Chapter 1

Formal Stochastic Differential Equations

The goal of this first chapter is to establish the Chen-Strichartz formula which, in a way, is a cornerstone of this book. This formula is universal and determines very precisely and explicitly the local structure of any stochastic flow. To derive this formula, it is quite convenient to work in an abstract and formal setting, in which we do not have to care about convergence questions.

The reader which is not so familiar with the theory of stochastic differential equations and vector fields is invited read the Appendices A and B which are included at the end of the book.

1.1 Motivation

Let us consider a stochastic differential equation on $\mathbb{R}^n$ of the type

$$X_t^{x_0} = x_0 + \sum_{i=1}^{d} \int_0^t V_i(X_s^{x_0}) \circ dB_s^i,\ t \geq 0, \tag{1.1}$$

where:

(1) $x_0 \in \mathbb{R}^n$;
(2) $V_1, ..., V_d$ are C^∞ bounded vector fields on $\mathbb{R}^n$;
(3) $\circ$ denotes Stratonovitch integration;
(4) $(B_t)_{t\geq 0} = (B_t^1, ..., B_t^d)_{t\geq 0}$ is a d-dimensional standard Brownian motion.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and denote by $(X_t^{x_0})_{t\geq 0}$ the solution of (2.1) with initial condition $x_0 \in \mathbb{R}^n$. First, by Itô's formula, we have

$$f(X_t^{x_0}) = f(x_0) + \sum_{i=1}^{d} \int_0^t (V_i f)(X_s^{x_0}) \circ dB_s^i, \ t \geq 0.$$

Now, a new application of Itô's formula to $V_i f(X_s^x)$ leads to

$$f(X_t^{x_0}) = f(x_0) + \sum_{i=1}^{d} (V_i f)(x_0) B_t^i + \sum_{i,j=1}^{d} \int_0^t \int_0^s (V_j V_i f)(X_u^{x_0}) \circ dB_u^j \circ dB_s^i.$$

We can continue this procedure to get after N steps

$$f(X_t^{x_0}) = f(x_0) + \sum_{k=1}^{N} \sum_{I=(i_1,...i_k)} (V_{i_1}...V_{i_k} f)(x_0) \int_{\Delta^k[0,t]} \circ dB^I + \mathbf{R}_N(t),$$

for some remainder term $\mathbf{R}_N$, where we used the notations:

(1)

$$\Delta^k[0,t] = \{(t_1, ..., t_k) \in [0,t]^k, t_1 \leq ... \leq t_k\};$$

(2) If $I = (i_1, ...i_k) \in \{1, ..., d\}^k$ is a word with length k,

$$\int_{\Delta^k[0,t]} \circ dB^I = \int_{0 \leq t_1 \leq ... \leq t_k \leq t} \circ dB_{t_1}^{i_1} \circ ... \circ dB_{t_k}^{i_k}.$$

If we dangerously do not care about convergence questions (these questions are widely discussed in [Ben Arous (1989b)]), it is tempting to let $N \to +\infty$ and to assume that $\mathbf{R}_N \to 0$. We are thus led to the nice (but formal!) formula

$$f(X_t^{x_0}) = f(x_0) + \sum_{k=1}^{+\infty} \sum_{I=(i_1,...i_k)} (V_{i_1}...V_{i_k} f)(x_0) \int_{\Delta^k[0,t]} \circ dB^I. \tag{1.2}$$

We can rewrite this formula in a more convenient way. Let Φ_t be the stochastic flow associated with the stochastic differential equation (2.1). There is a natural action of Φ_t on smooth functions: The pull-back action given by

$$(\Phi_t^* f)(x_0) = (f \circ \Phi_t)(x_0) = f(X_t^{x_0}).$$

The formula (1.2) shows then that we have the following formal development for this action

$$\Phi_t^* = \mathbf{Id} + \sum_{k=1}^{+\infty} \sum_{I=(i_1,\dots i_k)} V_{i_1}\dots V_{i_k} \int_{\Delta^k[0,t]} \circ dB^I. \tag{1.3}$$

Though this formula does not make sense from an analytical point of view, at least, it shows that the *probabilistic information* contained in the stochastic flow associated with the stochastic differential equation (1.1) is given by the set of Stratonovitch chaos $\int_{\Delta^k[0,t]} \circ dB^I$. What is a priori less clear is that the *algebraic information* which is relevant for the study of Φ_t^* is given by the structure of the Lie algebra generated by the V_i's, and this is precisely this aspect we want to stress in this chapter which is devoted to the study of formal objects like

$$\mathbf{Id} + \sum_{k=1}^{+\infty} \sum_{I=(i_1,\dots i_k)} V_{i_1}\dots V_{i_k} \int_{\Delta^k[0,t]} \circ dB^I.$$

Such objects and their relations with flows seem to appear the first time in the works of K.T. Chen [Chen (1957)], [Chen (1961)].

1.2 The signature of a Brownian motion

Let us denote by $\mathbb{R}[[X_1,\dots,X_d]]$ the **non-commutative** algebra of formal series with d indeterminates.

