we compute the Laplacian (log $\phi(z)$), i.e.,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)[\log 2-\log(1+x^2+y^2)]=-\left[\frac{2(1-x^2+y^2)}{(1+x^2+y^2)^2}+\frac{2(1+x^2-y^2)}{(1+x^2+y^2)^2}\right]$$

$$=\frac{-4}{(1+x^2+y^2)^2}$$

$$=-(\phi(z))^2$$

Hence $k = +1$.

4 THE TWO-DIMENSIONAL CONFORMAL FIELD THEORY

As the title suggests we shall give in this section a simplified version of conformal field theory—in the sense that we shall limit our discussions of the theory on the Minkowskian/Euclidean space of two dimensions. In later chapters (Chapters. 5 and 11), we shall see that conformal groups play a dominant role in string theory via the Virasoro operators (see Sec. 5.2) L_n–which happen to be the generators of this group.

4.1 Conformal Group

Given an n-dimensional Minkowski/Euclidean space M with coordinates x^μ ($\mu = 1, \cdots, n$), a *conformal transformation* is a diffeomorphism $x^\mu \to \tilde{x}^\mu$ such that the line (metric) element is preserved up to a scale factor, i.e.:

$$d\tilde{s}^2 = d\tilde{x}^\mu\, d\tilde{x}^\nu\, \tilde{\eta}_{\mu\nu} = \Omega(x) dx^\mu\, dx^\nu\, \eta_{\mu\nu} \tag{1.4.1)(a}$$

In the case of an infinitesimal transformation $x^\mu \to x^\mu + \in^\mu$, the infinitesimal distance ds^2 transforms as:

$$ds^2 \to ds^2 + (\partial_\mu \in_\nu + \partial_\nu \in_\mu)\, dx^\mu\, dx^\nu. \qquad \in_\mu = \in^\nu \eta_{\mu\nu} \tag{1.4.1)(b}$$

These two taken together lead to:

$$\partial_\mu \in_\nu + \partial_\nu \in_\mu = (2/n)\, \eta_{\mu\nu}\, \partial^\rho \in_\rho, \tag{1.4.1)(c}$$

The collection of all such transformations given in (1.4.1) is the *conformal group* of M. It is known that for all dimensions $n > 2$, it is finite-dimensional, whereas for $n = 2$ it is infinite-dimensional. However, it is in this case that models in statistical mechanics can be related to string theory since whatever be the dimension of the embedding space, the theory in effect described by the world-sheet swept by string is 2-dimensional. And this brings the two great theories—the quantum and the string—much closer and renders them more comprehensible (see the original work of Belavin Polyakov and Zamolodchikov [4] and the references there, and 7.[21]). Since this group is infinite-dimensional, it has an infinite number of generators as we shall see in Subsec 4.

To begin with, we consider a particular case of this group—the Lorentz group—the group of those transformations where the scale factor is ± 1. This group is abelian and thus has one generator and its irreducible representations are one-dimensional (see Chapter 2). Thus all representations of this group, such as tensors, can be decomposed into one-dimensional representations. To achieve this end, we shall use the light-cone formalism [20]:

4.2 Light-cone Formalism and the Lorentz Group

Define

$$z \equiv x^0 + x^1,\ \bar{z} \equiv x^0 - x^1 \tag{1.4.2}$$

where z and $\bar{z}$ are independent coordinates. The coordinate transformation inverse to (1.4.2) gives back the Minkowski coordinates (x_0, x_1):

$$x^0 = \frac{1}{2}(z + \bar{z}),\ x^1 = \frac{1}{2}(z - \bar{z}) \tag{1.4.3}$$

Under the Lorentz group with (single element) $\omega = \omega^{01}$ (see Remark (2.2.12)), the transformations for light-cone coordinates (resulting from $\delta x^0 = \omega x^1$, $\delta x^1 = \omega x^0$) are

$$\delta z = \omega z,\ \delta \bar{z} = -\omega \bar{z}. \tag{1.4.4}$$

The Minkowski line element in these coordinates becomes:

$$ds^2 = -(dx^0)^2 + (dx^1)^2 = -dz\, d\bar{z} \tag{1.4.5}$$

and written out with metric tensor this becomes:

$$ds^2 = g_{zz} dz dz + g_{z\bar{z}}\, dz\, d\bar{z} + g_{\bar{z}z}\, d\bar{z} dz + g_{\bar{z}\bar{z}}\, d\bar{z}\, d\bar{z}.^{21} \tag{1.4.6}$$

