Chapter 1

Introduction

Classical mechanics is one of the most familiar of scientific theories. Its basic concepts — mass, acceleration, force, and so on — have become very much a part of our everyday modes of thought. So we may easily regard their physical meaning as more obvious than it really is. For this reason, a large part of this introductory chapter will be devoted to a critical examination of the fundamental concepts and principles of mechanics.

Every scientific theory starts from a set of hypotheses, which are suggested by our observations, but represent an idealization of them. The theory is then tested by checking the predictions deduced from these hypotheses against experiment. When persistent discrepancies are found, we try to modify the hypotheses to restore the agreement with observation. If many such tests are made and no serious disagreement emerges, then the hypotheses gradually acquire the status of 'laws of nature'. When results that apparently contradict well-established laws appear, as they often do, we tend to look for other possible explanations — for simplifying assumptions we have made that may be wrong, or neglected effects that may be significant.

It must be remembered however that, no matter how impressive the evidence may be, we can never claim for these laws a universal validity. We may only be confident that they provide a good description of that class of phenomena for which their predictions have been adequately tested. One of the earliest examples is provided by Euclid's axioms. On any ordinary scale, they are unquestionably valid, but we are not entitled to assume that they should necessarily apply on either a cosmological or a sub-microscopic scale. Indeed, they have been modified in Einstein's theory of gravitation ('general relativity').

The laws of classical mechanics are no exception. Since they were first formulated by Galileo and by Newton in his *Principia*, their range of known validity has been enormously extended, but in two directions they have been found to be inadequate. For the description of the small-scale phenomena of atomic and nuclear physics, classical mechanics has been superseded by quantum mechanics, and for phenomena involving speeds approaching that of light, by relativity.

This is not to say that classical mechanics has lost its value. Indeed both quantum mechanics and the special and general theories of relativity are extensions of classical mechanics in the sense that they reproduce its results in appropriate limiting cases. Thus the fact that these theories have been confirmed actually reinforces our belief in the correctness of classical mechanics within its own vast range of validity. Indeed, it is a remarkably successful theory, which provides a coherent and satisfying account of phenomena as diverse as the planetary orbits, the tides and the motion of a gyroscope. Moreover, even outside this range, many of the results of classical mechanics still apply. In particular, the conservation laws of energy, momentum and angular momentum are, so far as we yet know, of universal validity.

1.1 Space and Time

The most fundamental assumptions of physics are probably those concerned with the concepts of space and time. We assume that space and time are continuous, that it is meaningful to say that an event occurred at a specific point in space and a specific instant of time, and that there are universal standards of length and time (in the sense that observers in different places and at different times can make meaningful comparisons of their measurements). These assumptions are common to the whole of physics, and, though all are being challenged, there is as yet no compelling evidence that we have reached the limits of their range of validity.

In ‘classical’ physics, we assume further that there is a universal time scale (in the sense that two observers who have synchronized their clocks will always agree about the time of any event), that the geometry of space is Euclidean, and that there is no limit in principle to the accuracy with which we can measure all positions and velocities. These assumptions have been somewhat modified in quantum mechanics and relativity. Here, however, we shall take them for granted, and concentrate our attention on the more specific assumptions of classical mechanics.

The relativity principle

In Aristotle's conception of the universe, the fact that heavy bodies fall downwards was explained by supposing that each element (earth, air, fire, water) has its own appointed sphere, to which it tends to return unless forcibly prevented from so doing. The element *earth*, in particular, tends to get as close as it can to the centre of the Universe, and therefore forms a sphere about this point. In this kind of explanation, the central point plays a special, distinguished role, and position in space has an absolute meaning.

In Newtonian mechanics, on the other hand, bodies fall downward because they are attracted towards the *Earth*, rather than towards some fixed point in space. Thus position has a meaning only relative to the Earth, or to some other body. In just the same way, velocity has only a relative significance. Given two bodies moving with constant relative velocity, it is impossible in principle to decide which of them is at rest, and which moving. This statement, which is of fundamental importance, is the *principle of relativity.*

Acceleration, however, still retains an absolute meaning, since it is experimentally possible to distinguish between motion with uniform velocity (*i.e.*, constant in magnitude and direction) and accelerated motion. If we are sitting inside an aircraft, we can easily detect its acceleration, but we cannot measure its velocity — though by looking out we can estimate its velocity *relative* to objects outside. (In Einstein's theory of general relativity, even acceleration becomes a relative concept, at least on a small scale. This is made possible by the fact that, to an observer in a confined region of space, the effects of being accelerated and of being in a gravitational field are indistinguishable.)