Definition 1.1 The signature of a d-dimensional standard Brownian motion $(B_t)_{t\geq 0}$ is the element of $\mathbb{R}[[X_1,\dots,X_d]]$ defined by

$$S(B)_t = \mathbf{1} + \sum_{k=1}^{+\infty} \sum_{I=(i_1,\dots i_k)} X_{i_1}\dots X_{i_k} \int_{\Delta^k[0,t]} \circ dB^I, \ t \geq 0.$$

Remark 1.1 *We define the signature by using Stratonovitch's integrals because we keep in mind the connection with stochastic flows which appeared with formula (1.3). Nevertheless, it is possible to define a signature by using Itô's integrals. The link between these two signatures is given in Proposition 1.2 below.*

Remark 1.2 *In the same way, it is of course also possible to define the signature of a general semimartingale.*

Observe that the signature hence defined is the solution of the *formal* stochastic differential equation

$$S(B)_t = 1 + \sum_{i=1}^{d} \int_0^t S(B)_s X_i \circ dB^i_s, \ t \geq 0. \tag{1.4}$$

Such linear equations appear in the study of Brownian motions on Lie groups. Indeed, let $\mathbb{G}$ be a Lie group with Lie algebra $\mathfrak{g}$.

Definition 1.2 A process $(X_t)_{t\geq 0}$ with values in $\mathbb{G}$ is called a (left) Brownian motion on $\mathbb{G}$ if:

(1) $(X_t)_{t\geq 0}$ is continuous;
(2) for each $s \geq 0$, the process $\left(X_s^{-1} X_{t+s}\right)_{t\geq 0}$ is independent of the process $(X_u)_{0\leq u\leq s}$;
(3) for each $s \geq 0$, the processes $\left(X_s^{-1} X_{t+s}\right)_{t\geq 0}$ and $(X_t)_{t\geq 0}$ are identical in law.

In a general way, one can construct Brownian motions on Lie groups by solving differential equations. Let us consider $V_1, ..., V_d \in \mathfrak{g}$. As explained in Appendix B, $V_1, ..., V_d \in \mathfrak{g}$ can be seen as left invariant vector fields on $\mathbb{G}$, so that we can consider the following stochastic differential equation

$$X_t = 1_{\mathbb{G}} + \sum_{i=1} \int_0^t V_i(X_s) \circ dB^i_s, \ t \geq 0, \tag{1.5}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion on $\mathbb{R}^d$. For instance, if $\mathbb{G}$ is a linear group of matrices, equation (1.5) can be rewritten

$$X_t = 1_{\mathbb{G}} + \sum_{i=1} \int_0^t X_s V_i \circ dB^i_s.$$

It is easily seen that there is a unique solution $(X_t)_{t\geq 0}$ to the stochastic differential equation (1.5), and this solution is a (left) Brownian motion on $\mathbb{G}$. The process $(X_t)_{t\geq 0}$ is called a lift of $(B_t)_{t\geq 0}$ in $\mathbb{G}$. It is interesting to note that, conversely, each Brownian motion on $\mathbb{G}$ is solution of a stochastic differential equation

$$X_t = X_0 + \int_0^t V_0(X_s) ds + \sum_{i=1} \int_0^t V_i(X_s) \circ dB^i_s, \ t \geq 0,$$

where $V_0, V_1, ..., V_d$ are left-invariant vector fields on $\mathbb{G}$; for further details on this, we refer to [Hunt (1958)] and [Yosida (1952)].

With this in mind, we interpret now $\mathbb{R}[[X_1, ..., X_d]]$ as the universal enveloping algebra of the free Lie algebra with d generators $\mathfrak{f}_d$. So, with this interpretation, at the formal level the signature of $(B_t)_{t \geq 0}$ can be interpreted as a lift of $(B_t)_{t \geq 0}$ in the formal object $\exp(\mathfrak{f}_d)$.

On the other hand, the first section of this chapter has shown that the pull-back action on functions of the stochastic flow $(\Phi_t)_{t \geq 0}$ associated with the stochastic differential equation

$$X_t^{x_0} = x_0 + \sum_{i=1}^{d} \int_0^t V_i(X_s^{x_0}) \circ dB_s^i, \ t \geq 0,$$

solves formally the stochastic differential equation

$$\Phi_t^* = \mathbf{Id} + \sum_{i=1} \int_0^t \Phi_s^* V_i \circ dB_s^i,$$

so that $(\Phi_t^*)_{t \geq 0}$ can formally be seen as a lift of $(B_t)_{t \geq 0}$ in the formal object $\exp(\mathfrak{L}(V_1, ..., V_d))$ where $\mathfrak{L}(V_1, ..., V_d)$ is the Lie algebra generated by $V_1, ..., V_d$.

Therefore, since $\mathfrak{f}_d$ is a universal object in the theory of Lie algebras, the signature appears as a universal object in the theory of stochastic flows. In particular, if we do not care about convergence questions, any algebraic formula concerning the signature of $(B_t)_{t \geq 0}$ can be applied to study the stochastic flow associated with **any** stochastic differential equation driven by $(B_t)_{t \geq 0}$. As it will be seen in the next chapters, one of the most illuminating example in this direction is certainly the universality of the Chen-Strichartz formula; an other example is given by the expectation of the signature (see the end of the chapter), a purely algebraic object, which *explains* in a different way than the usual one, the Markov property shared by any process that solves a stochastic differential equation driven by Brownian motions.

We have a fundamental flow property for the signature which stems directly from the following key but simple relations, known as the Chen's relations since the seminal work [Chen (1957)].

