_1/\partial\tau$) at $(0,2\pi)$ equals to zero. As the derivative of φ_0 in slow time $\partial\varphi_0/\partial\tau_1$ has to be considered as a constant at $(0,2\pi)$, integration of Eq. (1.1.55) from 0 to 2π gives

$$\frac{d\varphi_0}{d\tau_1}=\widetilde{\Phi}_0(\varphi_0,\varphi_0^*). \quad (1.1.59)$$

This equation is an equivalent of Eq. (1.1.20). Note that Eq. (1.1.59) follows from the condition

$$\int_0^{2\pi} \frac{\partial \varphi_1}{\partial \tau} d\tau = 0, \tag{1.1.60}$$

with account of $\varphi_k(0) = 0$, $\varphi_k(2\pi) = 0$, (k=1,2,...).

Equation (1.1.55) with account of (1.1.58), (1.1.59) takes the form

$$\frac{\partial \varphi_1}{\partial \tau} = \widetilde{F}(\varphi_0, \varphi_0^*, \tau) - \widetilde{\Phi}_0(\varphi_0, \varphi_0^*). \tag{1.1.61}$$

It allows us to determine $\varphi_1(\tau_1,...)$, i.e. to obtain the second approximation. After solving (1.1.62) we can consider Eq. (1.1.56) in order to obtain the dependence of φ_0 on the second order slow time τ_2 (if there is a need of the next approximation). Similar considerations result in the following equation for $\partial\varphi_0/\partial\tau_2$:

$$\frac{\partial \varphi_0}{\partial \tau_2} = \frac{1}{2\pi} \int_0^{2\pi} \left(F'_\varphi(\varphi_0, \varphi_0^*)\varphi_1 + F'_{\varphi^*}(\varphi_0, \varphi_0^*)\varphi_1^* - \frac{\partial \varphi_1}{\partial \tau_1} \right) d\tau . \tag{1.1.62}$$

Equation (1.1.62) with account of orthogonality conditions (1.1.57) determines $\varphi_2(\tau)$, and so on.

1.2 Forced Oscillations of a Nonlinear Oscillator

1.2.1 *Complex equations of motion and solution by the multiple scales method*

The next problem under consideration is the oscillator with cubic nonlinearity in the harmonic external field, i.e., the oscillatory system described by the equation (non-autonomous Duffing equation)

$$m\ddot{U} + 2n\dot{U} + c_1 U + c_3 U^3 = F\cos\Omega t, \tag{1.2.1}$$

with initial conditions $U(0) = U_0$, $\dot{U}(0) = V_0$. In dimensionless variables $\tau = \omega_0 t$, $u = U/U^*$ Eq. (1.2.1) and initial conditions take the form

$$\frac{d^2u}{d\tau^2} + 2\varepsilon\,\mu\frac{du}{d\tau} + u + 8\varepsilon\,\alpha\,u^3 = f\cos\overline{\omega}\tau\,, \tag{1.2.2}$$

$$\tau = 0: \quad u = u_0 \equiv \frac{U_0}{U^*};\ \frac{du}{d\tau} = \nu_0 \equiv \frac{V_0}{\omega_0 U^*}, \tag{1.2.3}$$

where $\omega_0 = \sqrt{c_1/m}$, U^* is a characteristic scale for u and

$$\varepsilon\,\mu = \frac{n}{\sqrt{c_1 m}},\ \ 8\alpha\,\varepsilon = \frac{c_3 U^{*2}}{c_1},\ \ f = \frac{F}{c_1 U^*},\ \ \overline{\omega} = \frac{\Omega}{\omega_0}. \tag{1.2.4}$$

(We do not yet assume the amplitude of the external force being small. This force will be considered small when considering the fundamental resonance).

The equivalent form of Eq. (1.2.2) is

$$\frac{dv}{d\tau} + u = -2\varepsilon\,\mu\,v - 8\varepsilon\,\alpha\,u^3 + f\cos\overline{\omega}\tau,$$

$$\frac{du}{d\tau} - v = 0.$$

In complex variables

$$\psi = \nu + iu \qquad \left(v = \frac{1}{2}(\psi + \psi^*),\quad u = \frac{1}{2i}(\psi - \psi^*)\right) \tag{1.2.5}$$

we come to the complex representation of the equations of motion:

$$\frac{d\psi}{d\tau} - i\psi = -\varepsilon\left[\mu\left(\psi + \psi^*\right) + i\alpha\left(\psi - \psi^*\right)^3\right] + \frac{f}{2}\left(e^{i\overline{\omega}\tau} + e^{-i\overline{\omega}\tau}\right) \tag{1.2.6}$$

We solve this equation by the multiple scales method with $\tau_n = \varepsilon^n\tau$ (n=0, 1,...). Using the differentiation rule (1.1.6) and expanding function $\psi(\tau)$ in the asymptotic series by ε

$$\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \ldots, \tag{1.2.7}$$

we have the following set of equations:

$$D_0\psi_0 - i\psi_0 = \frac{f}{2}\left(e^{i\overline{\omega}\tau} + e^{-i\overline{\omega}\tau}\right), \tag{1.2.8}$$

$$D_0\psi_1 - i\psi_1 = -D_1\psi_0 - \mu\left(\psi_0 + \psi_0^*\right) - i\alpha\left(\psi_0 - \psi_0^*\right)^3. \tag{1.2.9}$$

Solution of the nonhomogeneous linear equation (1.2.8) is

$$\psi_0 = Ae^{i\tau} + \frac{i}{2}\Lambda\left[(1+\bar{\omega})e^{i\bar{\omega}\tau} + (1-\bar{\omega})e^{-i\bar{\omega}\tau}\right], \tag{1.2.10}$$

where

$$\Lambda = \frac{f}{1-\bar{\omega}^2}, \tag{1.2.11}$$

and A depends on the "slow" time: $A = A(\tau_1, \ldots)$.

Substitution of (1.2.10) into the r.h.s. of Eq. (1.2.9) yields:

$$D_0\psi_1 - i\psi_1 = -\frac{dA}{d\tau_1}e^{i\tau} - \mu(Ae^{i\tau} + i\bar{\omega}\Lambda e^{i\bar{\omega}\nu}) -$$

$$-i\alpha(A^3e^{3i\tau} - 3|A|^2 Ae^{i\tau}) + 3\alpha\Lambda(A^2 e^{i(2+\bar{\omega})\tau} - 2|A|^2 e^{i\bar{\omega}\tau} +$$

$$+ A^{*2} e^{i(-2+\bar{\omega})\tau}) + 3i\alpha\Lambda^2 A(e^{i(1+2\bar{\omega})\tau} + 2e^{i\tau} + e^{i(1-2\bar{\omega})\tau}) -$$

$$-\alpha\Lambda^3(e^{3i\bar{\omega}\tau} + 3e^{i\bar{\omega}\tau}) + \text{c. c.} \tag{1.2.12}$$

where c.c. denotes complex conjugate terms. The condition of secular terms absence takes different forms depending on $\bar{\omega}$ value. That is why we consider various cases — the nonresonant case, secondary resonances $\bar{\omega} \approx 3$ and $\bar{\omega} \approx 1/3$ (the primary resonance $\bar{\omega} \approx 1$ will be considered separately).

1.2.2 *Nonresonant case*

In this case $\bar{\omega}$ is not close to 0, 1, 3, 1/3. Then the condition of secular terms absence is:

$$\frac{dA}{d\tau_1} + \mu A - 3i\alpha A(|A|^2 + 2\Lambda^2) = 0. \tag{1.2.13}$$

After the change of dependent variable

$$A = e^{-\mu\tau_1}\vartheta \tag{1.2.14}$$

we have

$$\frac{d\vartheta}{d\tau_1} = 3i\alpha\,\vartheta\,(|\vartheta|^2 e^{-2\mu\tau_1} + 2\Lambda^2\ ). \tag{1.2.15}$$

We can conclude from (1.2.15) that $|\vartheta|$ is constant because the derivative of ϑ is orthogonal to the vector ϑ. Hence the general solution of (1.2.15) is

$$\vartheta = \vartheta_0 \exp\left[3\,i\alpha\left(|\vartheta_0|^2 \frac{1-e^{-2\mu\tau_1}}{2\mu} + 2\,\Lambda^2\tau_1\right)\right].$$

where the constant ϑ_0 depends on the initial conditions. From (1.2.13) we have

$$A = \vartheta_0\, e^{-\mu\tau_1} \exp\left[3\,i\alpha\left(|\vartheta_0|^2 \frac{1-e^{-2\mu\tau_1}}{2\mu} + 2\,\Lambda^2\tau_1\right)\right]. \tag{1.2.16}$$

Then for the complex oscillation with account of Eq.(1.2.10) we obtain

$$\psi_0 = \vartheta_0\, e^{-\mu\tau_1} \exp\left[3\,i\alpha\left(|\vartheta_0|^2 \frac{1-e^{-2\mu\varepsilon\tau}}{2\mu} + 2\Lambda^2\varepsilon\tau\right)\right] e^{i\tau} +$$

$$+\frac{i}{2}\Lambda\left[(1+\overline{\omega})\,e^{i\overline{\omega}\tau} + (1-\overline{\omega})\,e^{-i\overline{\omega}\tau}\right]. \tag{1.2.17}$$