If two unaccelerated observers perform the same experiment, they must arrive at the same results. It makes no difference whether it is performed on the ground or in a smoothly travelling vehicle. However, an accelerated observer who performs the experiment may well get a different answer. The relativity principle asserts that all unaccelerated observers are equivalent; it says nothing about accelerated observers.

Inertial frames

It is useful at this point to introduce the concept of a frame of reference. To specify positions and time, each observer may choose a zero of the time scale, an origin in space, and a set of three Cartesian co-ordinate axes. We shall refer to these collectively as a *frame of reference.* The position and time of any event may then be specified with respect to this frame by the three Cartesian co-ordinates x, y, z and the time t. If we are located on a solid body, such as the Earth, we may, for example, choose some point of the body as the origin, and take the axes to be rigidly fixed to it (though, as we discuss later, this frame is not quite unaccelerated).

In view of the relativity principle, the frames of reference used by different unaccelerated observers are completely equivalent. The laws of physics expressed in terms of our x, y, z, t must be identical with those of someone else's x', y', z', t'. They are not, however, identical with the laws expressed in terms of the co-ordinates used by an accelerated observer. The frames used by unaccelerated observers are called *inertial* frames.

We have not yet said how we can tell whether a given observer is unaccelerated. We need a criterion to distinguish inertial frames from the others. Formally, an inertial frame may be defined to be one with respect to which any isolated body, far removed from all other matter, would move with uniform velocity. This is of course an idealized definition, since in practice we can never get infinitely far away from other matter. For all practical purposes, however, an inertial frame is one whose orientation is fixed relative to the 'fixed' stars, and in which the Sun (or more precisely the centre of mass of the solar system) moves with uniform velocity. It is an essential assumption of classical mechanics that such frames exist. Indeed, this assumption (together with a definition of inertial frames) is the real physical content of *Newton's first law* (a body acted on by no forces moves with uniform velocity in a straight line).

It is generally convenient to use only inertial frames, but there is no necessity to do so. Sometimes it proves convenient to use a non-inertial (in particular, rotating) frame, in which the laws of mechanics take on a more complicated form. For example, we shall discuss in Chapter 5 the use a frame rigidly fixed to the rotating Earth.

Vectors

It is often convenient to use a notation which does not refer explicitly to a particular set of co-ordinate axes. Instead of using Cartesian co-ordinates x, y, z we may specify the position of a point P with respect to a given origin O by the length and direction of the line OP. A quantity which is specified by a magnitude and a direction is called a *vector*; in this case the *position vector* $\boldsymbol{r}$ of P with respect to O. Many other physical quantities are also vectors: examples are velocity and force. They are to be distinguished from *scalars* — like mass and energy — which are completely specified by a magnitude alone.

We assume here that readers are familiar with the ideas of vector algebra; if not, they will find a discussion which includes all the results we shall need in Appendix A.

Throughout this book, vectors will be denoted by boldface letters (like $\boldsymbol{a}$). The magnitude of the vector will be denoted by the corresponding letter in ordinary italic type (a), or by the use of vertical bars ($|\boldsymbol{a}|$). The scalar and vector products of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ will be written $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{a} \wedge \boldsymbol{b}$ respectively. We shall use $\hat{\boldsymbol{r}}$ to denote the unit vector in the direction of $\boldsymbol{r}$, $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$. The unit vectors along the x-, y-, z-axes will be denoted by $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, so that

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}.$$

We shall use the vector notation in formulating the basic laws of mechanics, both because of the mathematical simplicity thereby attained, and because the physical ideas behind the mathematical formalism are often much clearer in terms of vectors.

1.2 Newton's Laws

Classical mechanics describes how physical objects move, how their positions change with time. Its basic laws may be applied to objects of any size (above the atomic level) and of any shape and internal structure, and, in classical hydrodynamics, to fluids too. It is not immediately obvious, however, what is meant by the 'position' of a large object of complex shape. Only in the idealized case of point particles (which do not exist in nature) does this concept have an intuitively obvious meaning. We shall therefore consider first only small bodies which can be effectively located at a point, so that the position of each, at time t, can be specified by a position vector $\boldsymbol{r}(t)$.