This eventually leads to the values of these metric tensor components:

$$g_{zz} = g_{\bar{z}\bar{z}} = 0,\ g_{z\bar{z}} = g_{\bar{z}z} = -\frac{1}{2} \tag{1.4.7}$$

in light-cone coordinates. Inverse metric can easily be verified as:

$$g^{zz} = g^{\bar{z}\bar{z}} = 0,\ g^{z\bar{z}} = g^{\bar{z}z} = -2. \tag{1.4.8}$$

An arbitrary contravariant tensor with components t^μ in (x^μ) system has components t^z and $t^{\bar{z}}$ given by the rule:

$$t^z = t^0 + t^1,\quad t^{\bar{z}} = t^0 - t^1 \tag{1.4.9(a)}$$

and a covariant tensor T_μ (in view of (1.4.3)) is given as

$$Tz = \frac{1}{2}(T_0 + T_1),\quad T_{\bar{z}} = \frac{1}{2}(T_0 - T_1) \tag{1.4.9(b)}$$

---

20. Light-cone coordinates (an accepted usage in literature) must not be confused with complex coordinates of previous sections.

21. $\overline{zz}$ in all expressions stands for $\bar{z}\,\bar{z}$.

The easiest examples of these tensors given by Eqs. (1.4.9)(a) and (1.4.9)(b) are

$$dz,\ d\bar{z} \text{ and } \frac{\partial}{\partial z},\ \frac{\partial}{\partial \bar{z}}$$

when written out in full they stand for:

$$dz = dx^0 + dx^1,\ d\bar{z} = dx^0 - dx^1 \tag{1.4.10)(a}$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}\right),\ \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1}\right) \tag{1.4.10)(b}$$

From (1.4.4), the following fact regarding the action of the Lorentz group on these tensors is immediate.

Fact 1.4.1: Each component of a tensor in the light-cone coordinates forms an irreducible tensor of the Lorentz group:

(a) $\delta t^z = \omega t^z,\quad \delta t^{\bar{z}} = -\omega t^{\bar{z}},$

(b) $\delta T_z = -\omega T_z,\quad \delta T_{\bar{z}} = \omega T_{\bar{z}}$ (1.4.11)

It is interesting to note that the scalar product in light-cone formalism is one of following

(a) $t \cdot T = t^\mu T_\mu = t^z T_z + t^{\bar{z}} T_{\bar{z}}$

(b) $= -2(t_{\bar{z}} T_z + t_z T_{\bar{z}})$ (1.4.12)

(c) $= -\frac{1}{2}(t^z T^{\bar{z}} + t^{\bar{z}} T^z).$

Equality (c) implies that sum of any tensor with two indices $z\bar{z}$, such as $T^{z\bar{z}} + T^{\bar{z}z}$, can be expressed as a divergence, i.e.,

$$T^{z\bar{z}} + T^{\bar{z}z} = -2\,T^\mu_\mu \tag{1.4.13}$$

Also, using the metric, one can express any tensor in terms of only upper and lower z ($\bar{z}$) indices. In conclusion we note that in light-cone formalism (1.4.1)(c) becomes:

$$\partial_0 \in_1 = -\,\partial_1 \in_0 \quad \text{and} \quad \partial_0 \in_0 + \partial_1 \in_1 = 0$$

4.3 Euclidean Space Formalism

We next show an analogue of the light-cone coordinates in Euclidean space. This is desirable for two reasons, namely (1) working in a space with positive definite metric allows one to have an access to the mathematical theory of Riemann surfaces; (2) conformal field theories associated with statistical models are in Euclidean space (in string theory they are in Minkowski space). The change to Euclidean space from Minkowskian space is achieved via the Wick's rotation principle:

$$x^0 \to -ix^0,\quad x^1 \to x^1 \tag{1.4.14}$$

Denoting the Euclidean coordinates as x^1, x^2 in place of Minkowskian x^0, x^1, the line element (1.4.5) becomes

$$ds^2 = (dx^1)^2 + (dx^2)^2 \tag{1.4.15}$$

The coordinates z and $\bar{z}$ in this format are chosen as

$$z \equiv x^1 + ix^2, \quad \bar{z} = x^1 - ix^2.$$

The inverse coordinate transformation is evidently

$$x^1 = \frac{1}{2}(z + \bar{z}), \quad x^2 = \frac{i}{2}(z - \bar{z}).$$

The coordinates $(z, \bar{z})$ are our familiar complex coordinates. The line element (1.4.15) now becomes:

$$ds^2 = dz\, d\bar{z} \tag{1.4.16}$$

and the metric tensor is given by:

(a) $$g_{zz} = g_{\bar{z}\bar{z}} = 0, \quad g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2},$$

(b) $$g^{zz} = g^{\bar{z}\bar{z}} = 0, \quad g^{z\bar{z}} = g^{\bar{z}z} = 2 \tag{1.4.17}$$

The group of transformations in the case of the Euclidean space is $SO(2)$, with the following rule of *Euclidean* rotation:

$$x^1 \to x^1 \cos\omega + x^2 \sin\omega$$
$$x^2 \to x^2 \cos\omega - x^1 \sin\omega \tag{1.4.18}$$

These transformations lead to:

$$z \to e^{-i\omega} z, \; \bar{z} \to e^{i\omega} \bar{z} \tag{1.4.19}$$

Given a tensor with components t^{μ} and T_{μ}, their counterpart in z, $\bar{z}$ coordinates is given by:

(a) $$t^z = t^1 + it^2, \quad t^{\bar{z}} = t^1 - it^2,$$

(b) $$T_z = \frac{1}{2}(T_1 - iT_2), \quad T_{\bar{z}} = \frac{1}{2}(T_1 + iT_2) \tag{1.4.20}$$

From these equations it is evident that all the conditions that are satisfied in light-cone formalism can be shown to hold good in this complex coordinate formalism by appropriate introduction of $\pm i$. We shall need this fact while studying Majorana and Weyl spinors (Chapter 7).

We also observe that Eq. (1.4.1)(c) in the Euclidean case gives:

$$\partial_1 \epsilon_2 + \partial_2 \epsilon_1 = 0 \quad \text{and} \quad \partial_1 \epsilon_1 - \partial_2 \epsilon_2 = 0 \tag{1.4.20)(a}$$

4.4 Two-dimensional Conformal Group

We now return (although briefly) to the conformal transformation group in two dimensions. We first note that since the line element $ds^2 = -dzd\bar{z}$ is preserved up to a scale factor, this implies that we can find a smooth function $e^{\psi(z, \bar{z})}$ such that

$$ds^2 = -e^{\psi(z, \bar{z})}\, dz\, d\bar{z} \tag{1.4.21}$$

Accordingly the metric in $(z, \bar{z})$ coordinates is given by:

$$\text{(a)} \qquad g_{zz} = g_{\bar{z}\bar{z}} = 0, \; g_{z\bar{z}} = g_{\bar{z}z} = -\frac{1}{2} e^{\psi(z, \bar{z})},$$

$$\text{(b)} \qquad g^{zz} = g^{\bar{z}\bar{z}} = 0, \; g^{\bar{z}z} = g^{z\bar{z}} = -2e^{-\psi(z, \bar{z})} \qquad (1.4.22)$$

We shall soon use these to discuss the conformal tensor calculus, but first we establish the claim (made in the introduction of this section) that the conformal group for 2-dimensional spaces has infinitely many generators. However, the good part is that the Lie algebra formed by them is of great value in physical theories.

Note that the transformations of the type:

$$\text{(a)} \qquad z \to f(z) \quad \bar{z} \to g(\bar{z})$$

$$\text{(b)} \qquad z \to h(\bar{z}) \quad \bar{z} \to k(z) \qquad (1.4.23)$$

with f, g, h, k as smooth functions will preserve (1.4.21), in the case of (a) where for instance $e^{\psi(z, \bar{z})}$ will be $f'(z)g'(\bar{z})\, e^{\psi(z, \bar{z})}$. The second transformation, though, will change the orientation in view of (1.4.10)(b).