From initial conditions (1.2.3) $\vartheta_0 = v_0 + i\,(u_0 - \Lambda)$. Then we can easily obtain the real oscillation with account of (1.2.5). In dimensionless variables we have

$$u = e^{-\mu\varepsilon\tau}\left[(u_0 - \Lambda)\cos(1+\beta\,\varepsilon)\tau + v_0 \sin(1+\beta\,\varepsilon)\tau\right] + \Lambda\cos\overline{\omega}\,\tau\,, \tag{1.2.18}$$

$$\beta = 3\alpha\left(\left[(u_0-\Lambda)^2 + v_0^2\right]\frac{1-e^{-2\mu\,\varepsilon\,\tau}}{2\mu\,\varepsilon\,\tau} + 2\Lambda^2\right). \tag{1.2.19}$$

In dimensional variables with account of (1.2.4):

$$U = e^{-\frac{n}{m}t}\left[(U_0 - \Lambda^0)\cos\omega_0(1+\beta^0)t + V_0 \sin\omega_0(1+\beta^0)t\right] + \Lambda^0\cos\Omega t, \tag{1.2.20}$$

where

$$\beta^0 = \frac{3c_3}{8c_1}\alpha\left(\left[(U_0-\Lambda^0)^2+\left(\frac{V_0}{\omega_0}\right)^2\right]\left(1-\exp(-\frac{2nt}{m})\right)\frac{m}{2nt}+2\left(\Lambda^0\right)^2\right)$$

$$\Lambda^0 = \frac{F}{c_1\left(1-\dfrac{\Omega^2}{\omega_0^2}\right)} . \qquad (1.2.21)$$

Expressions (1.2.18), (1.2.20) have a clear and transparent sense. The first terms describe the "natural" oscillation decreasing due to damping. The second ones give the "purely forced oscillation". Expressions (1.2.19), (1.2.21) determine the "natural" oscillation frequency change due to nonlinearity and damping. In the case of forced oscillation this change of frequency depends on energies both of "natural" and forced oscillations. The only stationary regime is the purely forced oscillation with frequency Ω.

Note that the nonresonant problem with *weak* excitation was considered in (Bogolyubov, Mitropolsky, 1963). However the effect of the external force in such a case can be obtained only in the second approximation. Analysis of *hard* excitation in real variables was carried out in (Nayfeh, Mook, 1979).

1.2.3 *Subharmonic resonance*

Consider the case $\bar{\omega} = \Omega/\omega_0 \approx 3$. Let

$$\Omega/\omega_0 = 3+\varepsilon\sigma , \qquad (1.2.22)$$

where σ is a detuning parameter. Then an additional secular term appears in the r.h.s. of Eq. (1.2.11), and the condition of secular terms absence takes the form (accounting that $\tau_1 = \varepsilon\tau$):

$$\frac{dA}{d\tau_1}+\mu A-3i\alpha\left(A|A|^2+2\Lambda^2 A-i\Lambda A^{*2}e^{i\sigma\tau_1}\right)=0 . \qquad (1.2.23)$$

Putting

$$A = a e^{i\theta}, \qquad (1.2.24)$$

after separation of real and imaginary parts in (1.2.23) we obtain the following set of two first order differential equations in slow time:

$$\frac{da}{d\tau_1} = -\mu a + 3\alpha \Lambda a^2 \cos(3\theta - \sigma\tau_1), \qquad (1.2.25)$$

$$a\frac{d\theta}{d\tau_1} = 3\alpha a\left[a^2 + 2\Lambda^2 - \Lambda a \sin(3\theta - \sigma\tau_1)\right]. \qquad (1.2.26)$$

These equations describe the amplitude and phase modulations. Introducing a new variable

$$\gamma = 3\theta - \sigma\tau_1, \qquad (1.2.27)$$

rewrite the set (1.2.25), (1.2.26) in an autonomous form:

$$\frac{da}{d\tau_1} = -\mu a + 3\alpha \Lambda a^2 \cos\gamma, \qquad (1.2.28)$$

$$a\frac{d\gamma}{d\tau_1} = 9\alpha a(a^2 + 2\Lambda^2 - \Lambda a \sin\gamma) - \sigma a. \qquad (1.2.29)$$

At first we study the ***steady-state oscillations*** (a = const, γ = const), when the r.h.s. in (1.2.28), (1.2.29) equal to zero. The trivial solution $a = 0$ corresponds to purely forced oscillation (see (1.2.10)). For $a \neq 0$ we have the set of equations

$$-\mu + 3\alpha \Lambda a \cos\gamma = 0, \qquad (1.2.30)$$

$$9\alpha(a^2 + 2\Lambda^2 - \Lambda a \sin\gamma) - \sigma = 0. \qquad (1.2.31)$$

The steady-state subharmonic oscillation ($a \neq 0$) is impossible at exact subharmonic resonance $\Omega/\omega_0 = 3$ (the set (1.2.30), (1.2.31) has no solution at $\sigma = 0$, as can be easily seen after excluding γ, with account of inequality $(a^2 + 2\Lambda^2)^2 > (a\Lambda)^2$).

It is expedient to introduce the following dimensionless parameters:

$$\xi = -\frac{a}{\Lambda}, \quad e^* = \frac{\alpha \Lambda^2}{\sigma}, \quad \mu^* = \frac{\mu}{\sigma}. \qquad (1.2.32)$$

Parameter ξ characterizes the amplitude ratio for the free oscillation and forced oscillation terms in (1.2.10) (sign "–" is taken in order ξ to be positive, since Λ (1.2.11) in this case is negative: $\Lambda \approx -f/8$); e^* is a generalized parameter proportional to the energy of excitation. Now the set (1.2.30), (1.2.31) takes the form

$$-\mu^* - 3e^*\xi \cos\gamma = 0, \tag{1.2.33}$$

$$9e^*(\xi^2 + 2 + \xi \sin\gamma) - 1 = 0. \tag{1.2.34}$$

Hence

$$\cos\gamma = -\frac{\mu^*}{3e^*\xi}, \tag{1.2.35}$$

and the steady-state motions ($a \neq 0$) exist only at values of ξ satisfying the inequality

$$\xi \geq \frac{\mu^*}{3e^*}, \quad \text{or} \quad |\Lambda a| \geq \frac{\mu}{3\alpha}. \tag{1.2.36}$$

In particular, they do not exist at sufficiently small values of the excitation force or the amplitude of the free oscillation term (for $\mu \neq 0$).

Excluding γ from (1.2.33), (1.2.34) we obtain following equation for ξ:

$$(3\mu^*)^2 + \left[9\, e^*(\xi^2 + 2) - 1\right]^2 = (9e^*\xi)^2. \tag{1.2.37}$$

(this equation coincides with the equation presented, in other notations, in (Nayfeh, 1981)).

Equation (1.2.37) has solutions only for positive e^*, i.e., when signs of α and σ coincide (in this case only the frequency of free oscillation at finite amplitudes can be equal exactly to $\Omega/3$).

Equation (1.2.37) is a quadratic equation with respect to ξ^2. Its roots

$$\xi^2 = \frac{1}{e^*}\left(\frac{1}{9} - \frac{3}{2}e^* \pm \frac{1}{3}\sqrt{e^* - \frac{63}{4}e^{*2} - \mu^{*2}}\right). \tag{1.2.38}$$

are real and both positive, if $|\mu^*| < 1/\sqrt{63} \approx 0.126$, and e^* value falls in the interval (e_1^*, e_2^*), where

$$e_1^* = \frac{2}{63}\left(1 - \sqrt{1 - 63\mu^{*2}}\right), \quad e_2^* = \frac{2}{63}\left(1 + \sqrt{1 - 63\mu^{*2}}\right). \quad (1.2.39)$$

Expressions (1.2.39) determine bifurcational values of the excitation parameter e^* at which two steady-state oscillations appear and disappear. These oscillations include nonzero free oscillation term, i.e., they are two-frequency oscillations, with frequencies Ω and $\Omega/3$. Indeed, the free oscillation term in (1.2.10) with account of (1.2.22), (1.2.24) and (1.2.27) can be written as follows:

$$a e^{\frac{i}{3}\left(\frac{\Omega}{\omega_0}\tau + \gamma\right)}.$$

The frequency of this oscillation (in nondimensional time τ) equals

$$\frac{1}{3}\left(\frac{\Omega}{\omega_0} + \frac{d\gamma}{d\tau}\right). \quad (1.2.40)$$

So the frequency of the steady-state oscillations (γ = const) in real time t equals to $\Omega/3$.

As will be shown below, only oscillation with amplitudes ratio parameter ξ corresponding to the upper sign "+" in (1.2.38) is stable.