When we come to deal with large extended bodies, in Chapter 8, we shall make the additional assumption that any such body may be divided up into a large number of very small bodies, each of which may be treated as a point particle. (We shall also need to make some assumptions about the nature of the internal forces between these particles.) We shall then find that if we are interested in the overall motion of even a very large object, such as a planet, we may often legitimately treat it as a point particle located at the *centre of mass* of the body. The laws themselves prescribe the meaning of the 'position' of an extended body.

We shall begin by simply stating Newton's laws, and defer to the following section a discussion of the physical significance of the concepts involved, particularly those of *mass* and *force*.

Let us consider an isolated system comprising N bodies, which we label by an index $i = 1, 2, \dots, N$. By saying that the system is *isolated*, we mean that all other bodies are sufficiently remote to have a negligible influence on it. Each of the N bodies is assumed to be small enough to be treated as a point particle. The position of the ith body with respect to a given inertial frame will be denoted by $\boldsymbol{r}_i(t)$. Its velocity and acceleration are

$$\begin{aligned}\boldsymbol{v}_i(t) &= \dot{\boldsymbol{r}}_i(t),\\ \boldsymbol{a}_i(t) &= \dot{\boldsymbol{v}}_i(t) = \ddot{\boldsymbol{r}}_i(t),\end{aligned}$$

where the dots denote differentiation with respect to the time t. For example

$$\dot{\boldsymbol{r}} \equiv \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}.$$

Each body is characterized by a scalar constant, its *mass* m_i. Its momentum $\boldsymbol{p}_i$ is defined to be mass × velocity:

$$\boldsymbol{p}_i = m_i \boldsymbol{v}_i.$$

The equation of motion, which specifies how the body will move is *Newton's second law* (mass × acceleration = force):

$$\dot{\boldsymbol{p}}_i = m_i \boldsymbol{a}_i = \boldsymbol{F}_i, \tag{1.1}$$

where $\boldsymbol{F}_i$ is the total force acting on the body. This force is composed of a sum of forces due to each of the other bodies in the system. If we denote the force *on* the ith body *due to* the jth body by $\boldsymbol{F}_{ij}$, then

$$\boldsymbol{F}_i = \boldsymbol{F}_{i1} + \boldsymbol{F}_{i2} + \cdots + \boldsymbol{F}_{iN} = \sum_{j=1}^{N} \boldsymbol{F}_{ij}, \tag{1.2}$$

where of course $\boldsymbol{F}_{ii} = \boldsymbol{0}$, since there is no force on the ith body due to itself. Note that since the sum on the right side of (1.2) is a vector sum, this equation incorporates the 'parallelogram law' of composition of forces.

The two-body forces $\boldsymbol{F}_{ij}$ must satisfy *Newton's third law*, which asserts that 'action' and 'reaction' are equal and opposite,

$$\boldsymbol{F}_{ji} = -\boldsymbol{F}_{ij}. \tag{1.3}$$

Moreover, $\boldsymbol{F}_{ij}$ is a function of the positions and velocities (and internal structure) of the ith and jth bodies, but is unaffected by the presence of the other bodies. (It can be argued that this is an unnecessarily restrictive assumption. It would be perfectly possible to include also, say, three-body forces, which depend on the positions and velocities of three particles simultaneously. However, within the realm of validity of classical mechanics, no such forces are known, and their inclusion would be an inessential complication.) Because of the relativity principle, the force can in fact depend only on the *relative* position

$$\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$$

(see Fig. 1.1), and the *relative* velocity

$$\boldsymbol{v}_{ij} = \boldsymbol{v}_i - \boldsymbol{v}_j.$$

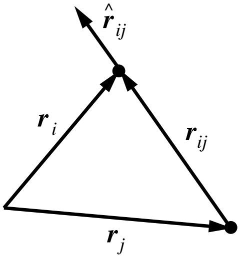

Fig. 1.1

If the forces are known, as functions of the positions and velocities, then from (1.1) we can predict the future motion of the bodies. Given their initial positions and velocities, we can solve these equations (analytically or numerically) to find their positions at a later time.

