$$w = \lambda i \frac{1-z}{1+z}$$

and that when $\lambda = \frac{i}{2}$ in (1.1.16), it becomes the Möbius transformation (see Sec. 3).

Example 1.1.3: The line integral $(dx^2 + dy^2)$ in complex coordinates can be expressed as a product $dz d\bar{z}$. This is immediate from the definition $z = x + iy$ which implies $dz = dx + idy$ and $d\bar{z} = dx - idy$.

1.7 Elliptic Curves

Let $\Gamma = \Gamma_1 + i\Gamma_2$ be an element of $\mathbb{C}$ with $\Gamma_2 > 0$. The set of points defined by:

$$\mathbb{C}/(2\pi\mathbb{Z} + 2\pi\Gamma\mathbb{Z}) \qquad (1.1.19)$$

is called an elliptic curve (denoted E_Γ) associated to the complex number Γ. Given below is a simple diagram pertaining to the area enclosed by E_Γ which can easily be seen to be $4\pi^2\Gamma_2$ (see p. 15 in 5.[14] for its use in string path integrals).

The area of the parallelogram $OACB$ is evidently the same as that of $B'C'CB$ with arm lengths 2π and $2\pi\Gamma_2$, respectively.

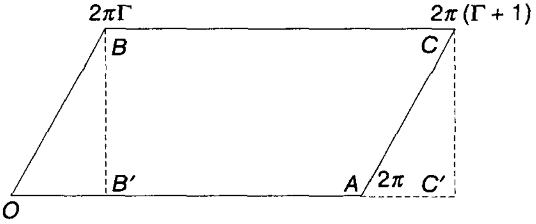

Fig. 1.4

2 COMPLEX STRUCTURES ON A MANIFOLD, KÄHLER METRIC

2.1 Complex Manifold *M*

Definition 1.2.1: In layman's language, an n-dimensional complex manifold is a real manifold of dimension $2n$ if there can (always) be found complex coordinates on it with holomorphic (i.e., analytic on real manifold) transition functions. More precisely, let M be a complex manifold of (complex) dimension n and z^λ, $(\lambda = 1, 2, \cdots, n)$ a system of complex local coordinates on an open subset U of M. Let $z^\lambda = x^\lambda + iy^\lambda$; then $(x^1, y^1, \cdots, x^n, y^n)$ is a system of (real) local coordinates of the differentiable manifold M on U.

2.2 Complex Structure on M

For each $\mathbf{x} \in U$ we define a linear transformation $J_\mathbf{x}$ of the tangent space $T_\mathbf{x}(M)$ that transforms the pair

$$\left(\frac{\partial}{\partial x^\lambda}, \frac{\partial}{\partial y^\lambda}\right)$$

in following manner:

$$J_{\mathbf{x}}\left(\frac{\partial}{\partial x^{\lambda}}\right)=\frac{\partial}{\partial y^{\lambda}},\ J_{\mathbf{x}}\left(\frac{\partial}{\partial y^{\lambda}}\right)=-\frac{\partial}{\partial x^{\lambda}}. \tag{1.2.1}$$

The linear transformation $J_{\mathbf{x}}$ satisfies the condition:

$$J_{\mathbf{x}}^{2}=-1 \tag{1.2.2}$$

and the assignment $J:\mathbf{x}\rightarrow J_{\mathbf{x}}$ defines a tensor field of type (1,1) on the differentiable manifold M.[1] The tensor J is called a *complex structure* of M.