Lemma 1.1 *For any word* $(i_1, ..., i_n) \in \{1, ..., d\}^n$ *and any* $0 < s < t$,

$$\int_{\Delta^n[0,t]} \circ dB^{(i_1,...,i_n)} = \sum_{k=0}^{n} \int_{\Delta^k[0,s]} \circ dB^{(i_1,...,i_k)} \int_{\Delta^{n-k}[s,t]} \circ dB^{(i_{k+1},...,i_n)},$$

where we used the following notations:

(1)

$$\int_{\Delta^k[s,t]} \circ dB^{(i_1,\dots,i_k)} = \int_{s\le t_1\le\dots\le t_k\le t} \circ dB^{i_1}_{t_1} \circ \dots \circ dB^{i_k}_{t_k};$$

(2) if I is a word with length 0, then $\int_{\Delta^0[0,t]} \circ dB^I = 1$.

Proof. It follows readily by induction on n by noticing that

$$\int_{\Delta^n[0,t]} \circ dB^{(i_1,\dots,i_n)} = \int_0^t \left(\int_{\Delta^{n-1}[0,t_n]} \circ dB^{(i_1,\dots,i_{n-1})} \right) \circ dB^{i_n}_{t_n}.$$

□

Proposition 1.1 *For $0 < s < t$,*

$$S(B)_t = S(B)_s \left(1 + \sum_{k=1}^{+\infty} \sum_{I=(i_1,\dots i_k)} X_{i_1}\dots X_{i_k} \int_{\Delta^k[s,t]} \circ dB^I \right).$$

Proof. We have, thanks to the previous lemma,

$$\begin{aligned}
&S(B)_s \left(1 + \sum_{k=1}^{+\infty} \sum_{I} X_{i_1}\dots X_{i_k} \int_{\Delta^k[s,t]} \circ dB^I \right) \\
=&1 + \sum_{k,k'=1}^{+\infty} \sum_{I,I'} X_{i_1}\dots X_{i_k} X_{i'_1}\dots X_{i'_{k'}} \int_{\Delta^k[s,t]} \circ dB^I \int_{\Delta^{k'}[0,s]} \circ dB^{I'} \\
=&1 + \sum_{k=1}^{+\infty} \sum_{I} X_{i_1}\dots X_{i_k} \int_{\Delta^k[0,t]} \circ dB^I \\
=&S(B)_t.
\end{aligned}$$

□

Remark 1.3 *Observe that if $I \in \{1,\dots,d\}^k$ is a word with length k then for any $0 < s < t$:*

(1) $\int_{\Delta^k[s,t]} \circ dB^I$ is independent from $(B_u)_{u\le s}$;
(2)

$$\int_{\Delta^k[s,t]} \circ dB^I =^{law} \int_{\Delta^k[0,t-s]} \circ dB^I.$$

Therefore, we can roughly conclude that:

(1) for each $s \ge 0$, the process $\left(S(B)_s^{-1} S(B)_{t+s}\right)_{t\ge 0}$ is independent of the process $(S(B)_u)_{0\le u\le s}$;

(2) *for each $s \geq 0$, the processes $\left(S(B)_s^{-1} S(B)_{t+s}\right)_{t \geq 0}$ and $(S(B)_t)_{t \geq 0}$ are identical in law.*

By using the relation between Stratonovitch's and Itô's integral (see Appendix A), it is possible to give a formula for the signature of a Brownian motion which only involves Itô's iterated integrals.

Proposition 1.2 *We have*

$$S(B)_t = \mathbf{1} + \sum_{k=1}^{+\infty} \sum_{I \in \{0,1,\dots,d\}^k} X_{i_1} \dots X_{i_k} \int_{\Delta^k[0,t]} dB^I, \ t \geq 0,$$

where we used the following notations:

(1)

$$X_0 = \frac{1}{2} \sum_{i=1}^{d} X_i^2, \ B_t^0 = t;$$

(2)

$$\int_{\Delta^k[0,t]} dB^I = \int_{0 \leq t_1 \leq \dots \leq t_k \leq t} dB_{t_1}^{i_1} \dots dB_{t_k}^{i_k}.$$

Proof. Let $I = (i_1, \dots, i_k) \in \{1, \dots, d\}^k$. From the definition of Stratonovitch's integrals, we have

$$\begin{aligned}\int_{\Delta^k[0,t]} \circ dB^I = &\int_0^t \left(\int_{\Delta^{k-1}[0,t_k]} \circ dB^{i_1,\dots,i_{k-1}} \right) dB_{t_k}^{i_k} \\ &+ \frac{1}{2} \tau_{i_{k-1},i_k} \int_0^t \left(\int_{\Delta^{k-2}[0,t_{k-1}]} \circ dB^{i_1,\dots,i_{k-2}} \right) dt_{k-1},\end{aligned}$$

where

$$\begin{aligned}\tau_{i_{k-1},i_k} &= 0 \text{ if } i_{k-1} \neq i_k \\ &= 1 \text{ if } i_{k-1} = i_k.\end{aligned}$$

Consider now the smallest set $\mathcal{I}$ of words which satisfies the following properties:

(1) $I \in \mathcal{I}$;
(2) if $J = (j_1, \dots, j_l) \in \mathcal{I}$ and if $j_m = j_{m+1} \neq 0$ for some $1 \leq m \leq l-1$, then $(j_1, \dots, j_{m-1}, 0, j_{m+2}, \dots, j_l) \in \mathcal{I}$.