There is here an implicit assumption of perfect knowledge and infinite precision of calculation. It is now recognized (see Chapters 13, 14) that this assumption is, in general, false, leading to a loss of predictability. However, for the time being, we shall assume that our solution can be effected.

All that then remains is to specify the precise laws by which the two-body forces are to be determined. The most important class of forces consists of the *central, conservative* forces, which depend only on the relative positions of the two bodies, and have the form

$$\boldsymbol{F}_{ij} = \hat{\boldsymbol{r}}_{ij} f(r_{ij}), \tag{1.4}$$

where, as usual, $\hat{\boldsymbol{r}}_{ij}$ is the unit vector in the direction of $\boldsymbol{r}_{ij}$ and $f(r_{ij})$ is a scalar function of the relative distance r_{ij}. If $f(r_{ij})$ is positive the force $\boldsymbol{F}_{ij}$ is a *repulsive* force directed outwards along the line joining the bodies; if $f(r_{ij})$ is negative, it is an *attractive* force, directed inwards.

According to *Newton's law of universal gravitation*, there is a force of this type between *every* pair of bodies, proportional in magnitude to the product of their masses. It is given by (1.4) with

$$f(r_{ij}) = -\frac{G m_i m_j}{r_{ij}^2}, \tag{1.5}$$

where G is Newton's gravitational constant, whose value is

$$G = 6.673 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}.$$

Since the masses are always positive, this force is always attractive.

In addition, if the bodies are electrically charged, there is an electrostatic Coulomb force given by

$$f(r_{ij}) = \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}^2}, \tag{1.6}$$

where q_i and q_j are the electric charges, and ϵ_0 is another constant,

$$\epsilon_0 = 8.854\,19 \times 10^{-12}\,\mathrm{F\,m^{-1}}.$$

Note that the analogue of Newton's constant G is

$$1/4\pi\epsilon_0 = 8.987\,55 \times 10^{9}\,\mathrm{N\,m^2\,C^{-2}}.$$

Electric charges may be of either sign, and therefore the electrostatic force may be repulsive or attractive according to the relative sign of q_i and q_j.

Note the enormous difference in the orders of magnitude of the constants G and $1/4\pi\epsilon_0$ when expressed in SI units. This serves to illustrate the fact that gravitational forces are really exceptionally weak. They appear significant to us only because we happen to live close to a body of very large mass. Correspondingly large charges never appear, because positive and negative charges largely cancel out, leaving macroscopic bodies with a net charge close to zero.

In bodies with structure, central, conservative forces between their constituent parts can evidently give rise to forces which are still conservative (*i.e.*, which are independent of velocity, and satisfy some further conditions that need not worry us here — see §3.1 and §A.6), but no longer central (*i.e.*, not directed along the line joining the bodies). This can happen, for example, if there is a distribution of electric charge within each body.

They can also give rise, in a less obvious way, to non-conservative, velocity-dependent forces, as we shall see in Chapter 2. Many resistive and frictional forces can be understood as macroscopic effects of forces which are really conservative on a small scale. The chief distinguishing feature of conservative forces is the existence of a quantity which is *conserved*, *i.e.*, whose total value never changes, namely the *energy* of the system. Frictional forces have the effect of transferring some of this energy from the large-scale motion of the bodies to small-scale movements in their interior, and therefore appear non-conservative on a large scale.

In a sense, therefore, we may regard central, conservative forces as the norm. It would be wrong to conclude, however, that we can explain everything in terms of them. In the first place, the concepts of classical mechanics cannot be applied to the really small-scale structure of matter. For that, we need quantum mechanics. More serious is the existence of *electromagnetic* forces, which are of great importance even in the realm of classical physics, but which cannot readily be accommodated in the framework of classical mechanics. The force between two charges in relative motion is neither central nor conservative, and does not even satisfy Newton's third law (1.3). This is a consequence of the finite velocity of propagation of electromagnetic waves. The force on one charge depends not only on the instantaneous position of the other, but on its past history. The effect of a disturbance of one charge is not felt immediately by the other, but after an interval of time sufficient for a light signal to propagate from one to the other. This particular difficulty may be resolved by introducing the concept of the electromagnetic *field*. Then we may suppose that one charge does not act directly on the other, but on the field in its immediate vicinity; this in turn affects the field further out, and so on. By supposing that the field itself can carry energy and momentum, we can reinstate the conservation laws, which are among the most important consequences of Newton's laws.

