Introduction: Pseudo Differential Operators and Markov Processes

This monograph is devoted to Markov processes with state space $\mathbb{R}^n$, in particular we will discuss jump processes. A brief, but very readable description of the development of the theory of stochastic processes until the late fifties can be found in the preface of the monograph of K. Itô and H. McKean [158]. In particular the different ways of constructing processes are pointed out: constructions using (functional) analysis (N. Wiener, A. Kolmogorov, W. Feller, K. Yosida, E.B. Dynkin) and pathwise constructions (P. Lévy, K. Itô).

Today, pathwise constructions via stochastic differential equations, for example, are in the centre of probabilists' interest and they led to the field of stochastic analysis, i.e. analysis on path spaces which is since the publication of P. Malliavin's paper [216] one of the central themes in modern probability theory. The reader is highly recommended to P. Malliavin's essays [218] to get first hand information.

However, after the publication of M. Fukushima's work [100] on Dirichlet forms and Markov processes the (functional) analytic approach to stochastic processes returned also into the centre of probabilists' interest.

In this monograph we will emphasise another point of view, namely the relation of Fourier analysis and Markov processes. This theme was first taken up by P. Lévy and in particular by S. Bochner when discussing stochastically continuous processes with stationary and independent increments, i.e. *Lévy processes*. Bochner's monograph [40] should be mentioned as the most inspiring source to these considerations. The observation is that every Lévy process $(X_t)_{t\geq 0}$ with state space $\mathbb{R}^n$ is completely determined by one and only one

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function $\psi : \mathbb{R}^n \to \mathbb{C}$ which is defined by the relation

$$\mathrm{E}\left(e^{i\mathrm{X}_t\cdot\xi}\right) = e^{-t\psi(\xi)}. \tag{0.1}$$

The function ψ, called the *characteristic exponent* of $(\mathrm{X}_t)_{t\geq 0}$, is a continuous negative definite function and contains all information about $(\mathrm{X}_t)_{t\geq 0}$. Some results for $(\mathrm{X}_t)_{t\geq 0}$ are best proven by looking directly at ψ, for example the process is conservative if and only if $\psi(0) = 0$. But for pathwise considerations it is useful to take the Lévy–Khinchin representation

$$\psi(\xi) = c + ib\cdot\xi + \mathrm{Q}(\xi) + \int_{\mathbb{R}^n\setminus\{0\}} \left(1 - e^{-iy\cdot\xi} - \frac{iy\cdot\xi}{1+|y|^2}\right) \times \frac{1+|y|^2}{|y|^2}\mu(\mathrm{d}y). \tag{0.2}$$

Our starting point is the following observation made in [169]. Let $\left((\mathrm{X}_t)_{t\geq 0}, P^x\right)_{x\in\mathbb{R}^n}$ be a (nice) Feller process with state space $\mathbb{R}^n$. Then the function

$$\mathrm{q}(x,\xi) := -\lim_{t\to 0} \frac{\mathrm{E}^x\left(e^{i(X_t - x)\cdot\xi}\right) - 1}{t} \tag{0.3}$$

completely characterises $\left((\mathrm{X}_t)_{t\geq 0}, P^x\right)_{x\in\mathbb{R}^n}$. In analogy to the theory of partial differential operators we will call $\mathrm{q}(x,\xi)$ the *symbol of the process* $\left((\mathrm{X}_t)_{t\geq 0}, P^x\right)_{x\in\mathbb{R}^n}$. Let us try to understand (0.3) heuristically from two different starting points.