By iterating the previous formula, we get

$$\int_{\Delta^k[0,t]} \circ dB^I = \sum_{J\in\mathcal{I}} \frac{1}{2^{k-|J|}} \int_{\Delta^{|J|}[0,t]} dB^J,$$

where $\mid J \mid$ denotes the length of the word J. The expected result follows readily. □

Remark 1.4 *Observe that if we write equation (1.4) in Itô's form, we get*

$$S(B)_t = \mathbf{1} + \frac{1}{2}\int_0^t S(B)_s \left(\sum_{i=1}^d X_i^2\right) ds + \sum_{i=1}^d \int_0^t S(B)_s X_i dB_s^i,$$

which explains intuitively formula of Proposition 1.2.

1.3 The Chen-Strichartz development formula

This section is devoted to the proof of the Chen-Strichartz development formula. The formula we give is actually a restatement of a result of [Chen (1957)] and [Strichartz (1987)], and can be seen as a deep generalization of the Baker-Campbell-Hausdorff formula (see Appendix B).

The Chen-Strichartz formula is an explicit formula for $\ln S(B)_t$. In particular, it appears that $\ln S(B)_t$ is a Lie element, a result which is far from being obvious at a first look. As it is illustrated in this book, the geometric consequences of this development are rather deep. Before we go into the heart of this formula, let us first try to understand a simple case: the commutative case.

We denote $\mathfrak{S}_k$ the group of the permutations of the index set $\{1, ..., k\}$ and if $\sigma \in \mathfrak{S}_k$, we denote for a word $I = (i_1, ..., i_k)$, $\sigma \cdot I$ the word $(i_{\sigma(1)}, ..., i_{\sigma(k)})$. Now, let us observe that if $X_1, ..., X_d$ were commuting, we would have

$$S(B)_t = \exp\left(\sum_{i=1}^d X_i B_t^i\right).$$

Indeed in that case, by symmetrization, we get

$$S(B)_t = \mathbf{1} + \sum_{k=1}^{+\infty} \sum_{I=(i_1,...,i_k)} X_{i_1}...X_{i_k} \left(\frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} \int_{\Delta^k[0,t]} \circ dB^{\sigma\cdot I}\right).$$

Now observe that

$$\sum_{\sigma\in\mathfrak{S}_k}\int_{\Delta^k[0,t]} \circ dB^{\sigma\cdot I} = B_t^{i_1}...B_t^{i_k},$$

which implies,

$$S(B)_t = \mathbf{1} + \sum_{k=1}^{+\infty}\frac{1}{k!}\sum_{I=(i_1,...,i_k)} X_{i_1}...X_{i_k}B_t^{i_1}...B_t^{i_k} = \exp\left(\sum_{i=1}^{d} X_i B_t^i\right).$$

Of course, in the general case, this formula does not hold anymore. But, we still have a nice formula for $\ln S(B)_t$ which involves iterated functionals of the commutators $X_iX_j - X_jX_i$.

We define the bracket between two elements U and V of $\mathbb{R}[[X_1,...,X_d]]$ by

$$[U,V] = UV - VU,$$

and it is easily checked that this bracket endows $\mathbb{R}[[X_1,...,X_d]]$ with a Lie algebra structure. If $I = (i_1,...,i_k) \in \{1,...,d\}^k$ is a word, we denote by X_I the commutator defined by

$$X_I = [X_{i_1},[X_{i_2},...,[X_{i_{k-1}},X_{i_k}]...].$$

If $\sigma \in \mathfrak{S}_k$, we denote $e(\sigma)$ the cardinality of the set

$$\{j \in \{1,...,k-1\}, \sigma(j) > \sigma(j+1)\}.$$

Theorem 1.1 *We have*

$$S(B)_t = \exp\left(\sum_{k\geq 1}\sum_{I=(i_1,...,i_k)} \Lambda_I(B)_t X_I\right), \quad t \geq 0,$$

where:

$$\Lambda_I(B)_t = \sum_{\sigma\in\mathfrak{S}_k}\frac{(-1)^{e(\sigma)}}{k^2\binom{k-1}{e(\sigma)}}\int_{\Delta^k[0,t]} \circ dB^{\sigma^{-1}\cdot I}.$$

Proof. We shall proceed in several steps.

Step 1. First, we write

$$S(B)_t = \mathbf{1} + \sum_{k=1}^{+\infty}\int_{\Delta^k[0,t]} \circ d\omega_{s_1}... \circ d\omega_{s_k},$$

where we used the notation

$$d\omega = \sum_{i=1}^{d} X_i dB^i.$$

Now we have,

$$S(B)_t = \exp\left(\ln S(B)_t\right) = \exp\left(\sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k}(S(B)_t - \mathbf{1})^k\right).$$

Therefore we get,

$$S(B)_t = \exp Z_t,$$

with

$$Z_t = \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k} \left(\sum_{n=1}^{+\infty} \int_{\Delta^n[0,t]} \circ d\omega_{s_1} \ldots \circ d\omega_{s_n} \right)^k. \tag{1.6}$$