However, this does not completely remove the difficulty, for there is still an apparent contradiction between this classical electromagnetic theory and the principle of relativity discussed in §1.1. This arises from the fact that if the speed of light is constant with respect to one inertial frame — as it should be according to electromagnetic theory — then the usual rules for combining velocities would lead to the conclusion that it is not constant with respect to a relatively moving frame, in contradiction with the statement that all inertial frames are equivalent. This paradox can only be resolved by the introduction of Einstein's theory of relativity (*i.e.*, 'special' relativity). Classical electromagnetic theory and classical mechanics *can* be incorporated into a single self-consistent theory, but only by ignoring the relativity principle and sticking to one 'preferred' inertial frame.

1.3 The Concepts of Mass and Force

It is an important general principle of physics (though not universally applied!) that no quantity should be introduced into the theory which cannot, at least in principle, be measured. Now, Newton's laws involve not only the concepts of velocity and acceleration, which can be measured by measuring distances and times, but also the new concepts of mass and force. To give the laws a physical meaning we have, therefore, to show that these are measurable quantities. This is not quite as trivial as it might seem, because any experiment designed to measure these quantities must necessarily involve Newton's laws themselves in its interpretation. Thus the operational definitions of mass and force — the prescriptions of how they may be measured — which are required to make the laws physically significant, are actually contained in the laws themselves. This is by no means an unusual or logically objectionable situation, but it may clarify the status of these concepts to reformulate the laws in such a way as to isolate their definitional element.

Let us consider first the measurement of mass. Since the units of mass are arbitrary, we have to specify a way of comparing the masses of two given bodies. It is important to realize that we are discussing here the *inertial* mass, which appears in Newton's second law, (1.1) and not the *gravitational* mass, which appears in (1.5). The two are of course proportional, but this *equivalence principle* is a physical law derived from experimental observation (in particular from Galileo's observations of falling bodies, from which he deduced that in a vacuum all bodies would fall equally fast) rather than an *a priori* assumption. To verify the law, we must be able to measure each kind of mass separately. This rules out, for example, the use of a balance, which compares gravitational masses.

Clearly, we can compare the inertial masses of two bodies by subjecting them to equal forces and comparing their accelerations, but this does not help unless we have some way of knowing that the forces *are* equal. However there is one case in which we *do* know this, because of Newton's third law. If we isolate the two bodies from all other matter, and compare their mutually induced accelerations, then according to (1.1) and (1.3),

$$m_1\boldsymbol{a}_1 = -m_2\boldsymbol{a}_2, \tag{1.7}$$

so that the accelerations are oppositely directed, and inversely proportional to the masses. If we allow two small bodies to collide, then during the collision the effects of more remote bodies are generally negligible in comparison with their effect on each other, and we may treat them approximately as an isolated system. (Such collisions will be discussed in detail in Chapters 2 and 7.) The mass ratio can then be determined from measurements of their velocities before and after the collision, by using (1.7) or its immediate consequence, the law of *conservation of momentum*,

$$m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2 = \text{constant}. \tag{1.8}$$

If we wish to separate the definition of mass from the physical content of equation (1.7), we may adopt as a fundamental axiom the following:

> In an isolated two-body system, the accelerations always satisfy the relation $\boldsymbol{a}_1 = -k_{21}\boldsymbol{a}_2$, where the scalar k_{21} is, for two given bodies, a constant independent of their positions, velocities and internal states.

If we choose the first body to be a standard body, and conventionally assign it unit mass (say $m_1 = 1$ kg), then we may *define* the mass of the second to be k_{21} in units of this standard mass (here $m_2 = k_{21}$ kg).

Note that for consistency, we must have $k_{12} = 1/k_{21}$. We must also assume of course that if we compare the masses of three bodies in this way, we obtain consistent results:

> For any three bodies, the constants k_{ij} satisfy $k_{31} = k_{32}k_{21}$.

It then follows that for *any* two bodies, k_{32} is the mass ratio: $k_{32} = m_3/m_2$.