To avoid complications of orientation change, we stick to the transformations Eq. (1.4.23)(a), and consider infinitesimal transformations[22]:

$$z \to z + \sum_n a_n z^{n+1}, \; \bar{z} \to \bar{z} + \sum_n \bar{a}_n \bar{z}^{n+1} \; n \in \mathbf{Z} \qquad (1.4.24)$$

In view of (1.4.13)(a) and (1.4.20)(a) it follows, that these transformations are generated by:

$$z^{n+1} \frac{\partial}{\partial z} \equiv L_n, \; \bar{z}^{n+1} \frac{\partial}{\partial \bar{z}} \equiv \bar{L}_n \; n \in \mathbf{Z}. \qquad (1.4.25)$$

Obviously the Lie algebra formed by them satisfies:

$$\text{(a)} \qquad [L_n, L_m] = -(n-m)\, L_{n+m},$$

$$\text{(b)} \qquad [L_n, \bar{L}_m] = 0,$$

$$\text{(c)} \qquad [\bar{L}_n, \bar{L}_m] = -(n-m)\, \bar{L}_{n+m} \qquad (1.4.26)$$

We shall return to these generators (operators) in Chapter 5 and of course shall use them in Chapter 11 (see in particular Sec. 11.6).

4.5 Möbius Transformation

In the previous section we have already come across Möbius transformations. We give below an important result on these transformations—which will also serve as an example to the theory discussed above.

22. In writing $\bar{a}_n$ in the second correspondence, we have followed the usual practice in literature. This does not mean that $\bar{a}_n$ is the complex conjugate of a_n. The common feature they share is that they are both infinitesimals independent of z and $\bar{z}$ respectively ($\bar{a}_n$ of z and a_n of $\bar{z}$).

Result 1.4.2: The most general transformation that maps the Riemann sphere onto itself is the Möbius transformation:

$$z \to z' = \frac{az+b}{cz+d}, \qquad \bar{z} \to \bar{z}' = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}} \tag{1.4.27}$$

where a, b, c, d are complex parameters satisfying the equality:

$$ad - bc = 1.$$

The mapping is one-one. The conformal group also known as Möbius group is a six real-parameter group (see Remark (2.2.12)).

Infinitesimal Möbius transformations can be written as[23]:

$$\begin{aligned} z' &= z + a_{-1} + a_0 z + a_1 z^2 \\ \bar{z}' &= \bar{z} + \bar{a}_{-1} + \bar{a}_0 \bar{z} + \bar{a}_1 \bar{z}^2 \end{aligned} \tag{1.4.28}$$

and in view of (1.4.25), they are generated by L_0, $L_{\pm 1}$, and $\bar{L}_0$, $\bar{L}_{\pm 1}$.

4.6 Conformal Tensor Calculus

Let the coordinate changes be written as:

$$z \to w(z), \quad \bar{z} \to \bar{w}(\bar{z})$$

then a general tensor $T^{i,\bar{j}}_{k,\bar{l}} \equiv T(z, \bar{z})$ will correspond to a tensor $T'(w, \bar{w})$ in $(w, \bar{w})$ coordinates, with transformation rule:

$$T(z, \bar{z}) \to T'(w, \bar{w}) = \left(\frac{dw}{dz}\right)^{i-k} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{\bar{j}-\bar{l}} T(z, \bar{z}) \tag{1.4.29}$$

The numerical quantities $(i - k) \equiv h$, $(\bar{j} - \bar{l}) \equiv \bar{h}$ are called the *conformal weights* or *dimensions* of tensor T. Similar to other metric theories, we can raise and lower the covariant and contravariant indices by using the appropriate metric tensor, thus for instance:

$$T_z = g_{z\bar{z}}\, T^{\bar{z}} = -\frac{1}{2}\, e^{\psi(z,\bar{z})}\, T^{\bar{z}} \text{ and } T_{\bar{z}} = -\frac{1}{2}\, e^{\psi(z,\bar{z})}\, T^{z}. \tag{1.4.30}$$

We now state some important results that relate our discussions to familiar pictures in physical theories.

Result 1.4.3: A time translation $x^0 \to x^0 + c$, where c is real, is induced by $z \to z + c$, $\bar{z} \to \bar{z} + c$ and so is generated by the sum of generators $(L_{-1} + \bar{L}_{-1})$ in view of (1.4.25). The generator $(L_{-1} + \bar{L}_{-1})$ is the Hamiltonian.

Result 1.4.4: A space shift $x^1 \to x^1 + \lambda$ is induced by $z \to z + \lambda$, $\bar{z} \to \bar{z} - \lambda$ and is generated by $L_{-1} - \bar{L}_{-1}$, which is the total momentum.