For each positive integer r, consider all ways of writing

$$r = p_1 + \ldots + p_m, \; m = 1, \ldots, r,$$

for p_j positive integers, and set $q_0 = 0$ and $q_j = p_1 + \ldots + p_j$, for $j \geq 1$. We can now expand out

$$\sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k} \left(\sum_{n=1}^{+\infty} \int_{\Delta^n[0,t]} \circ d\omega_{s_1} \ldots \circ d\omega_{s_n} \right)^k,$$

to obtain, thanks to Lemma 1.1,

$$Z_t = \sum_{r=1}^{+\infty} \sum_{m=1}^{r} \sum_{p_j} \frac{(-1)^{m-1}}{m} \int \circ d\omega_{s_1} \ldots \circ d\omega_{s_r},$$

where the integral is taken over the region given by the inequalities

$$\begin{aligned} &0 < s_1 < \ldots < s_{q_1} < t, \\ &0 < s_{q_1+1} < \ldots < s_{q_2} < t \\ &\ldots \\ &0 < s_{q_{m-1}+1} < \ldots < s_{q_m} < t. \end{aligned} \tag{1.7}$$

Step 2. By applying now the generalized Baker-Campbell-Hausdorff formula (B.4) of Appendix B, we obtain

$$Z_t = \sum_{r=1}^{+\infty} \sum_{m=1}^{r} \sum_{p_j} \frac{(-1)^{m-1}}{mr} \int [\ldots[\circ d\omega_{s_1}, \circ d\omega_{s_2}]\ldots], \circ d\omega_{s_r}], \tag{1.8}$$

where the integral is taken over the same region. The domain determined by the inequalities (1.7) can be written as the union of simplices obtained from the simplex $\Delta^r[0,t]$ by permuting the variables, actually

$$\int [\ldots[\circ d\omega_{s_1}, \circ d\omega_{s_2}]\ldots], \circ d\omega_{s_r}] = \sum \int_{\Delta^r[0,t]} [\ldots[\circ d\omega_{s_{\sigma(1)}}, \circ d\omega_{s_{\sigma(2)}}]\ldots], \circ d\omega_{s_{\sigma(r)}}],$$

where the inner sum is taken over the permutations $\sigma \in \mathfrak{S}_r$ that satisfy

$$\sigma(q_j+1) < \sigma(q_j+2) < \ldots < \sigma(q_{j+1}),\ j = 0, \cdots, m-1.$$

Therefore, by regrouping the terms in (1.8), we obtain that Z_t is equal to

$$\sum_{r=1}^{+\infty} \sum_{\sigma\in\mathfrak{S}_r} \sum_{m=1}^{r} \frac{(-1)^{m-1}}{mr} d(r,m,\sigma) \int_{\Delta^r[0,t]} [\ldots[\circ d\omega_{s_{\sigma(1)}}, \circ d\omega_{s_{\sigma(2)}}]\ldots], \circ d\omega_{s_{\sigma(r)}}],$$

where $d(r,m,\sigma)$ is the number of ways of choosing positive integers $p_1, \ldots, p_m$ with $p_1 + \ldots + p_m = r$ satisfying $\sigma(q_j+1) < \ldots < \sigma(q_{j+1})$, $j = 0, \ldots, m-1$.

Step 3. We claim now that

$$\sum_{m=1}^{r} \frac{(-1)^{m-1}}{mr} d(r,m,\sigma) = \frac{(-1)^{e(\sigma)}}{r^2 \binom{r-1}{e(\sigma)}}.$$

Indeed, a straightforward combinatorial argument shows that

$$d(r,m,\sigma) = \binom{r-e(\sigma)-1}{m-e(\sigma)-1},$$

so that we need to sum

$$\sum_{m=e(\sigma)+1}^{r} \frac{(-1)^{m-1}}{mr} \binom{r-e(\sigma)-1}{m-e(\sigma)-1}.$$

But let us observe that for $n \geq 0$ and $k > 0$,

$$\begin{aligned}\sum_{j=0}^{n} \frac{(-1)^j}{k+j} \binom{n}{j} &= \sum_{j=0}^{n} (-1)^j \binom{n}{j} \int_0^1 x^{j+k-1} dx \\ &= \int_0^1 (1-x)^n x^{k-1} dx \\ &= \frac{(k-1)!n!}{(k+n)!}.\end{aligned}$$

Therefore, by setting $k = e(\sigma) + 1$, $m = k + j$, and $n = r - k$ we obtain

$$\sum_{m=1}^{r} \frac{(-1)^{m-1}}{mr} d(r,m,\sigma) = \frac{(-1)^{e(\sigma)}}{r^2 \binom{r-1}{e(\sigma)}}.$$

Step 4. Putting things together, we get therefore

$$Z_t = \sum_{r=1}^{+\infty} \sum_{\sigma \in \mathfrak{S}_r} \frac{(-1)^{e(\sigma)}}{r^2 \binom{r-1}{e(\sigma)}} \int_{\Delta^r[0,t]} [\ldots[\circ d\omega_{s_{\sigma(1)}}, \circ d\omega_{s_{\sigma(2)}}]\ldots], \circ d\omega_{s_{\sigma(r)}}].$$