To complete the list of fundamental axioms, we need one which deals with systems containing more than two bodies, analogous to the law of composition of forces, (1.2). This may be stated as follows:

> The acceleration induced in one body by another is some definite function of their positions, velocities and internal structure, and is unaffected by the presence of other bodies. In a many-body system, the acceleration of any given body is equal to the sum of the accelerations induced in it by each of the other bodies individually.

These laws, which appear in a rather unfamiliar form, are actually completely equivalent to Newton's laws, as stated in the previous section. In view of the apparently fundamental role played by the concept of force in Newtonian mechanics, it is remarkable that we have been able to reformulate the basic laws without mentioning this concept. It can of course be introduced, by defining it through Newton's second law, (1.1). The utility of this definition arises from the fact that forces satisfy Newton's third law, (1.3), while accelerations satisfy only the more complicated law, (1.7). Since the mutually induced accelerations of two given bodies are always proportional, they are essentially determined by a single function, and it is useful to introduce the more symmetric concept of force, for which this becomes obvious.

It is interesting to note, finally, that one consequence of our basic laws is the additive nature of mass. Let us take a three-body system. Then, returning to the notation of the previous section, the equations of motion for the three bodies are

$$\begin{aligned} m_1\boldsymbol{a}_1 &= \boldsymbol{F}_{12} + \boldsymbol{F}_{13}, \\ m_2\boldsymbol{a}_2 &= \boldsymbol{F}_{21} + \boldsymbol{F}_{23}, \\ m_3\boldsymbol{a}_3 &= \boldsymbol{F}_{31} + \boldsymbol{F}_{32}. \end{aligned} \tag{1.9}$$

If we add these equations, then, in view of (1.3), the terms on the right cancel in pairs, and we are left with

$$m_1\boldsymbol{a}_1 + m_2\boldsymbol{a}_2 + m_3\boldsymbol{a}_3 = \boldsymbol{0}, \tag{1.10}$$

which is the generalization of (1.7). Now, if we suppose that the force between the second and third is such that they are rigidly bound together to form a composite body, their accelerations must be equal: $\boldsymbol{a}_2 = \boldsymbol{a}_3$. In that case, we get

$$m_1\boldsymbol{a}_1 = -(m_2 + m_3)\boldsymbol{a}_2,$$

which shows that the mass of the composite body is just $m_{23} = m_2 + m_3$.

1.4 External Forces

To find the motion of the various bodies in any dynamical system, we have to solve two closely interrelated problems. First, given the positions and velocities at any one instant of time, we have to determine the forces acting on each body. Second, given the forces acting, we have to compute the new positions and velocities after a short interval of time has elapsed. In a general case, these two problems are inextricably bound up with each other, and must be solved simultaneously. If, however, we are concerned with the motions of a small body, or group of small bodies, then we can often neglect its effect on other bodies, and in that case the two problems can be separated.

For example, in discussing the motion of an artificial satellite, we can clearly ignore its effect on the Earth. Since the motion of the Earth is already known, we can calculate the force on the satellite as a function of its position and (if atmospheric resistance is included) its velocity. Then, taking the force as known, we can solve separately the problem of its motion. In the latter problem, we are really concerned with the satellite alone. The Earth enters simply as a known external influence.

In many cases, therefore, it is useful to concentrate our attention on a small part of a dynamical system, and to represent the effect of everything outside this by external forces, which we suppose to be known in advance, as functions of position, velocity and time. This is the kind of problem with which we shall be mainly concerned in the next few chapters. Typically, we shall consider the motion of a particle under a known external force. In Chapter 6, we consider, for the gravitational and electrostatic cases, the complementary problem of determining the force from a knowledge of the positions of other bodies. Later, in Chapter 7, we return to the more complex type of problem in which the system of immediate interest cannot be taken to be merely a single particle.

1.5 Summary

To some extent, the selection of a group of basic concepts, in terms of which others are to be defined, is a matter of choice. We have chosen to regard position and time (relative to some frame of reference) as basic. From this point of view, Newton's laws must be regarded as containing definitions in addition to physical laws. The first law contains the definition of an inertial frame, together with the physical assertion that such frames exist, while the second and third laws contain definitions of mass and force. These laws, supplemented by the laws of force, such as the law of universal gravitation, provide the equations from which we can determine the motion of any dynamical system.