It follows from (1.2.38), that the steady-state subharmonic oscillations exist under condition

$$\mu^{*2} + \frac{63}{4}e^{*2} - e^* < 0.$$

At plane (e^*, μ^*) the domain of existence of the steady-state subharmonic oscillation is bounded by an ellipsis with center at the point (2/63; 0) and half-axes (2/63; $1/\sqrt{63}$); this domain in the first quadrant is presented in Fig. 1.1. Values of the amplitudes ratio parameter ξ for the bifurcational values of parameter e^* according to (1.2.38) are shown in Fig. 1.2. The lower part of the curve (Fig. 1.2) corresponds to the upper part of the curve for e^* (i.e. e_2^*) in Fig. 1.1, and vice versa.

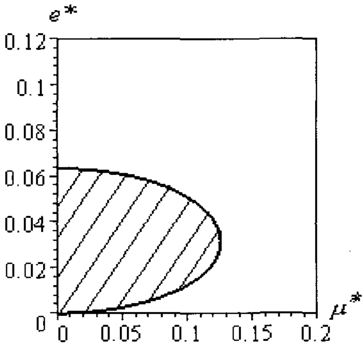


Fig. 1.1 Region of existence of the steady-state subharmonic oscillations.

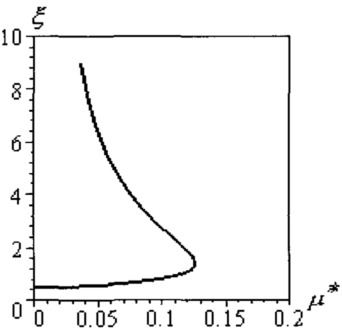


Fig. 1.2 Amplitudes ratio parameter ξ_* at bifurcational values of e^*.

When the detuning parameter σ increases (and other parameters are constant) the depicting point in the plane (e^*, μ^*) inevitably enters in the region of existence of steady-state subharmonic oscillation (it follows from (1.2.32)).

For the undamped system ($\mu = 0$) we have e_1^*=0, e_2^*=4/63, ξ_{min}=0.5. For $\mu = 0$ expression (1.2.35) gives $\cos\gamma = 0$, so the phase difference $\gamma = \pm\pi/2$.

It is seen from (1.2.38) and Fig. 1.2 that for subharmonic oscillations $\xi \geq 0.5$. Therefore appearance of these steady-state modes is not a result of branching from the force oscillation mode a=0. The frequency response curve, for certain values of μ and $\alpha\,\Lambda^2$, is presented in Fig. 1.3 (the frequency is specified by parameter σ). Calculations were made by formula (1.2.38) when considering ξ as a function of σ with account of (1.2.32). It is clear that the subharmonic oscillation appears when initial conditions fall in a certain "zone of attraction" of these modes.

The existence of steady-state free oscillation modes with frequency $\Omega/3$ (close to ω_0) means that there is a continuous energy transfer from the forced oscillation mode with frequency Ω. This transfer does fully compensate the dissipation of energy because of damping for every period (with a certain phase shift determined by (1.2.35)).

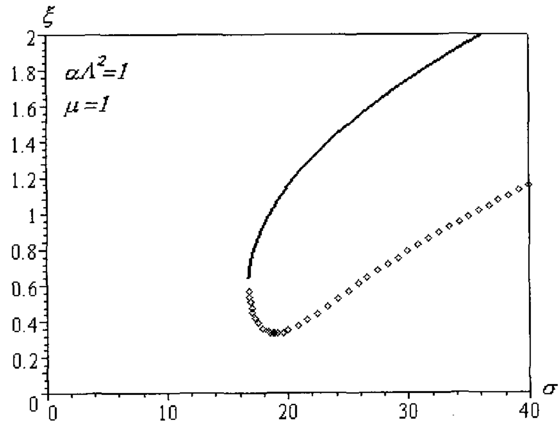


Fig. 1.3 Frequency response curve at subharmonic resonance.

To determine the response of the system at arbitrary initial conditions we have to consider ***nonstationary oscillations***. We return to set of equations (1.2.28), (1.2.29). These equations in dimensionless parameters (1.2.32) take the form

$$\frac{d\xi}{\sigma\, d\tau_1} = -\mu^*\xi - 3e^*\xi^2\cos\gamma\ , \tag{1.2.41}$$

$$\xi\frac{d\gamma}{\sigma\, d\tau_1} = 9e^*\xi\,(\xi^2 + 2 + \xi\sin\gamma) - \xi\ . \tag{1.2.42}$$

Undamped systems $\mu = 0$.

Dividing Eq. (1.2.41) by (1.2.42) we have

$$\frac{d\xi}{d\gamma} = \frac{-3e^*\xi^3\cos\gamma}{9e^*\xi\,(\xi^2 + 2 + \xi\sin\gamma) - \xi}\ . \tag{1.2.43}$$

This is an equation in full differentials. Its integral is

$$9e^*\left(\frac{\xi^4}{4} + \xi^2 + \frac{\xi^3}{3}\sin\gamma\right) - \frac{\xi^2}{2} = C\ . \tag{1.2.44}$$

This integral gives the connection between slow changes of the amplitude parameter ξ and phase difference parameter γ. Therefore we will call it integral of amplitude-frequency modulation (AFM). The integral curves in the plane (ξ ,γ) constitute an "amplitude-phase portrait" (APP) of the system at subharmonic resonance. Topology of the APP is determined by the single non-dimensional parameter e^* (1.2.32), which is proportional (at given α and σ) to the energy of forced oscillation. Two typical APPs are shown in Fig. 1.4 and 1.5. Fig. 1.4 corresponds to the case of "small energy": $e^* < e_2^*$ (1.2.39), Fig. 1.5 – the case of "large energy": $e^* > e_2^*$. We can restrict ourselves with the range γ (0, 2π) due to periodicity by γ. Note that the integral curves on both figures include line ξ =0 which corresponds to the forced oscillation ($a = 0$). For this line C =0.

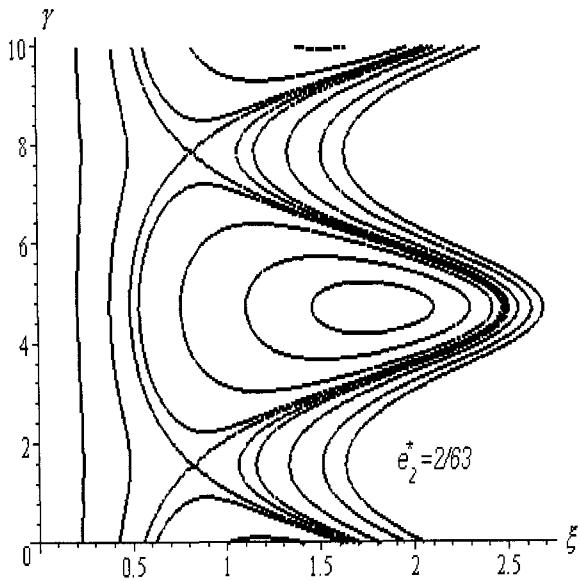


Fig. 1.4 APP for conservative system at subharmonic resonance; the case of "small energy" $e^* < e_2^*$.

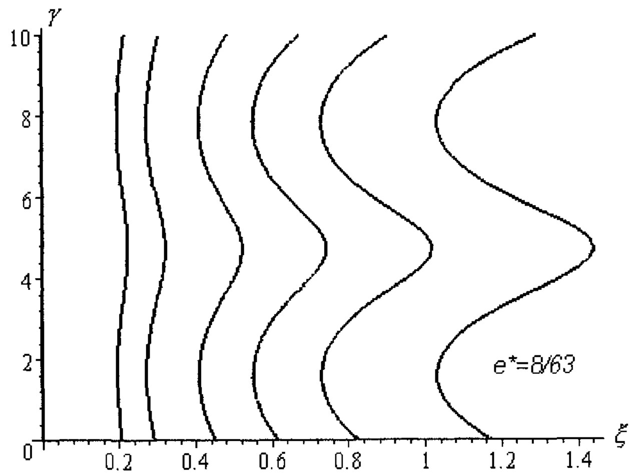


Fig. 1.5 APP for conservative system at subharmonic resonance; the case of "large energy" $e^* > e_2^*$.

Stationary points (steady-state oscillations) exist in the first case and are absent in the second one. The stationary points at $\gamma = -\pi/2 \pm 2k\pi$ ($k = 0, 1, \ldots$) are centers. Therefore they are stable (they are given by formula (1.2.38) with upper sign "+"). The stationary points at $\gamma = \pi/2 \pm 2k\pi$ ($k = 0, 1, \ldots$) are saddles, and are unstable (they are given by formula (1.2.38) with lower sign "–".) Integral curves crossing the unstable stationary points are separatrices. They divide the plane into regions of two types, with closed and unclosed integral curves. Closed curves depict motions with oscillating phase difference (about $\gamma = -\pi/2 \pm 2k\pi$). Unclosed curves correspond to motions with monotonously changing phase difference. In the case of "large energy" all curves are unclosed. The "depth" of amplitude modulation is determined by the ratio $\xi_{\min}/\xi_{\max}$, where $\xi_{\min}$ and $\xi_{\max}$ are minimal and maximal ξ -values on the integral curve, respectively. The modulation is very pronounced near the separatrices (if stationary points exist) and at large ξ -values (when stationary points are absent).