Now, observe that

$$\begin{aligned}&[\circ d\omega_{s_{\sigma(1)}}, [\ldots, [\circ d\omega_{s_{\sigma(r-1)}}, \circ d\omega_{s_{\sigma(r)}}]\ldots] \\ &= (-1)^{r-1} [\ldots[\circ d\omega_{s_{\sigma(r)}}, \circ d\omega_{s_{\sigma(r-1)}}]\ldots], \circ d\omega_{s_{\sigma(1)}}],\end{aligned}$$

and that if for $\sigma \in \mathfrak{S}_r$, σ^* denotes the permutation defined by $\sigma^*(k) = \sigma(r+1-k)$, then $e(\sigma^*) = r - 1 - e(\sigma)$. Therefore, we also have

$$Z_t = \sum_{r=1}^{+\infty} \sum_{\sigma \in \mathfrak{S}_r} \frac{(-1)^{e(\sigma)}}{r^2 \binom{r-1}{e(\sigma)}} \int_{\Delta^r[0,t]} [\circ d\omega_{s_{\sigma(1)}}, [\ldots, [\circ d\omega_{s_{\sigma(r-1)}}, \circ d\omega_{s_{\sigma(r)}}]\ldots].$$

By expanding out

$$\sum_{\sigma \in \mathfrak{S}_r} \frac{(-1)^{e(\sigma)}}{r^2 \binom{r-1}{e(\sigma)}} \int_{\Delta^r[0,t]} [\circ d\omega_{s_{\sigma(1)}}, [\ldots, [\circ d\omega_{s_{\sigma(r-1)}}, \circ d\omega_{s_{\sigma(r)}}]\ldots]$$

into

$$\sum_{\sigma\in\mathfrak{S}_r} \frac{(-1)^{e(\sigma)}}{r^2\binom{r-1}{e(\sigma)}} \sum_{I=(i_1,\dots,i_r)} X_I \int_{\Delta^k[0,t]} \circ dB^{\sigma^{-1}\cdot I},$$

we obtain finally the claimed formula. ☐

Remark 1.5 *Observe that the first terms in the Chen-Strichartz formula are:*

(1)

$$\sum_{I=(i_1)} \Lambda_I(B)_t X_I = \sum_{k=1}^{d} B_t^i X_i;$$

(2)

$$\sum_{I=(i_1,i_2)} \Lambda_I(B)_t X_I = \frac{1}{2} \sum_{1\le i<j\le d} [X_i, X_j] \int_0^t B_s^i \circ dB_s^j - B_s^j \circ dB_s^i;$$

it is interesting to note the above Stratonovitch integrals are also Itô integrals, that is

$$\int_0^t B_s^i \circ dB_s^j - B_s^j \circ dB_s^i = \int_0^t B_s^i dB_s^j - B_s^j dB_s^i.$$

Remark 1.6 *Actually, the Chen-Strichartz formula holds for the signature of any semimartingale: this is indeed a pathwise result.*

Remark 1.7 *The formal development for the action on functions of the stochastic flow* Φ_t^* *associated with a stochastic differential equation of the type (1.3) reads therefore*

$$\Phi_t^* = \exp\left(\sum_{k\ge1} \sum_{I=(i_1,\dots,i_k)} \Lambda_I(B)_t V_I\right).$$

It is also possible to obtain a formal development for the action of Φ_t *on smooth tensor fields. Indeed an iteration of the formula given in the Proposition A.6 of Appendix A leads, due to the Lie algebra homomorphism property of the Lie derivative, to*

$$\Phi_t^* = \exp\left(\sum_{k\ge1} \sum_{I=(i_1,\dots,i_k)} \Lambda_I(B)_t \mathcal{L}_{V_I}\right).$$

1.4 Expectation of the signature of a Brownian motion

It is interesting to note that it is possible to derive in a purely algebraic manner the semigroup P_t associated with the solution of a stochastic differential equation driven by Brownian motions .

Definition 1.3 The element of $\mathbb{R}[[X_1, ..., X_d]]$ defined by

$$P_t = \mathbf{1} + \sum_{k=1}^{+\infty} \sum_{I=(i_1,...i_k)} X_{i_1}...X_{i_k} \mathbb{E}\left(\int_{\Delta^k[0,t]} \circ dB^I\right), \ t \geq 0,$$

is called the expectation of the signature of the Brownian motion $(B_t)_{t\geq 0}$.

Proposition 1.3 *We have*

$$P_t = \exp\left(\frac{1}{2}t\sum_{i=1}^{d} X_i^2\right), \ t \geq 0.$$

Proof. By using proposition (1.2), we get

$$P_t = \mathbf{1} + \sum_{k=1}^{+\infty} \sum_{I\in\{0,1,...,d\}^k} X_{i_1}...X_{i_k} \mathbb{E}\left(\int_{\Delta^k[0,t]} dB^I\right), \ t \geq 0,$$

where:

$$X_0 = \frac{1}{2}\sum_{i=1}^{d} X_i^2, \text{ and } B_t^0 = t.$$

In the previous sum, the only terms whose expectation does not vanish are the terms

$$\int_{\Delta^k[0,t]} dB^I$$

where I is a word which contains only 0. Therefore,

$$P_t = \mathbf{1} + \sum_{k=1}^{+\infty} X_0^k \int_{\Delta^k[0,t]} dt_1...dt_k.$$