Problems

Note. Here and in later chapters, starred problems are somewhat harder.

1. An object A moving with velocity $\boldsymbol{v}$ collides with a stationary object B. After the collision, A is moving with velocity $\frac{1}{2}\boldsymbol{v}$ and B with velocity $\frac{3}{2}\boldsymbol{v}$. Find the ratio of their masses. If, instead of bouncing apart, the two bodies stuck together after the collision, with what velocity would they then move?
2. The two components of a double star are observed to move in circles of radii r_1 and r_2. What is the ratio of their masses? (*Hint*: Write down their accelerations in terms of the angular velocity of rotation, ω.)
3. Consider a system of three particles, each of mass m, whose motion is described by (1.9). If particles 2 and 3, even though not rigidly bound together, are regarded as forming a composite body of mass $2m$ located at the mid-point $\boldsymbol{r} = \frac{1}{2}(\boldsymbol{r}_2+\boldsymbol{r}_3)$, find the equations describing the motion of the two-body system comprising particle 1 and the composite body (2+3). What is the force on the composite body due to particle 1? Show that the equations agree with (1.7). When the masses are *un*equal, what is the correct definition of the position of the composite $(2 + 3)$ that will make (1.7) still hold?
4. Find the distance r between two protons at which the electrostatic repulsion between them will equal the gravitational attraction of the Earth on one of them. (Proton charge $= 1.6 \times 10^{-19}$ C, proton mass $= 1.7 \times 10^{-27}$ kg.)
5. Consider a transformation to a relatively uniformly moving frame of reference, where each position vector $\boldsymbol{r}_i$ is replaced by $\boldsymbol{r}'_i = \boldsymbol{r}_i - \boldsymbol{v}t$. (Here $\boldsymbol{v}$ is a constant, the relative velocity of the two frames.) How does a relative position vector $\boldsymbol{r}_{ij}$ transform? How do momenta and forces transform? Show explicitly that if equations (1.1) to (1.4) hold in the original frame, then they also hold in the new one.
6. A body of mass 50 kg is suspended by two light, inextensible cables of

lengths 15 m and 20 m from rigid supports placed 25 m apart on the same level. Find the tensions in the cables. (Note that by convention 'light' means 'of negligible mass'. Take $g = 10\,\mathrm{m\,s^{-2}}$. This and the following two problems are applications of vector addition.)

7. *An aircraft is to fly to a destination 800 km due north of its starting point. Its airspeed is $800\,\mathrm{km\,h^{-1}}$. The wind is from the east at a speed of $30\,\mathrm{m\,s^{-1}}$. On what compass heading should the pilot fly? How long will the flight take? If the wind speed increases to $50\,\mathrm{m\,s^{-1}}$, and the wind backs to the north-east, but no allowance is made for this change, how far from its destination will the aircraft be at its expected arrival time, and in what direction?
8. *The two front legs of a tripod are each 1.4 m long, with feet 0.8 m apart. The third leg is 1.5 m long, and its foot is 1.5 m directly behind the mid-point of the line joining the other two. Find the height of the tripod, and the vectors representing the positions of its three feet relative to the top. (*Hint*: Choose a convenient origin and axes and write down the lengths of the legs in terms of the position vector of the top.) Given that the tripod carries a weight of mass 2 kg, find the forces in the legs, assuming they are purely compressional (*i.e.*, along the direction of the leg) and that the legs themselves have negligible weight. (Take $g = 10\,\mathrm{m\,s^{-2}}$.)
9. *Discuss the possibility of using force rather than mass as the basic quantity, taking for example a standard weight (at a given latitude) as the unit of force. How should one then define and measure the mass of a body?
10. The first estimate of Newton's constant was made by the astronomer Nevil Maskelyne in 1774 by measuring the angle between the directions of the apparent plumb-line vertical on opposite sides of the Scottish mountain Schiehallion (height 1081 m, chosen for its regular conical shape). Find a rough estimate of the angle through which a plumb line is deviated by the gravitational attraction of the mountain, by modelling the mountain as a sphere of radius 500 m and density $2.7 \times 10^3\,\mathrm{kg\,m^{-3}}$, and assuming that its gravitational effect is the same as though the total mass were concentrated at its centre. (This latter assumption will be justified, for a spherical object, in Chapter 6.)