2.3 The Tangent and Cotangent Spaces to *M*

Let $T_{\mathbf{x}}(M)^{c}$ and $T_{\mathbf{x}}^{*}(M)^{c}$ be the complexifications of the tangent and the cotangent vector spaces respectively; then the elements

$$\frac{\partial}{\partial z^{\lambda}},\ \frac{\partial}{\partial \bar{z}^{\lambda}} \text{ and } dz^{\lambda},\ d\bar{z}^{\lambda}$$

belonging to $T_{\mathbf{x}}(M)^{c}$ and $T_{\mathbf{x}}^{*}(M)^{c}$ (respectively) are given by:

$$\text{(a)}\quad \frac{\partial}{\partial z^{\lambda}}=\frac{1}{2}\left(\frac{\partial}{\partial x^{\lambda}}-i\frac{\partial}{\partial y^{\lambda}}\right),\ \frac{\partial}{\partial \bar{z}^{\lambda}}=\frac{1}{2}\left(\frac{\partial}{\partial x^{\lambda}}+i\frac{\partial}{\partial y^{\lambda}}\right)$$

$$\text{(b)}\quad dz^{\lambda}=dx^{\lambda}+idy^{\lambda},\ d\bar{z}^{\lambda}=dx^{\lambda}-idy^{\lambda}. \tag{1.2.3}$$

Thus the endomorphism $J_{\mathbf{x}}$ of the vector space $T_{\mathbf{x}}(M)$ defines the endomorphism of $T_{\mathbf{x}}(M)^{c}$ and we have:

$$T_{\mathbf{x}}(M)^{c}=T_{\mathbf{x}}^{+}(M)\oplus T_{\mathbf{x}}^{-}(M),\ \overline{T_{\mathbf{x}}^{+}}=T_{\mathbf{x}}^{-}. \tag{1.2.4}$$

where $T_{\mathbf{x}}^{+}$ (respectively $T_{\mathbf{x}}^{-}$) consists of all $v\in T_{\mathbf{x}}(M)^{c}$ such that $J_{\mathbf{x}}v=iv$ (respectively $J_{\mathbf{x}}v=-iv$) and bar on $T_{\mathbf{x}}$ denotes the conjugation of $T_{\mathbf{x}}(M)^{c}$.

The elements

$$\left\{\frac{\partial}{\partial z^{\lambda}}\right\},\left\{\frac{\partial}{\partial \bar{z}^{\lambda}}\right\}\ (\lambda=1,2,\cdots,n)$$

form the bases of $T_{\mathbf{x}}^{+}$ and $T_{\mathbf{x}}^{-}$ respectively at each point $\mathbf{x}$ of the coordinate neighbourhood U.

2.4 Holomorphic Vector Fields and Holomorphic Forms on *M*

Definition 1.2.2: A complex vector field X on M is a map on M such that for every $\mathbf{x}\in M$ there is an element $X_{\mathbf{x}}\in T_{\mathbf{x}}(M)^{c}$. On a coordinate neighbourhood U, X can be uniquely expressed as:

$$X=\xi^{\lambda}\frac{\partial}{\partial z^{\lambda}}+\xi^{\bar{\lambda}}\frac{\partial}{\partial \bar{z}^{\lambda}} \tag{1.2.5}$$

---

[1] See the definition in Chapter 0.

where ξ^{λ} and $\xi^{\bar{\lambda}}$ are complex valued smooth functions on U. X is a real vector field (i.e., $X_x \in T_x(M)$) if and only if $\overline{\xi^{\lambda}} = \xi^{\bar{\lambda}}$ for $\lambda = 1, 2, \cdots, n$.

For any complex vector field X, let $\bar{X}$ be the vector field such that $(\bar{X})_{\mathrm{x}} = \bar{X}_{\mathrm{x}}$ at each x in M, then X is real if and only if $\bar{X} = X$. X is said to be of type (1, 0) (respectively of type (0, 1)) if $X_{\mathrm{x}} \in T_{\mathrm{x}}^{+}$ (respectively T_{x}^{-}) at each point $\mathbf{x}$. When X is of type (1, 0) we can write it locally as:

$$X = \xi^{\lambda}\frac{\partial}{\partial z^{\lambda}}.$$

If the components ξ^{λ} are holomorphic functions of the local coordinates (z^{λ}), then X is called a *holomorphic vector field on M.*

Definition 1.2.3: A differential r-form α is a map defined on M such that for every $\mathbf{x} \in M$, $\alpha(\mathbf{x})$ is an alternating r-linear function on $T_{\mathrm{x}}(M)^{c}$. We denote it as α_{x}.