Since

$$\int_{\Delta^k[0,t]} dt_1...dt_k = \frac{1}{k!}\int_{[0,t]^k} dt_1...dt_k = \frac{t^k}{k!},$$

we get

$$P_t = \mathbf{1} + \sum_{k=1}^{+\infty} \frac{t^k}{k!} X_0^k = e^{tX_0}.$$

□

Remark 1.8 *If we think $(S(B)_t)_{t \geq 0}$ as the solution of the formal stochastic differential equation*

$$S(B)_t = \mathbf{1} + \sum_{i=1}^{d} \int_0^t S(B)_s X_i \circ dB_s^i, \qquad (1.9)$$

and P_t as $\mathbb{E}\left(S(B)_t\right)$, then the above formula is rather intuitive. Indeed, by writing the Itô's form of (1.9), and by taking the expectation, we obtain the equation

$$P_t = 1 + \int_0^t P_s \left(\frac{1}{2} \sum_{i=1}^{d} X_i^2 \right) ds,$$

which directly implies

$$P_t = \exp\left(\frac{1}{2} t \sum_{i=1}^{d} X_i^2 \right).$$

Remark 1.9 *In the commutative case, we have*

$$S(B)_t = \exp\left(\sum_{i=1}^{d} X_i B_t^i \right),$$

and the formula for P_t reduces to the well-known Laplace transform formula

$$\mathbb{E}\left(\exp\left(\sum_{i=1}^{d} X_i B_t^i \right) \right) = \exp\left(\frac{1}{2} t \sum_{i=1}^{d} X_i^2 \right).$$

We stress the fact that the last formula only holds in the commutative case.

Observe that the semigroup property of P_t, that is

$$P_{t+s} = P_t P_s,$$

could have been directly derived from the identity

$$S(B)_{t+s} = S(B)_t \left(\mathbf{1} + \sum_{k=1}^{+\infty} \sum_{I=(i_1,\ldots i_k)} X_{i_1} \ldots X_{i_k} \int_{\Delta^k[t,t+s]} \circ dB^I \right).$$

Indeed, since $\int_{\Delta^k[t,t+s]} \circ dB^I$ is independent of $(B_u)_{u\leq t}$, we deduce

$$P_{t+s} = P_t \left(\mathbf{1} + \sum_{k=1}^{+\infty} \sum_{I=(i_1,\dots i_k)} X_{i_1}\dots X_{i_k} \mathbb{E}\left(\int_{\Delta^k[t,t+s]} \circ dB^I \right) \right).$$

Now observe that, due to the stationarity of the increments of a Brownian motion,

$$\mathbb{E}\left(\int_{\Delta^k[t,t+s]} \circ dB^I \right) = \mathbb{E}\left(\int_{\Delta^k[0,s]} \circ dB^I \right),$$

so that

$$P_{t+s} = P_t \left(1 + \sum_{k=1}^{+\infty} \sum_{I=(i_1,\dots i_k)} X_{i_1}\dots X_{i_k} \mathbb{E}\left(\int_{\Delta^k[0,s]} \circ dB^I \right) \right) = P_t P_s.$$

We have already pointed out that the signature is a universal object in the theory of stochastic flows, so let us see the analytic counterpart of the purely algebraic formula

$$\mathbf{1} + \sum_{k=1}^{+\infty} \sum_{I=(i_1,\dots i_k)} X_{i_1}\dots X_{i_k} \mathbb{E}\left(\int_{\Delta^k[0,t]} \circ dB^I \right) = \exp\left(\frac{1}{2} t \sum_{i=1}^{d} X_i^2 \right).$$

In the first section, we have seen that for the action on smooth functions of the stochastic flow Φ associated with the stochastic differential equation

$$X_t^{x_0} = x_0 + \sum_{i=1}^{d} \int_0^t V_i(X_s^{x_0}) \circ dB_s^i,$$

we had formally

$$\Phi_t^* = \mathbf{Id} + \sum_{k=1}^{+\infty} \sum_{I=(i_1,\dots i_k)} V_{i_1}\dots V_{i_k} \int_{\Delta^k[0,t]} \circ dB^I.$$

Therefore,

$$\mathbb{E}\left(\Phi_t^*\right) = \mathbf{Id} + \sum_{k=1}^{+\infty} \sum_{I=(i_1,\dots i_k)} V_{i_1}\dots V_{i_k} \mathbb{E}\left(\int_{\Delta^k[0,t]} \circ dB^I \right) = e^{\frac{1}{2}t\sum_{i=1}^{d} V_i^2}.$$

By coming back to the definition of Φ_t^* we deduce that if $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function,

$$\mathbb{E}\left(f(X_t^{x_0})\right) = \left(e^{\frac{1}{2}t\sum_{i=1}^{d} V_i^2} f\right)(x_0),$$

which exactly says that $(X_t^{x_0})_{t\geq 0}$ is a Markov process with generator $e^{\frac{1}{2}t\sum_{i=1}^{d} V_i^2}$. In the same way, by using the formal development of Φ_t on smooth tensor fields which reads

$$\Phi_t^* = \mathbf{Id} + \sum_{k=1}^{+\infty} \sum_{I=(i_1,...i_k)} \mathcal{L}_{V_{i_1}}...\mathcal{L}_{V_{i_k}} \int_{\Delta^k[0,t]} \circ dB^I,$$

where $\mathcal{L}$ denotes the Lie derivative, we obtain that if K is a smooth tensor field on $\mathbb{R}^n$,

$$\mathbb{E}\left[(\Phi_t^* K)(x_0)\right] = \left(e^{\frac{1}{2}t\sum_{i=1}^{d} \mathcal{L}_{V_i}^2} K\right)(x_0).$$

Of course, all this is only formal, but should convince the reader of the relevance of the formal calculus on the signature.