So the subharmonic resonance manifests itself not only in appearance (under certain conditions) of steady-state oscillation with frequency close to ω_0*, but also in nonstationary oscillations which are periodically modulated motions.*

The time characteristics for these motions, in particular, the modulation period, can be obtained by excluding γ from Eq. (1.2.41) and integral (1.2.44) and solving the resulting first-order ODE with separable variables.

Note that stability or instability of stationary points in general cases can be established by using a standard procedure. For conservative systems with integral $\Phi(\xi,\gamma) = C$ the stability of stationary points can be investigated simply by considering hessian H for function $\Phi(\xi,\gamma)$

$$H = \frac{\partial^2\Phi}{\partial\xi^2}\cdot\frac{\partial^2\Phi}{\partial\gamma^2} - \left(\frac{\partial^2\Phi}{\partial\xi\,\partial\gamma}\right)^2. \qquad (1.2.45)$$

A stationary point is stable if this is an elliptic point of the surface $\Phi(\xi,\gamma) = C$ ($H>0$), and is unstable if this is a hyperbolic point ($H<0$). The stability of points $\gamma = -\pi/2 \pm 2k\pi$ and instability of points $\gamma = \pi/2 \pm 2k\pi$ can be easily proved by calculating hessian at stationary points (1.2.28) ($\mu^* = 0$). But we would like to note that construction of amplitude-phase portraits eliminates the need in the investigation of stationary points stability.

Damped systems $\mu \neq 0$.

Set (1.2.41), (1.2.42) gives the following equation in plane (ξ, γ)

$$\frac{d\xi}{d\gamma} = \frac{-\mu^*\xi - 3e^*\xi^2\cos\gamma}{9e^*(\xi^2 + 2 + \xi\sin\gamma) - 1}. \qquad (1.2.46)$$

Determined by this equation direction field depends on parameters e^* and μ^*. In Fig. 6 (*a*)–(*d*) direction fields for four μ^* values at "small" e^*=2/63 are shown, and integral curves at certain initial conditions are given.

Stationary points are present in Fig. 1.6 (*a*)–(*c*) and absent in Fig. 1.6 (*d*) ($\mu^*>1/\sqrt{63}$). The centers become stable focuses at nonzero μ^*, but as μ^* increases their zone of attraction diminishes. The force oscillation mode — "stationary line" ξ $=0$ (γ value for this oscillation has no meaning) — is stable at any values of the excitation parameter e^* (or Λ) and damping parameter μ^*.

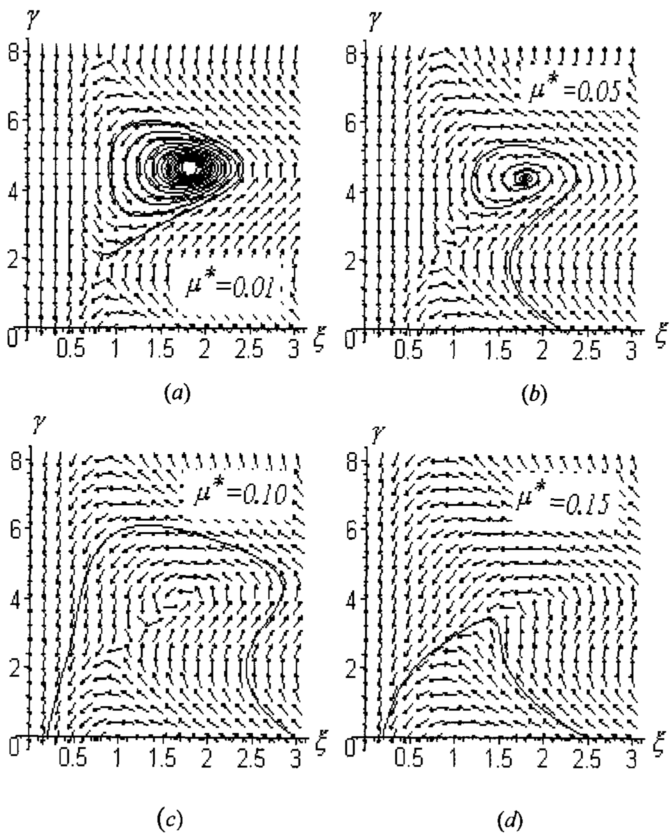
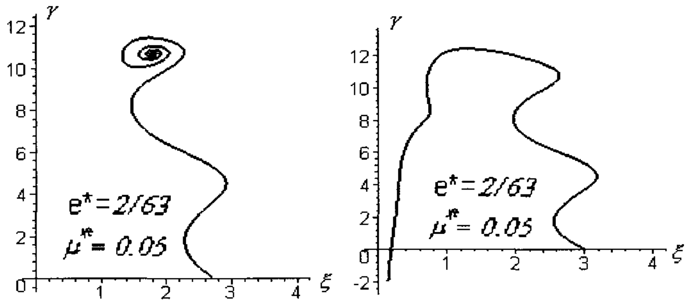


Fig. 1.6 Direction fields for nonstationary oscillations with various damping parameters at subharmonic resonance, and some integral curves; the case of "small energy" $e^*=2/63$.

The integral curves for $e^*=2/63$ and $\mu^*=0.05$ are presented in Fig. 1.7 (*a*), (*b*) for different initial points. Integral curves approach either the stable stationary points or the "stationary line" $\xi=0$ (depending on the initial conditions). In the case of "large damping" ($\mu^*>1/\sqrt{63}$) stable stationary points are absent.



(*a*) (*b*)

Fig. 1.7 Integral curves at subharmonic resonance with different initial conditions; the case of "small energy".

Thus, the response of real (dissipative) systems can be separated into two time ranges under subharmonic resonance condition. In the first range a nonstationary (transient) motion occurs. This motion includes both the free oscillation term (with pronounced amplitude-frequency modulation) and the forced oscillation term in (1.2.10).

In the second range oscillation reaches the steady state. This *steady-state oscillation can be either purely forced mode with frequency* Ω *or a superposition of this forced oscillation and free oscillation with frequency* $\Omega/3$. The latter case is possible if:

- damping parameter μ^* (1.2.32) satisfies the condition $\mu^*<1/\sqrt{63}$;
- parameter of excitation e^* (1.2.32) is in the range (e_1^*, e_2^*) (1.2.39).

If both these conditions are satisfied, the appearance of free oscillation mode depends on the initial conditions. Attractors in the plane (ξ , γ) are the line ξ =0 and the stable stationary points determined by (1.2.35) and (1.2.38) (with sign "+"). The free oscillation term retains if initial point is into the zone of attraction of any stable stationary point, and vanishes if the initial point is into the "zone of attraction" of the forced mode line ξ =0.

In the original dimensional parameters the conditions of existence of the free oscillation mode can be written as follows (with an account of notations (1.2.4)):

1. $|\Omega-3\omega_0|>\sqrt{63}\,n/m$; (1.2.47)

2. $F_1^2<F^2<F_2^2$ (1.2.48)

where

$$F_1^2=\frac{16c_1^3}{|c_3|}\left|\frac{\Omega}{\omega_0}-3\right|(1-\eta),\qquad F_2^2=\frac{16c_1^3}{|c_3|}\left|\frac{\Omega}{\omega_0}-3\right|(1+\eta),$$

$$\eta=\sqrt{1-63\left(\frac{n}{m(\Omega-3\omega_0)}\right)^2}$$

(here we use condition $\Omega/\omega_0\approx 3$ and assume $64/63\approx 1$).

Summarizing, we can write the complex oscillation in the first approximation using (1.2.10), (1.2.32):

$$\psi_0=\Lambda\left\{-\xi\, e^{\frac{i(\bar{\omega}\tau+\gamma)}{3}}+\frac{i}{2}\left[(1+\bar{\omega})e^{i\bar{\omega}\tau}+(1-\bar{\omega})e^{-i\bar{\omega}\tau}\right]\right\} \tag{1.2.49}$$

and the real oscillation

$$u=\Lambda\left[-\xi\sin\frac{1}{3}(\bar{\omega}\tau+\gamma)+\cos\bar{\omega}\tau\right] \tag{1.2.50}$$

where ξ and γ are determined by (1.2.41), (1.2.42). For the steady-state oscillation ξ and γ are given by (1.2.38) and (1.2.35) respectively. In real time and original dimensional parameters for the steady-state oscillation we have

$$U=\Lambda^0\left[-\xi_1\sin\frac{1}{3}(\Omega t+\gamma)+\cos\Omega t\right] \tag{1.2.51}$$

$$\Lambda^0=\frac{F}{c_1\left(1-\dfrac{\Omega^2}{\omega_0^2}\right)}\approx-\frac{F}{8c_1},$$

$$\xi_1 = \frac{1}{\sqrt{e^*}}\left(\frac{1}{9}-\frac{3}{2}e^* + \sqrt{\frac{1}{9}e^* - \frac{7}{4}e^{*2} - \frac{1}{9}\mu^{*2}}\right)^{1/2},$$

$$\gamma = \arccos\left(-\frac{\mu}{3\alpha\Lambda^2\xi_1}\right) = \arccos\left(-\frac{512nc_1^3}{mc_3\Omega F^2\xi_1}\right).$$

In Fig. 1.8 we compare the asymptotic solution (1.2.51) and direct numerical solution of the differential equation (1.2.1) for some parameters (they satisfy conditions (1.2.47), (1.2.48)). At the numerical integration the initial conditions corresponding to the stable stationary point were chosen. The same solution could be obtained for arbitrary initial conditions being in the zone of attraction of this point, after certain time interval with transient regime of motion. Both solutions practically coincide.