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23. Note that while coefficients a_n, $\bar{a}_n$ in (1.4.24) can stand for light-cone coordinates or for their analogue in Euclidean space, in this case they are in second formalism (see the paragraph below (1.4.13)).

Result 1.4.5: The rotations $\delta x^0 = -\phi x^1$, $\delta x^1 = +\ \phi x^0$ are the consequence of the transformations $z \to e^{i\phi}z$, $\bar{z} \to e^{-i\phi}\bar{z}$. The resulting generator $L_0 - \bar{L}_0$ is the angular momentum generator. This is nothing but the generator of the 2-dimensional Poincaré group (see Exercise 4.1.7). Finally, the dilation $z \to \lambda z$, $\bar{z} \to \lambda\bar{z}$ for λ real is generated by $L_0 + \bar{L}_0$.

It can be checked that under a dilation a tensor $T \to \lambda^{(h+\bar{h})}\,T$, while under a rotation $T \to e^{i(h-\bar{h})\phi}T$. The sum $(h+\bar{h})$ and the difference $(h-\bar{h})$ is respectively called the dialation weight of T and the spin of T. Finally, we list a few results dealing with conformally invariant two-dimensional theories. These will be pursued in detail in later chapters (Chapters 6 and 11).

4.7 Conserved Currents

Result 1.4.6: If the theory is Poincaré-invariant, then there exists an energy momentum tensor $T_{\alpha\beta}$ (which can always be chosen to be symmetric)—which is a conserved current, i.e.,[24]

$$\partial^\alpha T_{\alpha\beta} = 0 \tag{1.4.31}$$

and whose charge generates translations (see Sec. 6.3).

The current corresponding to Lorentz rotations is a moment of energy-momentum tensor:

$$x_\alpha T_{\beta\delta} - x_\beta T_{\alpha\delta} \tag{1.4.32}$$

which is conserved on its δ index due to the symmetry of $T_{\alpha\beta}$ and due to its conservation.

Result 1.4.7: If the theory is dilation-invariant, i.e., it is invariant under $x^\alpha \to \lambda x^\alpha$, and the associate current j_β is given by a moment of the energy momentum tensor:

$$j_\beta = x^\alpha T_{\alpha\beta} \tag{1.4.33}$$

then j_β is conserved provided $T^\alpha_\alpha = 0$

The above relation leads to constructions of further conserved currents, for instance, define an arbitrary current

$$f^\alpha(x) T_{\alpha\beta} \tag{1.4.34}$$

and demand that

$$\partial^\beta f^\alpha + \partial^\alpha f^\beta - \phi\eta^{\alpha\beta} = 0 \tag{1.4.35}$$

where ϕ is an arbitrary function of x and $\eta^{\alpha\beta}$ is the metric tensor, then it can be checked that $f^\alpha(x)\,T_{\alpha\beta}$ is conserved. A simple example of (1.4.34) is given by

$$x_\alpha x^\delta T_{\delta\beta} - x^2 T_{\alpha\beta} \tag{1.4.36}$$

This generates the special translations of the conformal group. Moreover, these additional conserved currents define corresponding generators and together with Poincaré and dilation generators, they have the conformal group as their algebra.

Returning to $(z, \bar{z})$-coordinates in a two-dimensional Euclidean invariant theory, the above statements lead to the following realizations: If the energy momentum tensor is symmetric and traceless, it has only two components, say T_{00} and T_{01}. The traceless condition in coordinates $(z, \bar{z})$ implies:

$$T_{z\bar{z}} = 0 \tag{1.4.37}$$

24. Equations (1.4.31), etc., are in arbitrary dimension.

thus there are only two components, T_{zz} and $T_{\bar{z}\bar{z}}$. The conservation condition leads to

$$\partial_{\bar{z}} T_{zz} = 0, \quad \partial_z T_{\bar{z}\bar{z}} = 0 \tag{1.4.38}$$

showing that T_{zz} and $T_{\bar{z}\bar{z}}$ are functions of only z and $\bar{z}$, respectively.