In view of decomposition (1.2.4) we can think of it as an element of:

$$\sum_{p+q=r} \Lambda^{p}(T_{\mathrm{x}}^{+})^{*} \otimes \Lambda^{q}(T_{\mathrm{x}}^{-})^{*} \tag{1.2.6}$$

where Λ^{p}, (respectively Λ^{q}) are p-th power (q-th power) of exterior product on cotangent space $(T_{\mathrm{x}}^{+})^{*}$ (respectively $(T_{\mathrm{x}}^{-})^{*}$). Since the bases for these spaces are $\{dz^{\lambda}\}$ and $\{d\bar{z}^{\lambda}\}$, we can write:

$$\alpha = \sum_{p+q=r} \alpha_{p,q} \tag{1.2.7}$$

with $\alpha_{p,q}$ expressed as:

$$\alpha_{p,q} = \sum_{\substack{\lambda_1<\lambda_2\cdots<\lambda_p \\ \mu_1<<\mu_2\cdots<\mu_q}} \alpha_{\lambda_1\cdots\lambda_p\mu_1\cdots\mu_q\cdots},\ dz^{\lambda_1}\wedge\cdots\wedge dz^{\lambda_p}\wedge d\bar{z}^{\mu_1}\wedge\cdots\wedge d\bar{z}^{\mu_q}. \tag{1.2.8}$$

The differential form $\alpha_{p,q}$ is said to be of type (p, q). It can be easily checked that if α is of type (p, q), then the conjugate $\bar{\alpha}$ is of type (q, p). A form α is *holomorphic* if its components $\alpha_{p,q}$ are holomorphic functions of z^{λ}.

2.5 Some Calculus on *M*

Definition 1.2.4: A (complex) tensor field T of order (r, s) is a map on M that assigns to each $\mathbf{x}$ in M an element of the tensor product of

$$\otimes_r\, T_{\mathrm{x}}(M)^{c} \text{ and } \otimes_s\, T_{\mathrm{x}}^{*}(M)^{c}.$$

Evidently T is contravariant of order r, and covariant of order s.

We now give a few elementary facts on the calculus of differential forms. But before this, we would like to note that unlike tensor fields, the order of differential forms is restricted by $2n$—the real dimensionality of M—and this natural phenomenon provides a rich mathematical structure to differential forms in the form of Hodge theory and Chern classes, etc., which in turn play an important role in modern physical theories, e.g., Yang Mills and string theory.

Similar to real differential forms, we define two differential operators d' and d'' on complex differential forms in the following manner:

$$d'\alpha = \sum_{p,q} d'\,\alpha_{p,q},\quad d''\,\alpha = \sum_{p,q} d''\alpha_{p,q} \tag{1.2.9}$$

$$\text{(a)}\qquad d'\alpha_{p,q} = \sum_{\Lambda\bar{\mu}} \frac{\partial\alpha_{\Lambda\bar{\mu}}}{\partial z^{\lambda}}\, dz^{\lambda} \wedge dz^{\Lambda} \wedge d\bar{z}^{\mu}$$

$$\text{(b)}\qquad d''\alpha_{p,q} = \sum_{\Lambda\bar{\mu}} \frac{\partial\alpha_{\Lambda\bar{\mu}}}{\partial \bar{z}^{\nu}}\, d\bar{z}^{\nu} \wedge dz^{\Lambda} \wedge d\bar{z}^{\mu}. \tag{12.10}$$

where we have used Λ and $\bar{\mu}$ to denote the sets $(\lambda_1, \cdots, \lambda_p)$ and $(\bar{\mu}_1, \cdots, \bar{\mu}_q)$. Evidently $d'\alpha_{p,q}$ and $d''\alpha_{p,q}$ are forms of type $(p + 1, q)$ and $(p, q + 1)$, respectively. It can be easily checked that:

$$\text{(a)}\quad (d')^2 = (d'')^2 = 0, \qquad \text{(b)}\ d = d' + d'' \text{ and} \qquad \text{(c)}\ \overline{d'\alpha} = d''\bar{\alpha} \tag{1.2.11}$$

Fact 1.2.5: A p-form α of type $(p, 0)$ is holomorphic if and only if $d''\,\alpha = 0$.