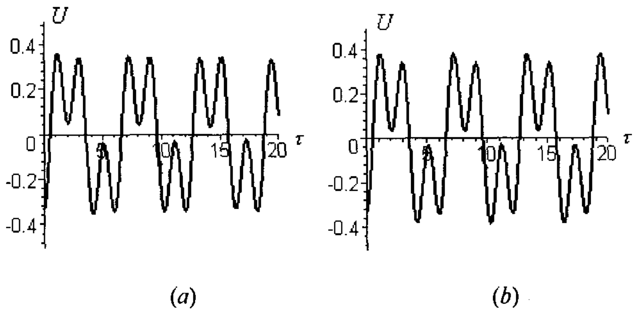


Fig. 1.8 Comparison of the response at subharmonic resonance obtained (*a*) by numerical integration of differential equation (1.2.1) and (*b*) by asymptotic solution (1.2.51) (for m=1, n=0.005, c_1=1, c_3=0.8, F=1.78885, Ω =3.1; ε was assumed be equal to 0.05, U^*=1). Nondimensional parameters: μ=0.1, α=2, e^*=0.05, σ=2, μ^*=0.05, Λ=–0.2236, ξ_1=1.1503, γ_1=–1.2768. Initial conditions: u_0=–0.3298, v_0=0.2421.

In the end of this paragraph we would like to note that the external force does not directly excite the free oscillation at subharmonic resonance. It excites the forced oscillation; the latter excites the oscillation with frequency $\Omega/3$ due to the cubic term in the elastic restoring force and partially transfers its energy to the free oscillation mode. So one may consider the subharmonic resonance as an internal resonance, as a result of energy exchange between different oscillations. Such a conclusion may be made although we have 1DOF system and the energy for the free oscillation term is after all supplied by the external force.

1.2.4 *Superharmonic resonance*

In this case $\bar{\omega} = \Omega/\omega_0 \approx 1/3$. We introduce the detuning parameter σ by the expression

$$\frac{\Omega}{\omega_0} = \frac{1}{3} + \varepsilon\,\sigma\,. \tag{1.2.52}$$

Then the term $-\alpha\,\Lambda^3 e^{3i\bar{\omega}\tau}$ in (1.2.11) becomes a secular one in addition to the terms presented in (1.2.13), and condition of secular terms absence takes the form:

$$-\frac{dA}{d\tau_1} - \mu\,A + 3i\alpha\,A(|A|^2 + 2\Lambda^2) - \alpha\,\Lambda^3 e^{3i\sigma\tau_1}) = 0\,. \tag{1.2.53}$$

Introducing polar coordinates $A = a\,e^{i\theta}$ and separating real and imaginary parts in (1.2.53), we obtain

$$\frac{da}{d\tau_1} = -\mu\,a - \alpha\,\Lambda^3 \cos(\theta - 3\sigma\,\tau_1)\,, \tag{1.2.54}$$

$$a\frac{d\theta}{d\tau_1} = 3\alpha\,a(a^2 + 2\Lambda^2) + \alpha\,\Lambda^3 \sin(\theta - 3\sigma\,\tau_1)\,. \tag{1.2.55}$$

By setting

$$\gamma = \theta - 3\sigma\,\tau_1 \tag{1.2.56}$$

these equations are transformed into autonomous form:

$$\frac{da}{d\tau_1} = -\mu\,a - \alpha\,\Lambda^3 \cos\gamma\,, \tag{1.2.57}$$

$$a\frac{d\gamma}{d\tau_1} = 3\alpha\,a(a^2 + 2\Lambda^2) + \alpha\,\Lambda^3 \sin\gamma - 3\sigma\,a\,. \tag{1.2.58}$$

Introducing dimensionless amplitude $\xi = a/\Lambda$ we rewrite the set (1.2.57), (1.2.58) in the form:

$$\frac{d\xi}{d\tau_1} = -\mu\xi - \alpha\Lambda^2 \cos\gamma , \tag{1.2.59}$$

$$\xi \frac{d\gamma}{d\tau_1} = 3\alpha\Lambda^2 \xi(\xi^2 + 2) + \alpha\Lambda^2 \sin\gamma - 3\sigma\xi . \tag{1.2.60}$$

Consider first ***steady-state oscillations*** (a = const, γ = const), corresponding to solutions of the set

$$\mu\xi + \alpha\Lambda^2 \cos\gamma = 0, \tag{1.2.61}$$

$$3\alpha\Lambda^2 \xi(\xi^2 + 2) + \alpha\Lambda^2 \sin\gamma - 3\sigma\xi = 0 . \tag{1.2.62}$$

From (1.2.62) we have

$$\cos\gamma = -\frac{\mu\xi}{\alpha\Lambda^2} . \tag{1.2.63}$$

Excluding γ from (1.2.61), (1.2.62) we obtain the equation for stationary points ξ :

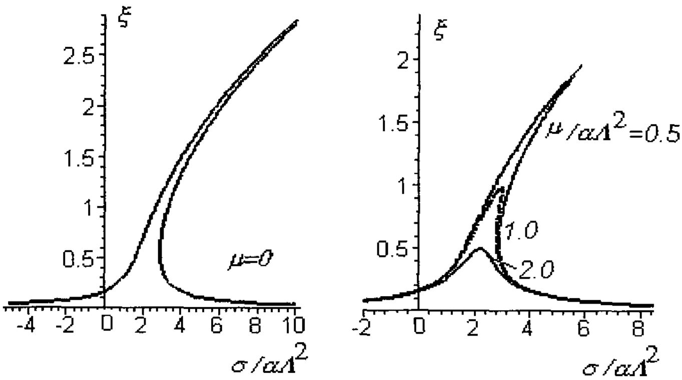
$$\xi^2 \left\{\mu^2 + 9\left[\alpha\Lambda^2 (\xi^2 + 2) - \sigma\right]^2\right\} = (\alpha\Lambda^2)^2 . \tag{1.2.64}$$

For undamped systems μ=0 we have $\cos\gamma = 0$, $\gamma = (2k+1)(\pi/2)$, k=0,±1,..., and Eq. (1.2.64) is reduced to:

$$3\xi\left(\xi^2 + 2 - \frac{\sigma}{\alpha\Lambda^2}\right)^2 \mp 1 = 0 . \tag{1.2.65}$$

A generalized frequency response curve in coordinates ξ , $\sigma/\alpha\Lambda^2$ for μ=0 is presented in Fig. 1.9 (a).

For damped systems the frequency response curves (1.2.64) depend on generalized parameter $\mu/\alpha\Lambda^2$. In Fig. 1.9 (b) these curves are drawn for values $\mu/\alpha\Lambda^2$ =0.5, 1.0 and 2.0, respectively. Curves for any values of this parameter lie between two branches of the curve for undamped systems.



(*a*) (*b*)

Fig. 1.9 Generalized frequency response curves for superharmonic resonance.

Note that the free oscillation term in (1.2.10) with account of (1.2.24), (1.2.52) and (1.2.56) can be written as follows:

$$a e^{i\left(\frac{3\Omega}{\omega_0}\tau+\gamma\right)} .$$

So the frequency of the steady-state oscillations (γ = const) in real time t is exactly 3Ω .

Further we will be interested in behavior of the system as the excitation energy increases, that can be characterized by parameter $e^* = \alpha\,\Lambda^2/\sigma$. The value of this parameter determines the number of real roots ξ in equation (1.2.65) (one or three). The "boundary" (bifurcational) value of e^* separating intervals with different numbers of real roots can be found by solving (1.2.65) together with equation $\partial\Psi/\partial\xi = 0$ where $\Psi(\xi, e^*)$ is the left hand side of (1.2.65). We obtain the following values:

$$e_b^* = (2+\sqrt[3]{0.75})^{-1} \approx 0.3438, \quad \xi_b^* = 1/\sqrt[3]{6} \approx 0.55. \qquad (1.2.66)$$

If $e^* > e_b^*$, only one real root exists (a single stationary point). If $e^* < e_b^*$, there are three real roots. In Fig. 1.9 value e_b^*=0.3438 corresponds to value $\sigma/\alpha\,\Lambda^2 = 1/e_b^*$=2.9086.

In the general case $\mu \neq 0$ the boundary value of e^* depends on damping parameter $\mu^* = \mu/\sigma$. Writing (1.2.64) in the form

$$\xi^2\left[\mu^{*2}+9e^{*2}\,(\xi^2+2-\frac{1}{e^*})^2\right]-e^{*2}=0$$

and following the same procedure we obtain an equation $f(e_b^*,\mu^*)=0$ (explicit expression for $f(e_b^*,\mu^*)$ is not written here because it is bulky). Its solution is presented in Fig. 1.10. The zone between the two branches of curve corresponds to the case of three real roots and the rest area to the case of single real root. The edge point is e_b^*=0.3675, $\mu_b^*=0.459$ (the corresponding ξ value is ξ_b=0.693). We see that the only stationary point exists for μ^*>0.459.