First suppose that $\left((\mathrm{X}_t)_{t\geq 0}, P^x\right)_{x\in\mathbb{R}^n}$ is given as a nice Feller process. For fixed $x \in \mathbb{R}^n$ we may consider the random variables X_t under P^x and look at their *characteristic functions*, i.e. we may consider

$$\lambda_t(x,\xi) := \mathrm{E}^x\left(e^{i(\mathrm{X}_t - x)\cdot\xi}\right) = e^{-ix\cdot\xi}\mathrm{E}^x\left(e^{iX_t\cdot\xi}\right). \tag{0.4}$$

Denoting by $(\mathrm{T}_t)_{t\geq 0}$ the semigroup associated with $\left((\mathrm{X}_t)_{t\geq 0}, P^x\right)_{x\in\mathbb{R}^n}$ we find

$$\mathrm{T}_t\mathrm{u}(x) = \mathrm{E}^x(\mathrm{u}(\mathrm{X}_t)) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\lambda_t(x,\xi)\hat{\mathrm{u}}(\xi)\,\mathrm{d}\xi, \tag{0.5}$$

which says that $(T_t)_{t\geq 0}$ is a family of pseudo–differential operators and the symbol of T_t is $\lambda_t(x,\xi)$. By assumption $(T_t)_{t\geq 0}$ is a Feller semigroup. Hence we may look at its generator

$$\mathrm{A}u = \lim_{t\to 0} \frac{T_t u - u}{t}, \tag{0.6}$$

where the limit is taken in the strong sense in the space $C_\infty(\mathbb{R}^n;\mathbb{R})$. Substituting (0.5) into (0.6) we arrive at

$$\mathrm{A}u(x) = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x,\xi)\hat{u}(\xi)\, d\xi \tag{0.7}$$

with $q(x,\xi)$ as in (0.3). Thus the generator of the semigroup (and of the process) is also a pseudo–differential operator. The function $\xi \mapsto \lambda_t(x,\xi)$ must be positive definite (and continuous). Supposing for simplicity for a moment that the process is conservative we have $\lambda_t(x,0) = 1$ and from the general theory of negative definite functions it follows that $\xi \mapsto -\frac{(\lambda_t(x,\xi)-1)}{t}$ must be negative definite, which must also hold for the limit. Hence, we find that the symbol $q(x,\xi)$ of the process must be a continuous negative definite function with respect to ξ.

There is another way to understand (0.7) (or (0.3)). As generator of a Feller semigroup A has to satisfy the positive maximum principle, i.e.

$$\sup_{x\in\mathbb{R}^n} u(x) = u(x_0) \geq 0 \text{ implies } \mathrm{A}u(x_0) \leq 0. \tag{0.8}$$

In [62] Ph. Courrège characterises the operators satisfying the positive maximum principle. In particular he proved under the reasonable assumption that $C_0^\infty(\mathbb{R}^n;\mathbb{R}) \subset D(\mathrm{A})$, then for $u \in C_0^\infty(\mathbb{R}^n;\mathbb{R})$ the operator A has the representation (0.7), where $q(x,\xi)$ is a locally bounded function in x which is continuous and negative definite in ξ.

Clearly, the generator completely characterises the semigroup, and further, the family of characteristic functions $(\lambda_t(x,\xi))_{t\geq 0}$ completely characterise the process. Now we are in the analogous situation as S. Bochner was in case of Lévy processes: The study of the process is reduced to the study of its symbol!

As in the case of Lévy processes it is sometimes more advantageous to use $q(x,\xi)$ directly, but sometimes it is better to use its Lévy–Khinchin decomposition

$$-\mathrm{q}(x,\xi) = c(x) + \sum_{j=1}^{n} \mathrm{b}_j(x)\xi_j + \sum_{k,l=1}^{n} a_{kl}(x)\xi_k\xi_l$$
$$+ \int_{\mathbb{R}^n\setminus\{0\}} \left(1 - e^{-iy\cdot\xi} - \frac{iy\cdot\xi}{1+|y|^2}\right)\frac{1+|y|^2}{|y|^2}\,\mathrm{N}(x,\mathrm{d}y). \qquad (0.9)$$

Now we may state the aim of the monograph:

Following Bochner: Construct and study Markov processes by systematically making use of their symbols.

It turns out that in doing so a lot of analysis is to be developed which is not covered by standard theories of pseudo–differential operators, of function spaces etc. This is due to the fact that the symbols under considerations do in general not belong to classical symbol classes. Thus, although stochastic processes are our aim, we have to develop our analytic tools first. We divide our presentation into three parts:

1. Fourier analysis and semigroups.

2. Generators and their potential theory.

3. Markov processes and applications.

For each part we will give a separate introduction.