Definition 1.2.6: A form α on M is called *closed* if its differential $d\alpha$ vanishes on M.

Definition 1.2.7: A symmetric covariant 2-tensor field T is said to be *Hermitian* if for any real vector fields V_1, V_2 on M, T satisfies:

$$T(JV_1, JV_2) = T(V_1, V_2). \tag{1.2.12}$$

Given a Hermitian tensor field T, define

$$w_T(V, V') = T(V, JV') \tag{1.2.13}$$

then w_T is a (real) differential form of type (1,1) and w_T can be written locally as:

$$w_T = -iT_{\lambda\bar{\mu}}\, dz^{\lambda} \wedge dz^{\bar{\mu}} \tag{1.2.14}$$

where

$$T_{\lambda\bar{\mu}} = T\left(\frac{\partial}{\partial z^{\lambda}}, \frac{\partial}{\partial \bar{z}^{\mu}}\right). \tag{1.2.15}$$

Conversely, if there is a real differential 2-form θ of type (1,1) such that for any vector fields V_1, V_2 on M

$$T(V_1, V_2) = \theta(JV_1, V_2) \tag{1.2.16}$$

then T is an Hermitian symmetric covariant 2-tensor field such that $\theta = w_T$[2] and locally

$$\theta_{\lambda\bar{\mu}} = -iT_{\lambda\bar{\mu}} \tag{1.2.17}$$

where

$$\theta_{\lambda\bar{\mu}} = \theta\left(\frac{\partial}{\partial z^{\lambda}}, \frac{\partial}{\partial \bar{z}^{\mu}}\right). \tag{1.2.18}$$

---

[2] Note the palcement of J in (1.2.13) and (1.2.16). This explains the factor $(-i)$ in (1.2.17).

2.6 Kähler Manifold

Definition 1.2.8: A Riemannian metric g on a complex manifold is said to be *Hermitian*, if it is Hermitian as a symmetric covariant 2-tensor field. A Hermitian metric g is called *Kählerian* if the real differential 2-form w of type (1,1) associated to g is a closed 2-form. The form w is called the *fundamental* form of the Kähler metric g. Thus a Kähler manifold is a complex manifold coupled with a Kähler metric.

We now state few results without proof on Kähler metric and Kähler manifolds (see ref. [10] for proofs).

Result 1.2.9: A real closed 2-form θ of type (1,1) is the fundamental form of a Kähler metric if and only if $\theta > 0$, i.e., the Hermitian matrix $(i\theta_{\lambda\bar{\mu}})$ given by relation (1.2.18) has a positive determinant.

Result 1.2.10: Let ∇ denote the operator of covariant differentiation with respect to a Hermitian metric g on complex manifold M; then M is Kählerian if and only if any of the following statements holds good:

$$\text{(i)}\ \nabla w = 0, \qquad \text{(ii)}\ \nabla J = 0, \qquad \text{(iii)}\ \nabla_{\lambda} Z_{\bar{\mu}} = \nabla_{\bar{\lambda}} Z_{\mu} = 0 \tag{1.2.19}$$

where

$$Z_{\lambda} = \frac{\partial}{\partial z^{\lambda}}, \qquad Z_{\bar{\mu}} = \frac{\partial}{\partial \bar{z}^{\mu}}, \qquad \nabla_{\lambda} = \nabla_{Z_{\lambda}}, \qquad \nabla_{\bar{\lambda}} = \nabla_{Z_{\bar{\lambda}}}$$

and λ, μ take all values $1, 2, \cdots, n$.