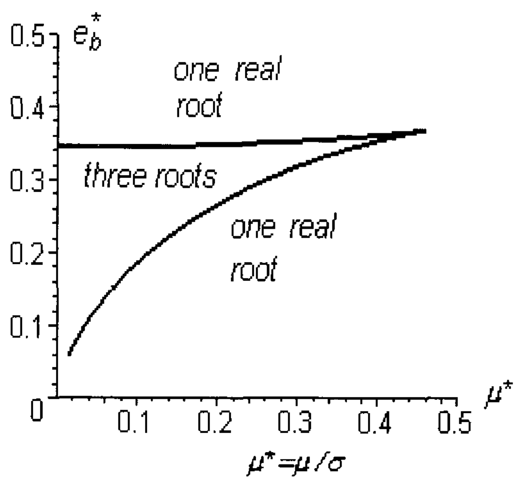


Fig. 1.10 Regions of existence of one or three steady-state superharmonic oscillations in damped systems.

Thus, *in distinction from the subharmonic resonance, at least one steady-state mode exists always under condition of superharmonic resonance.*

Consider now ***nonstationary oscillations***. We return to the set of equations (1.2.59), (1.2.60) and consider at first *undamped systems* $\mu=0$. Using parameter $e^*=\alpha\,\Lambda^2/\sigma$ we have:

$$\frac{d\xi}{\sigma\, d\tau_1} = -e^* \cos\gamma \ , \tag{1.2.67}$$

$$\xi \frac{d\gamma}{\sigma\, d\tau_1} = 3e^* \xi\, (\xi^2 + 2) + e^* \sin\gamma - 3\,\xi \,. \tag{1.2.68}$$

Dividing Eq. (1.2.67) by (1.2.68) we obtain

$$\frac{d\xi}{d\gamma} = \frac{-e^* \xi \cos\gamma}{3\,\xi \left[e^* (\xi^2 + 2) - 1\right] + e^* \sin\gamma} \,. \tag{1.2.69}$$

This equation in full differentials has AFM–integral

$$3e^* \left(\frac{\xi^4}{4} + \xi^2 \right) - \frac{3\xi^2}{2} + e^* \xi \sin\gamma = C \,. \tag{1.2.70}$$

The integral curves constitute a periodical in γ APP in the plane (ξ, γ). Topology of the APP is determined by the single nondimensional parameter e^*, similarly to the subharmonic resonance. In Fig. 1.11 (*a*)–(*d*) the APPs for four values e^*=0.2, 0.3, 0.4 and ∞ are presented.

The APPs presented in Fig. 1.11 (*a*), (*b*), are typical for values $e^* < e_b^* = 0.3438$ (see (1.2.66)). Here in every strip of width 2π ($0 < \gamma \le 2\pi$ and so on) three stationary points exist (two points at $\gamma = \pi/2 \pm 2k\pi$ and one point at $\gamma = 3\pi/2 \pm 2k\pi$). Two of these points are stable (centers, elliptic points of the surface (1.2.70)), the third point is unstable (a saddle, a hyperbolic point of the surface (1.2.70)). Of course, stability of these points could be easily investigated by standard procedure. These stationary points correspond to three branches of the amplitude-frequency curve for $\sigma/\alpha\, \Lambda^2 > 2.9086$ in Fig. 1.9.

The APPs presented in Fig. 1.11, (*c*), (*d*), are typical for values $e^* > e_b^*$. In this case in every strip of width 2π the only stationary point exists. This point is a center, therefore it is stable. It corresponds to the single branch of the amplitude-frequency curve in Fig. 1.9 for $\sigma/\alpha\, \Lambda^2 <$ 2.9086.

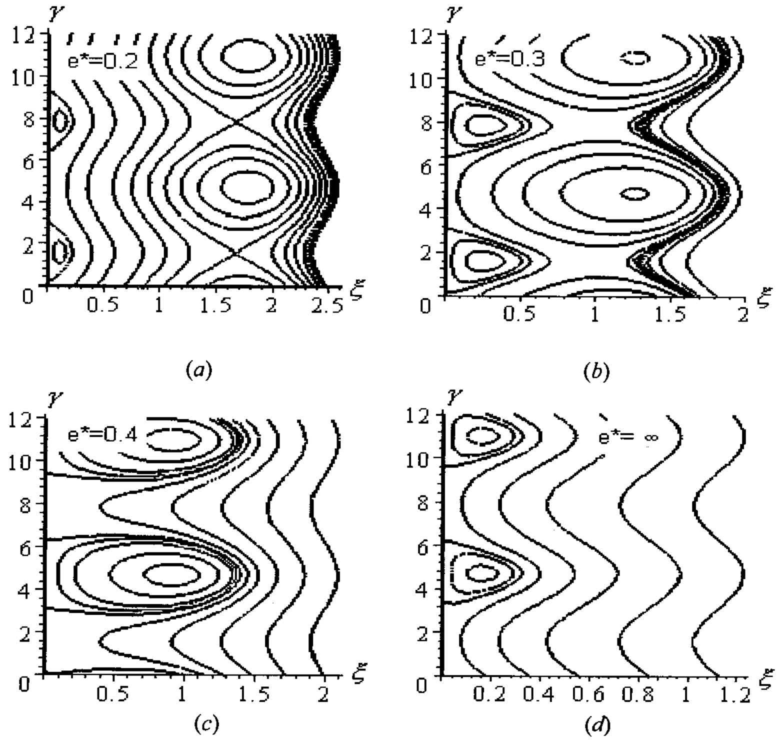


Fig. 1.11 Amplitude-phase portraits at superharmonic resonance (nonstationary oscillations of conservative systems).

The integral curves on both figures include the line $\xi=0$ corresponding to the forced oscillation ($a=0$); for this line C=0. But comparison of APPs in Fig. 1.11 and Figs 1.4, 1.5 shows an essential difference between superharmonic and subharmonic resonance. *At the superharmonic resonance, in contrast to the subharmonic one, separatrices surrounding the stable stationary points (nearest to the line* $\xi=0$*) intersect this line. It means that the forced oscillation* $\xi=0$ *(*$a=0$*) is always unstable (no integral curve in the vicinity of the line* $\xi=0$ *is closeat every point to this line). So the appearance of the free oscillation term at superharmonic resonance may be considered as loss of stability of the forced oscillation.*

We see that the superharmonic resonance, similarly to the subharmonic resonance, manifests itself not only in appearance of steady-state oscillation with frequency close to ω_0, but also in periodically modulated motions. The time dependence for these motions can be obtained, if needed, by excluding γ from Eq. (1.2.67) and integral (1.2.70) and by solving the resulting first-order differential equation with separable variables. The "depth" of amplitude modulation and the modulation period depend on initial conditions and essentially increase when the initial point approaches the separatrices.

Consider now briefly *nonstationary oscillations of damped systems.* From (1.2.59), (1.2.60) we have

$$\frac{d\xi}{d\gamma} = \frac{-\mu^*\xi^2 - e^*\xi\cos\gamma}{3e^*\xi(\xi^2+2) + e^*\sin\gamma - 3\xi}. \qquad (1.2.71)$$

In Fig. 1.12 (*a*)-(*d*) the direction fields in plane (ξ , γ) for $\mu^* = 0.2$ and four values of e^* are given, and integral curves at certain initial conditions are presented. As it is seen from Fig. 1.10, only value $e^*=0.3$ is in the zone where three stationary points exist in each strip of width 2π ; for other e^* values the only stationary point exists in such a strip. At nonzero μ^* the centers become stable focuses. Attractors are stable stationary points (not the line $\xi=0$, in distinction from the subharmonic resonance).