With the help of the above discussions, we can now define an infinite set of conserved currents, for instance, consider any two functions $f(z)$ and $g(\bar{z})$ and form a new pair:

$$f(z)T_{zz}, \qquad g(\bar{z})\, T_{\bar{z}\bar{z}} \tag{1.4.39}$$

which evidently satisfies $\partial_{\bar{z}}\, (f(z)T_{zz}) = 0$ and $\partial_z (g(\bar{z})\, T_{\bar{z}\bar{z}} = 0$. The corresponding generators are:

$$L_n = \frac{1}{2\pi i} \oint \frac{dz}{z}\, z^{n+2}\, T_{zz}$$

$$\bar{L}_n = \frac{1}{2\pi i} \oint \frac{d\bar{z}}{z}\, \bar{z}^{n+2}\, T_{\bar{z}\bar{z}} \tag{1.4.40}$$

This shows that a theory for which $T_{z\bar{z}} = 0$, carries an infinite-dimensional conformal group. As mentioned above, we shall return to these generators in later chapters.

Exercise 1.4

1. Find the light-cone components of tensors of type (2, 0) and (0, 2), i.e., components of contravariant and covariant tensors of degree 2, and then write its generalized version for a mixed tensor of type (r, s).
2. Obtain the Lorentz group transformation for tensors of type (2, 0) and (0, 2) in light-cone formalism and then write its generalized version.
3. Show that for the line element (1.4.21) the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ in z, $\bar{z}$ coordinates satisfy:

   $$\Gamma^{z}_{zz} = \frac{\partial}{\partial z}\psi, \quad \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = \frac{\partial}{\partial \bar{z}}\psi$$

   and the remaining ones are zero.
4. Establish the results (1.4.6) and (1.4.7).
5. Show that for the line element (1.4.21), the current defined by (1.4.36) is a conserved current that generates special translations of the conformal group.
6. Establish the generators of 2-dimensional conformal field theory as given in (1.4.40).
7. Show that for a free spin-zero 2-dimensional field theory, the energy momentum tensor components are:

   $$T_{z\bar{z}} = 0, \quad T_{zz} = \partial_z \psi \partial_z \psi, \quad T_{\bar{z}\bar{z}} = \partial_{\bar{z}} \psi \partial_{\bar{z}} \psi.$$

Hints to Exercise 1.4

1. The components of a contravariant tensor of degree 2 are obtained by writing the elements of tensor product of contravariant vector spaces. Thus denoting the tensor by s, we have

(a)
$$\begin{aligned} s^{z\bar{z}} &= t^z \otimes t^{\bar{z}} \\ &= (t^0 + t^1) \otimes (t^0 - t^1) \\ &= t^0 \otimes t^0 - t^0 \otimes t^1 + t^1 \otimes t^0 - t^1 \otimes t^1 \\ &= t^{00} - t^{01} + t^{10} - t^{11}. \end{aligned}$$

Similarly

$$s^{zz} = t^{00} + t^{01} + t^{10} + t^{11}$$

and $s^{\bar{z}z}$ and $s^{\overline{zz}}$ are respectively:

$$t^{00} - t^{10} + t^{01} - t^{11} \text{ and } t^{00} - t^{01} - t^{10} + t^{11}.$$

In the case of covariant tensor, we have:

(b)
$$\begin{aligned} S_{z\bar{z}} &= \frac{1}{2}(T_0 + T_1) \otimes \frac{1}{2}(T_0 - T_1) \\ &= \frac{1}{4}(T_{00} - T_{01} + T_{10} - T_{11}). \end{aligned}$$

And the other three are:

(c)
$$S_{zz} = \frac{1}{4}(T_{00} + T_{01} + T_{10} + T_{11})$$

$$S_{\bar{z}z} = \frac{1}{4}(T_{00} - T_{10} + T_{01} - T_{11})$$

$$S_{\overline{zz}} = \frac{1}{4}(T_{00} - T_{01} - T_{10} + T_{11}).$$

We denote the general tensor of type (r, s) by T and note that its components could look like:

(d)
$$T^{\overbrace{z\cdots z}^{i}\,\overbrace{\bar{z}\cdots\bar{z}}^{j}}_{\underbrace{z\cdots z}_{k}\,\underbrace{\bar{z}\cdots\bar{z}}_{l}} = T^{i\bar{j}}_{k\bar{l}} \equiv T(z, \bar{z})$$

where (i, j) and (k, l) are some partitions of r and s respectively, and z, $\bar{z}$ have been suppressed in $T^{i\bar{j}}_{k\bar{l}}$. (This is also denoted $T^{m\bar{m}}_{n\bar{n}}$ or $T(z, \bar{z})$ in literature).