1.5 Expectation of the signature of other processes

As already observed, the notion of signature can be defined for other processes than Brownian motions and there is a corresponding notion of expectation for the signature. Let us for instance mention the example of the signature of a fractional Brownian motion. A d-dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ is a Gaussian process

$$B_t = (B_t^1, ..., B_t^d),\ t \geq 0,$$

where $B^1, ..., B^d$ are d independent centered Gaussian processes with covariance function

$$R(t,s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

It can be shown that such a process admits a continuous version whose paths are locally p-Hölder for $p < H$. Therefore, if $H > \frac{1}{2}$, the integrals

$$\int_{\Delta^k[0,t]} dB^I$$

can be understood in the sense of Young's integration; see [Young (1936)] and [Zähle (1998)]. We define then the signature of $(B_t)_{t\geq 0}$ by

$$S(B)_t = \mathbf{1} + \sum_{k=1}^{+\infty} \sum_{I=(i_1,\dots i_k)} X_{i_1}\dots X_{i_k} \int_{\Delta^k[0,t]} dB^I,\ t \geq 0,$$

and associate with $S(B)$ the family of operators

$$P_t = \mathbf{1} + \sum_{k=1}^{+\infty} \sum_{I=(i_1,\dots i_k)} X_{i_1}\dots X_{i_k} \mathbb{E}\left(\int_{\Delta^k[0,t]} dB^I\right),\ t \geq 0.$$

The increments of $(B_t)_{t\geq 0}$ are not independent (they are however stationary), and $(P_t)_{t\geq 0}$ is therefore not a semigroup. Nevertheless, as shown in [Baudoin and Coutin (2004)], when $t \to 0$,

$$P_t = \mathbf{1} + \frac{1}{2}t^{2H}\left(\sum_{i=1}^{d} X_i^2\right) + t^{4H} \sum_{i,j,k,l=1}^{d} a_{i,j,k,l} X_i X_j X_k X_l + O(t^{6H}),$$

where,

$$\begin{aligned} a_{i,j,k,l} &= \frac{1}{2}\delta_{k,l}\delta_{j,i}\left[\frac{1}{4} - 2H\beta(2H, 2H-1)\right] + \frac{1}{2}\delta_{i,k}\delta_{j,l}\frac{2H-1}{4H(4H-1)} \\ &+ \frac{H(2H-1)}{8}\delta_{j,k}\delta_{i,l}\left[\beta(2H, 2H-2) + \frac{1}{4H-1} - \frac{1}{2H-1}\right], \end{aligned}$$

with $\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$, and $\delta_{i,j}$ is the Kronecker's symbol. A development for P_t which leads to development in small times of expressions of the type $\mathbb{E}\left(f(X_t^{x_0})\right)$, where $(X_t^{x_0})_{t\geq 0}$ denotes the solution of the equation

$$X_t^{x_0} = x_0 + \sum_{i=1}^{d} \int_0^t V_i(X_s^{x_0}) dB_s^i,\ t \geq 0,$$

which is understood in Young's sense (see [Nualart and Rascanu (2002)] for theorems concerning the existence and the uniqueness for the solution of such an equation).

Observe that when $H \to \frac{1}{2}$, then the above development tends to

$$P_t = \mathbf{1} + \frac{1}{2}t\left(\sum_{i=1}^{d} X_i^2\right) + \frac{1}{8}t^2\left(\sum_{i=1}^{d} X_i^2\right)^2 + O(t^3),$$

which is the development of the P_t corresponding to the Brownian motion.

Also observe that the fourth order operator

$$\sum_{i,j,k,l=1}^{d} a_{i,j,k,l} X_i X_j X_k X_l$$

can not be simply expressed from

$$\sum_{i=1}^{d} X_i^2.$$

Such a discussion can obviously be generalized to any stochastic differential equation driven by Gaussian processes whose paths are more than $\frac{1}{2}$ locally Hölder continuous and this is actually an interesting open question to decide what is the smallest sub-algebra of $\mathbb{R}[[X_1, ..., X_d]]$ that contains P_t.

Let us finally mention another type of processes for which the expectation of the signature can be explicitly computed. Let us consider the process

$$Z_t = B_{\sigma t}, \quad t \geq 0,$$

where $(B_t)_{t \geq 0}$ is a d-dimensional standard Brownian motion and σ a non negative random variable independent of $(B_t)_{t \geq 0}$ which satisfies $\mathbb{E}\left(\sigma^k\right) < +\infty$, $k \geq 0$. In that case, the expectation of the signature of $(Z_t)_{t \geq 0}$ is easily seen to be given by

$$P_t = \sum_{k=0}^{+\infty} \frac{1}{2^k k!} \mathbb{E}\left(\sigma^k\right) t^k \left(\sum_{i=1}^{d} X_i^2\right)^k,$$

and observe that, like in the Brownian case, the smallest algebra containing P_t is given by $\mathbb{R}\left[\sum_{i=1}^{d} X_i^2\right]$. For instance, by taking for σ an exponential law with parameter 1, that is

$$\mathbb{P}(\sigma \in dx) = e^{-x} \mathbf{1}_{\mathbb{R}_{\geq 0}}(x),$$

we get

$$P_t = \frac{1}{1 - \frac{1}{2} t \left(\sum_{i=1}^{d} X_i^2\right)}.$$