Thus, the response of a real (dissipative) system under superharmonic resonance condition (1.2.52) is different in two time ranges (similarly to the subharmonic resonance). In the first range a nonstationary (transient) motion depending on initial conditions occurs. In the second time range the oscillation reaches the steady state. This steady-state oscillation is a superposition of the forced mode with frequency Ω and the free oscillation with frequency 3Ω . *Influence of initial conditions in the second range can display itself only in realizing one of two possible attractors (in the case of two stable stationary points).*

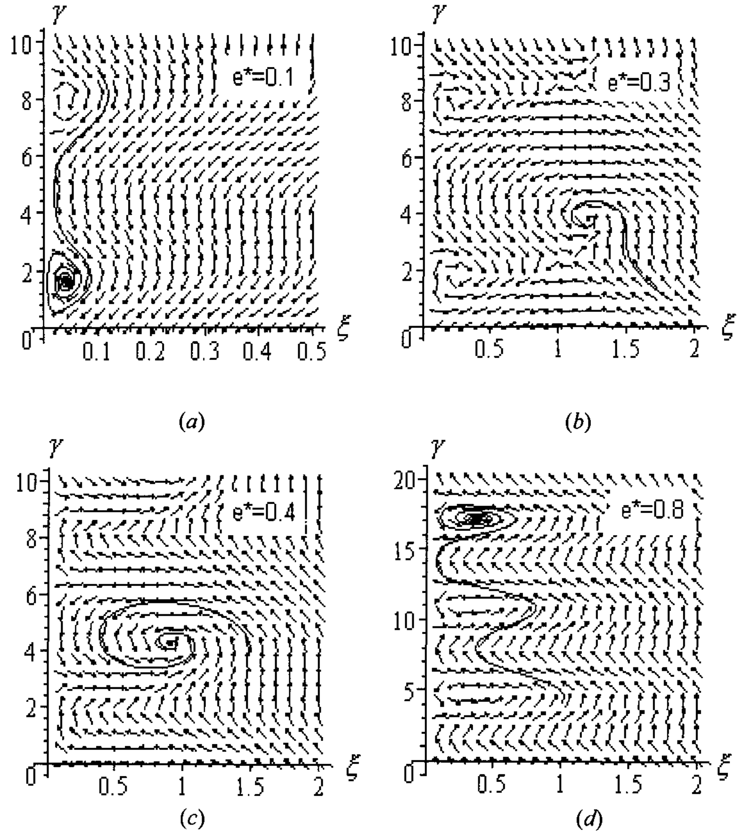


Fig. 1.12 Direction fields for nonstationary oscillations with various values of the excitation parameter e^* at superharmonic resonance, and integral curves; $\mu^* = 0.2$.

Summarizing, we can write the complex oscillation in the first approximation using (1.2.10), (1.2.24) and (1.2.52):

$$\psi_0 = \Lambda\left\{\xi\, e^{i(3\overline{\omega}\tau+\gamma)} + \frac{i}{2}\left[(1+\overline{\omega})e^{i\overline{\omega}\tau} + (1-\overline{\omega})e^{-i\overline{\omega}\tau}\right]\right\}, \quad (1.2.72)$$

and the real oscillation

$$u = \Lambda\left[\xi \sin(3\overline{\omega}\tau + \gamma) + \cos\overline{\omega}\tau\right], \quad (1.2.73)$$

where ξ and γ are determined by (1.2.59), (1.2.60). For the steady-state oscillation ξ is given by (1.2.64) (one or two values for stable stationary points) and γ is given by (1.2.63). In the real time and original dimensional parameters, for the steady-state oscillation we have :

$$U = \Lambda^0[\xi \sin(3\Omega t + \gamma) + \cos\Omega t], \qquad (1.2.74)$$

where

$$\Lambda^0 = \frac{F}{c_1\left(1 - \frac{\Omega^2}{\omega_0^2}\right)} \approx \frac{9F}{8c_1},$$

$$\gamma = \arccos\left(-\frac{\mu\xi}{\alpha\Lambda^2}\right) = \arccos\left(-\frac{512nc_1^3\xi}{243mc_3\Omega F^2}\right).$$

In Fig. 1.13 asymptotic solution (1.2.74) and the direct numerical solution of the differential equation (1.2.1) are compared for some parameters. At the numerical integration the initial conditions corresponding to the stable stationary point were chosen. Both solutions practically do coincide.

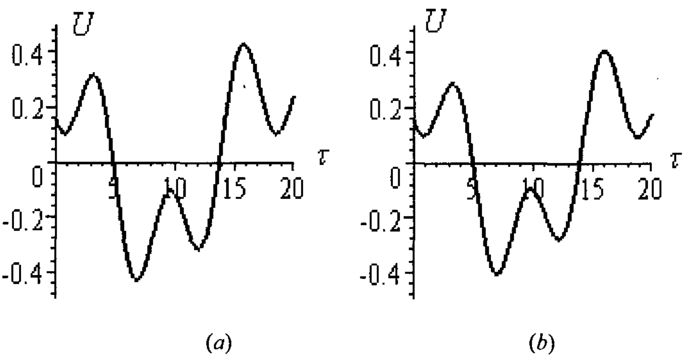


Fig. 1.13 Comparison of responses at the superharmonic resonance obtained (*a*) by asymptotic solution (1.2.74) and (*b*) by numerical integration of differential equation (1.2.1) (for m=1, n=0.005, c_1=1, c_3=0.8, F=0.2767, Ω =0.3533; ε was assumed be equal to 0.1, U^*=1). Nondimensional parameters: μ=0.1, α=1, e^*=0.5, σ=0.2, μ^*=0.5, Λ= 0.31623, ξ_1=0.63598, γ_1=4.0199. Initial conditions: u_0=0.16143, v_0=-0.1361.

In conclusion we would like to note that the superharmonic resonance, similarly to the subharmonic resonance, is caused by an energy exchange between different oscillations. The external force excites the forced oscillation; the latter one excites the oscillation with frequency 3Ω due to the cubic term in the elastic restoring force. So the superharmonic resonance may be also considered as an internal resonance phenomenon.

1.2.5 *Primary resonance*

Finally, let us consider the primary resonance, i.e., the case $\Omega \approx \omega_0$. We assume now that the amplitude of the external force in Eq. (1.2.2) is of the order of ε, and this equation takes the form:

$$\frac{d^2u}{d\tau^2} + 2\varepsilon\,\mu\frac{du}{d\tau} + u + 8\varepsilon\,\alpha\,u^3 = \varepsilon\, f\cos\overline{\omega}\tau\,. \tag{1.2.75}$$

($\overline{\omega} = \Omega/\omega_0$). Introducing a detuning parameter

$$\overline{\omega} = 1 + \varepsilon\,\sigma\,, \tag{1.2.76}$$

and using the complex variables (1.2.5), we obtain the complex representation of the equation of motion

$$\frac{d\psi}{d\tau} - i\psi = -\varepsilon\left[\mu\left(\psi + \psi^*\right) + i\alpha\left(\psi - \psi^*\right)^3 + \frac{f}{2}\left(e^{i\overline{\omega}\tau} + e^{-i\overline{\omega}\tau}\right)\right]. \tag{1.2.77}$$

Then the standard multiple scales procedure yields

$$D_0\psi_0 - i\psi_0 = 0\,, \tag{1.2.78}$$

$$D_0\psi_1 - i\psi_1 = -D_1\psi_0 - \mu\left(\psi_0 + \psi_0^*\right) - i\alpha\left(\psi_0 - \psi_0^*\right)^3 + \frac{f}{2}\left(e^{i\overline{\omega}\tau} + e^{-i\overline{\omega}\tau}\right). \tag{1.2.79}$$

Substituting the general solution of Eq. (1.2.78)

$$\psi_0 = Ae^{i\tau} \tag{1.2.80}$$

into the r.h.s. of Eq. (1.2.79) we obtain the following condition of secular terms absence:

$$-\frac{dA}{d\tau_1} - \mu\,A + 3i\alpha\,A\left|A\right|^2 + \frac{f}{2}e^{i\sigma\tau_1} = 0\,. \tag{1.2.81}$$

Using polar coordinates $A = a\exp(i\theta)$ and separating real and imaginary parts in (1.2.81), we obtain an autonomous system

$$\frac{da}{d\tau_1} = -\mu a + \frac{f}{2}\cos\gamma , \tag{1.2.82}$$

$$a\frac{d\gamma}{d\tau_1} = 3\alpha a^3 - a\sigma - \frac{f}{2}\sin\gamma , \tag{1.2.83}$$

where

$$\gamma = \theta - \sigma\tau_1 . \tag{1.2.84}$$

Consider first ***stationary oscillations***, when the right hand sides in (1.2.82), (1.2.83) vanish:

$$-\mu a + \frac{f}{2}\cos\gamma = 0 , \tag{1.2.85}$$

$$3\alpha a^3 - a\sigma - \frac{f}{2}\sin\gamma = 0 . \tag{1.2.86}$$

Hence

$$\cos\gamma = \frac{2\mu a}{f} . \tag{1.2.87}$$

Excluding γ from (1.2.85), (1.2.86) we obtain

$$(\mu a)^2 + a^2(\sigma - 3\alpha a^2)^2 = \frac{f^2}{4} . \tag{1.2.88}$$

Note that in the linear case ($\alpha = 0$)

$$a = \frac{f}{2\sqrt{\mu^2 + \sigma^2}} ,$$

with the maximum a value (at $\sigma = 0$) $a_{\max} = f/2\mu$.

In the nonlinear case, after introducing parameters (for generality of the analysis)

$$\xi = \frac{\alpha a^2}{\mu} , \quad \sigma^* = \frac{\sigma}{\mu} , \quad Q = \frac{\alpha f^2}{\mu^3} , \tag{1.2.89}$$

equation (1.2.88) is reduced to

$$\xi + \xi(\sigma^* - 3\xi)^2 = \frac{Q}{4}. \qquad (1.2.90)$$

Considering σ^* as a frequency parameter we obtain generalized frequency response curves presented in Fig. 1.14 for Q=1, 2.0528, 5 and 10 (ξ may be considered as oscillation energy parameter and Q as external force parameter).