2. Note that $\delta(t^z \otimes t^z) = \delta t^z \otimes t^z + t^z \otimes \delta t^z$. Similarly, $\delta(t^{\bar{z}} \otimes t^{\bar{z}}) = \delta t^{\bar{z}} \otimes t^{\bar{z}} + t^{\bar{z}} \otimes \delta t^{\bar{z}}$. Writing $t^z \otimes t^z = T^{zz}$ and $t^{\bar{z}} \otimes t^{\bar{z}} = T^{\bar{z}\bar{z}}$ and using (1.4.11)(a), we obtain:

(a)
$$\delta T^{zz} = 2\omega T^{zz} \text{ and } \delta T^{\overline{zz}} = -2\omega T^{\overline{zz}}$$

In the case of the combinations $t^{z} \otimes t^{\bar{z}}$ and $t^{\bar{z}} \otimes t^{z}$, it can be easily checked that $\delta(t^{z} \otimes t^{\bar{z}})$ and $\delta(t^{\bar{z}} \otimes t^{z})$ are zero. Using the same procedure to write $\delta(T_z \otimes T_z)$ and $\delta(T_{\bar{z}} \otimes T_{\bar{z}})$ with the help of (1.4.11)(b), we have:

(b) $$\delta S_{zz} = -2\omega\, S_{zz} \text{ and } \delta\, S_{\bar{z}\bar{z}} = 2\omega\, S_{\bar{z}\bar{z}}$$

and $\delta\, S_{z\bar{z}} = \delta S_{\bar{z}z} = 0$. In view of these computations it follows that for a general tensor, the Lorentz transformation rule would be:

(c) $$\delta T^{i\bar{j}}_{k\bar{l}} = \omega\, [(i - k) - (\bar{j} - \bar{l})]\, T^{i\bar{j}}_{k\bar{l}}.$$

3. Recall that

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\, g^{\alpha\delta}\, (\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma}).$$

Write α, β, γ as z and δ as z, as well as $\bar{z}$ for summation, then since $g^{zz} = 0 = g_{zz}$, it becomes

$$\Gamma^{z}_{zz} = \frac{1}{2}\, g^{z\bar{z}}\, (\partial_z\, g_{z\bar{z}} + \partial_z\, g_{z\bar{z}}) = e^{-\psi}\, (\partial_z\, e^{\psi}) = \partial_z \psi.$$

Similarly for $\Gamma^{\bar{z}}_{\bar{z}\bar{z}}$, we have $\partial_{\bar{z}}\, \psi$. The mixed component

$$\Gamma^{\bar{z}}_{zz} = \frac{1}{2} g^{\bar{z}z}\, (\partial_z\, g_{zz} + \partial_z g_{zz} - \partial_z\, g_{zz})$$

is evidently zero.

(A good source for the remaining four exercises is Ref. 7.[21].)

References

1. L. V. Ahlfors, *Complex Analysis* (New York: McGraw-Hill, 1979).
2. L. V. Ahlfors and L. Sario, *Riemann Surfaces* (Princeton University Press, 1960).
3. A. F. Beardon, *A Primer on Riemann Surfaces* (New York: Cambridge University Press, 1984).
4. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Infinite Conformal Symmetry in Two-dimensional Quantum Field Theory, *Nucl. Phys.* **241**, 333–380 (1984).
5. R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology* (Springer-Verlag, 1982).
6. P. Buser, *Geometry and Spectra of Compact Riemann Surfaces* (Birkhäuser, 1992).
7. R. V. Churchill and J. Brown, *Complex Variables and Applications* (McGraw-Hill, 1990).
8. H. M. Farkas and I. Kra, *Riemann Surfaces* (New York: Springer-Verlag, 1991).
9. O. Forster, *Lectures on Riemann Surfaces* (Springer-Verlag, 1981).
10. S. Kabayashi and K. Nomizu, *Foundations of Differential Geometry* (Interscience Publishers, 1969).
11. H. Weyl, *The Concept of a Riemann Surface* (3rd ed., Reading, MA: Addison-Wesley, 1964).