Result 1.2.11: Let M be a Kähler manifold and S the Ricci tensor of M (see Chapter 8), then S is a Hermitian symmetric covariant 2-tensor field.

The real 2-form s of type (1,1) that results from equality (1.2.13) is called the Ricci form of M. Locally s can be written as:

$$s = -iS_{\lambda\bar{\mu}}\, dz^{\lambda} \wedge d\bar{z}^{\mu}, \quad S_{\lambda\bar{\mu}} = S(Z_{\lambda}, Z_{\bar{\mu}}). \tag{1.2.20}$$

2.7 Harmonic Forms on a Kähler Manifold

We see next that when M is compact, we can define another operator on the collection of forms on M. To this end we assume that θ and η are any two forms of type (p, q), then the inner product (,) between θ and η is given by:

$$(\theta, \eta) = \int_M \langle \theta, \eta \rangle * 1 \tag{1.2.21}$$

where $* 1$ denotes the volume element with respect to metric g on M and $\langle \theta, \eta \rangle (x) = \langle \theta(x), \eta(x) \rangle$ is the scalar product on M (i.e., a smooth function $\forall\, x \in M$). Let $D^{p,q}$ denote the space of differential forms of type (p, q); then there exists an operator ∂'': $D^{p,q} \to D^{p,q-1}$ $(q \geq 1)$, $\partial'' D^{p,0} = 0$ such that

$$(d''\theta, \eta) = (\theta, \partial''\eta) \tag{1.2.22}$$

for any $\theta \in D^{p,q-1}$ and $\eta \in D^{p,q}$.

Define another operator:

$$\square'' = d''\,\partial'' + \partial''\,d''. \tag{1.2.23}$$

The zeros of $\Box''$ are called *harmonic* forms of M.[3]

We now state a few results that involve these operators. Recall that on a smooth compact Riemannian manifold M we have the so-called Laplace operator[4]:

$$\Delta = d\delta + \delta d \tag{1.2.24}$$

This operator can be viewed as operating on complex differential forms, when we assume that M is Kähler. The operators Δ and $\Box''$ (denoted Δ'') obey the fundamental relation:

$$2\Delta'' = \Delta \tag{1.2.25}$$

The following results are easy to prove.

Result 1.2.12: If M is a compact Kähler manifold and $\theta \in D^{p,\,0}$, then the three statements given below are equivalent:

(i) $d\theta = 0$, (ii) θ is holomorphic, (iii) θ is harmonic (1.2.26)

Result 1.2.13: If M is the same as above, and $\theta \in D^{0,\,q}$, then the statements given below are equivalent:

(i) $d\theta = 0$, (ii) θ is antiholomorphic, (iii) θ is harmonic (1.2.27)

Exercise 1.2

1. Prove Result (1.2.12).
2. Prove Result (1.2.13).

Hints to Exercise 1.2

Both of these exercises are easy to establish. One can use the procedures suggested in Exercises 6 and 7 of the next section or by first principles, using the definitions of these objects.

3 RIEMANN SURFACES

In the theory of strings, Riemann surfaces are important ingredients. In this section we define them and state some of the well known results related to their topology and geometry (differential as well as algebraic).

3.1 Riemann Surface *M*

Definition 1.3.1: A *Riemann surface* is a connected complex analytical manifold of complex dimension one, i.e., it is a two-real-dimensional manifold, with a maximal set of charts $\{U_\alpha, Z_\alpha\}$that satisfies:

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[3] We have introduced $\Box''$ and ∂''in a very simplistic manner—our definition avoids the complex line bundles (see [8] and [9] for sophisticated approach). Operator $\Box''$ and ∂'' are denoted Δ'' and δ''also.

[4] See Chapter 3.