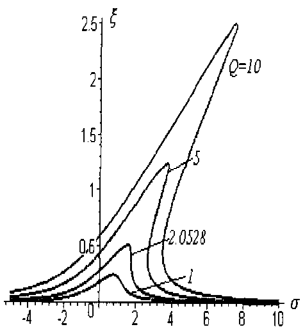


Fig. 1.14 Generalized frequency response curves for primary resonance at four values of external force parameter Q.

Note that the maximum ξ value coincides with that for the linear problem: $\xi_{max} = Q/4$; it is reached at $\sigma^* = 3Q/4 = 3\xi_{max}$, so the nonlinearity yields only certain inclination of the linear frequency response curve without stretching along the ξ-axis.

At low Q values the only real root (1.2.90) exists (curve 1 in Fig. 1.14). At large Q values there exist three real roots (curves 3, 4). The bifurcational value of Q can be easily found by elementary analysis of (1.2.90): $Q_b = 32\sqrt{3}/27 \approx 2.0528$. Curve 2 in Fig.1.14 is obtained for this Q; corresponding bifurcational σ^* value is $\sigma_b^* = \sqrt{3}$ and ξ value is $\xi_b^* = 2\sqrt{3}/9 \approx 0.3849$.

The case μ=0 has to be considered separately as parameters (1.2.89) are not applicable. For this case $\gamma = \pm\pi/2 + 2k\pi$ (from (1.2.87)), and Eq.(1.2.86) gives

$$3\alpha\, a^3 - a\sigma \mp \frac{f}{2} = 0. \qquad (1.2.91)$$

This equation determines ratio a/f as a function of parameters αf^2 and σ. Ratios a/f via σ for two values αf^2=0.5; 2 are presented in Fig. 1.15.

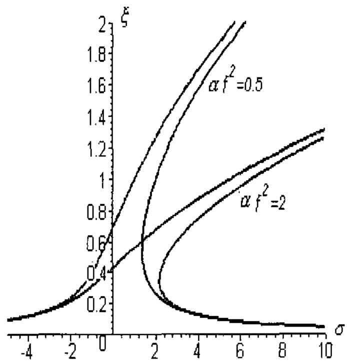


Fig. 1.15 Generalized frequency response curves for primary resonance at μ=0.

For a given σ , bifurcational value of parameter αf^2 (the change of the number of real roots in (1.2.91)) and corresponding value of amplitude a are:

$$(\alpha f^2)_b = \frac{16}{81}\sigma^3, \quad a_b = \frac{1}{3}\sqrt{\frac{\sigma}{\alpha}}. \qquad (1.2.92)$$

Now we return to the case of ***nonstationary oscillations***, described by Eqs. (1.2.82), (1.2.83). Consider first the *undamped system* μ =0:

$$\frac{da}{d\tau_1} = \frac{f}{2}\cos\gamma\ , \qquad (1.2.93)$$

$$a\frac{d\gamma}{d\tau_1} = 3\alpha\, a^3 - a\sigma - \frac{f}{2}\sin\gamma\ . \qquad (1.2.94)$$

Dividing Eq. (1.2.93) by (1.2.94) we obtain

$$\frac{da}{d\gamma}=\frac{2af\cos\gamma}{2a(3\alpha a^2-\sigma)-f\sin\gamma}\,.$$

This equation in full differentials has AFM–integral

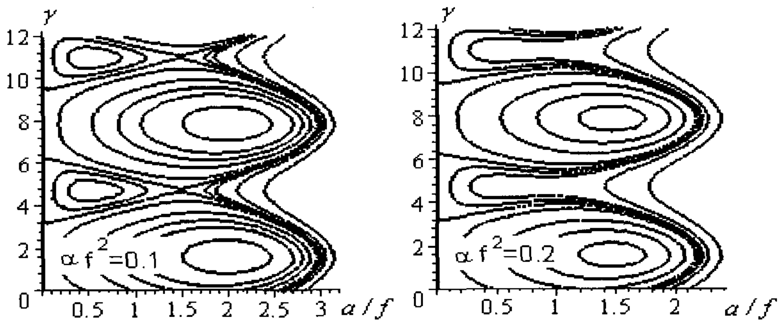
$$\frac{3\alpha a^4}{2}-\sigma a^2-fa\sin\gamma=C. \qquad (1.2.95)$$

Writing this equation in the form

$$\frac{3(\alpha f^2)}{2}\left(\frac{a}{f}\right)^4-\sigma\left(\frac{a}{f}\right)^2-\frac{a}{f}\sin\gamma=C_1,$$

we see that the integral curves in coordinates (a/f, γ) depend on two parameters αf^2 and σ (in contrast with the secondary resonances where integral curves depend on the only parameter).

The APPs in the plane (a/f, γ) formed by the integral curves are presented in Figures 1.16 (*a*), (*b*) for $\sigma=1.0$ and two values of αf^2. For $\sigma=1.0$, according to (1.2.92), the bifurcational value $(\alpha f^2)_b$=0.1975. The first plot, Fig. 1.16 (*a*), calculated for αf^2=0.1, represents the case of two stable and one unstable stationary points in each strip of width 2π ($0<\gamma\le 2\pi$). The second graph, Fig. 1.16 (*b*), calculated for αf^2=0.2, represents the case of one stable stationary point.



(*a*) (*b*)

Fig. 1.16 Amplitude-phase portraits at primary resonance (nonstationary oscillations of undamped systems); $\sigma=1.0$.

In general case of *nonstationary oscillations with damping* (Eqs. (1.2.82), (1.2.83)), the APPs view is apparent. Attractors are the stable stationary points determined by (1.2.88) or (1.2.90). Increase of damping parameter μ leads to decrease of parameters ξ, σ^* and Q (1.2.89). When σ^* becomes less than the bifurcational value $\sigma_b^* = \sqrt{3}$ the only stationary point in each strip of width 2π retains.

1.3 Concluding Remarks

The asymptotic approach based on the complex representation of equations of motion has been presented. Efficiency of this approach is illustrated by application to some known problems (Duffing equation, Van der Pol equation).

A detailed analysis of the primary and secondary resonances at forced oscillations of nonlinear (cubic) oscillator has been performed, including both steady-state and nonstationary motions. It is shown, in particular, that appearance of the subharmonic or superharmonic resonance may be regarded as an exhibition of the internal resonance, as a result of energy exchange between the excited oscillation and free oscillation modes.

Simultaneously we would like to emphasize a principal difference between subharmonic and superharmonic resonances. At subharmonic resonance the purely forced oscillation (with frequency of the excitation) remains steady-state (it does not loss stability), and the free oscillation component of motion can appear only when initial conditions fall into a certain domain. The superharmonic resonance always means the loss of stability of the purely forced oscillation, and at least one free oscillation mode appears necessarily.

Historical remarks. The complex representation of classical equations of motion for a system of linear oscillators was firstly used in quantum mechanics and for the analysis of so-called coupled oscillations and waves in mechanics, electronics and solid state physics (Rabinovich, Trubetskov, 1984), (Louisell, 1960), (Ovchinnikov, 1970), (Scott, Lomdahl, 1985), (Kosevich, Kovalyov, 1989), (Pierce, 1954). The complex conjugate linear combinations v_j+iu_j and v_j-iu_j of displacements

u_j and velocities v_j of oscillators can be visually presented as vectors of equal length rotating in opposite directions. Actually it is enough to find only one complex function for each oscillator completely defining both displacement and velocity. Such a choice of variables, in particular, leads to a very simple and natural procedure of quantization: complex conjugate functions become operators of creation and annihilation, and their squared moduli — the number of elementary excitations (Rabinovich, Trubetskov, 1984), (Kosevich, Kovalyov, 1989), (Louisell, 1962).

If there is coupling between oscillators then both complex conjugate functions are included in each equation of motion, but in the theory of coupled oscillations it is supposed that the unidirectional rotations of oscillators are connected more strongly than the vectors with opposite directions (Louisell, 1960), (Ovchinnikov, 1970), (Scott, Lomdahl, 1985). This simplification reduces the order of equations of motion by the factor of two and ensures right expressions for the first two terms in expansion on a coupling parameter of exact solutions of the linearized equations (Rabinovich, Trubetskov, 1984), (Louisell, 1960). In the case of coupled system of nonlinear oscillators the possibility of comparison with exact solutions is absent. In this connection the equations of motion for unidirectional rotations are usually treated phenomenologically as the simplest mathematical model of a nonlinear oscillatory system (Kosevich, Kovalyov, 1989). Its validity is confirmed by comparison with the averaged equations for complex amplitudes (Louisell, 1960), (Scott, Lomdahl, 1985), (Kosevich, Kovalyov, 1974), qualitative reasons (Scott, Lomdahl, 1985) and asymptotic estimations (Kosevich, Kovalyov, 1975).

In papers (Manevitch L.I. 1997, 1999a, 1999b, 2001a) the scheme of perturbation theory was employed not only to justify the domination of coupling between unidirectional rotations but also to provide a possibility of construction of higher approximations. The complex representation of the equations of motion for the weakly coupled nonlinear oscillators was regarded as the natural form for efficient application of a method of two-scale expansions (Cole, 1968), (Nayfeh, 1981), (Tsyan, 1956).