Since this involves only terms *linear* in the u_i, we see on substitution in Eq. (1.4.13) that the original Green's function now reappears on the right-hand side, and we obtain the closed set of coupled differential equations

$$\left(-\frac{d^2}{dt^2} - \Omega_0{}^2\right) G_{ij}(t) = \frac{1}{M}\,\delta_{ij}\,\delta(t) + \lambda \sum_k D_{ik} G_{kj}(t). \tag{1.4.14}$$

In more general cases, differentiation of a Green's function will not reproduce the original function on the right-hand side, and instead of a closed set of equations we obtain an infinite chain of equations involving Green's functions of successively higher orders. We then have to approximate at some stage to break off the chain, usually by some sort of "decoupling" procedure in which a high-order Green's function is expressed approximately as a product of Green's functions of lower order. The simple closed result obtained in the present problem depends on the special oscillator form of the hamiltonian.

1.5. THE ITERATION SOLUTION FOR *G*

If we know the solution of the equation of motion (1.4.14) for $\lambda = 0$ (corresponding to the unperturbed hamiltonian), we can obtain the general solution by iteration in powers of λ. For $\lambda = 0$, Eq. (1.4.14) reduces to a differential equation, with a delta-function term representing a unit applied impulse, of a form familiar in elementary discussions of Green's functions. The solution of this is clearly diagonal in the suffices i and j, of the form $G^0(t)\,\delta_{ij}$. The differential equation

$$\left(\frac{d^2}{dt^2} + \Omega_0{}^2\right) G^0(t) = -\frac{1}{M}\,\delta(t) \tag{1.5.1}$$

by itself does not, however, determine $G^0(t)$ completely. It tells us that, in each sub-interval $t < 0$ and $t > 0$, G^0 is some linear combination of the elementary solutions $\exp(\pm i\Omega_0 t)$ of the homogeneous equation. It also follows from Eq. (1.5.1) that G^0 is continuous at $t = 0$ and that its first derivative has a discontinuity there of magnitude $(-1/M)$. The time-ordered, retarded and advanced Green's functions all satisfy Eq. (1.5.1) and have these properties, and the distinction between them comes from initial (and final) conditions at $t = \pm\infty$.

In fact, to obtain $G^0(t)$ explicitly it is simplest not to use the differential equation, but to go back to the original definition of the Green's function. Thus, to obtain the time-ordered function, we have to evaluate

$$G^0(t) = -i\langle T[\tilde{u}_i(t)\tilde{u}_i(0)]\rangle_{\lambda=0}, \qquad (1.5.2)$$

with

$$\tilde{u}_i(t) = e^{iH_0t}u_i\,e^{-iH_0t}, \qquad H_0 = \sum_i \left(\frac{p_i^2}{2M} + \tfrac{1}{2}M\Omega_0^2u_i^2\right).$$

This is a standard harmonic oscillator problem, and we introduce *annihilation* and *creation operators* B_i, $B_i^\dagger$ in the usual way [Messiah (1961, Chap. XII)]. Put $\xi_i = u_i\sqrt{(M\Omega_0)}$, $p_{\xi_i} = p_i/\sqrt{(M\Omega_0)}$, so that

$$H_0 = \sum_i \tfrac{1}{2}\Omega_0(\xi_i^2 + p_{\xi_i}^2),$$

and write

$$B_i = \frac{1}{\sqrt{2}}(\xi_i + ip_{\xi_i}), \qquad B_i^\dagger = \frac{1}{\sqrt{2}}(\xi_i - ip_{\xi_i}). \qquad (1.5.3)$$

These operators satisfy the boson commutation rules

$$[B_i, B_j^\dagger] = \delta_{ij}, \qquad [B_i, B_j] = [B_i^\dagger, B_j^\dagger] = 0, \qquad (1.5.4)$$

and the hamiltonian becomes

$$H_0 = \sum_i \Omega_0(B_i^\dagger B_i + \tfrac{1}{2}). \qquad (1.5.5)$$

The time dependence of the B_i, $B_i^\dagger$ is obtained from the Heisenberg equations of motion; thus

$$i\frac{dB_i}{dt} = [B_i, H_0] = \Omega_0 B_i, \quad \text{giving } B_i(t) = B_i\,e^{-i\Omega_0t}, \qquad (1.5.6)$$

and similarly

$$B_i^\dagger(t) = B_i^\dagger\,e^{i\Omega_0t}, \qquad (1.5.7)$$

where $B_i = B_i(0)$, $B_i^\dagger = B_i^\dagger(0)$. In terms of the B_i operators,

$$u_i = \frac{1}{\sqrt{(2M\Omega_0)}} (B_i + B_i^\dagger), \tag{1.5.8}$$

and thus

$$G^0(t) = -\frac{i}{2M\Omega_0} \langle 0 | T[(B_i(t) + B_i^\dagger(t))(B_i(0) + B_i^\dagger(0))] | 0 \rangle.$$

Here $| 0 \rangle$ is the normalized ground state of H_0, with the property $B_i | 0 \rangle = 0$ and $\langle 0 | B_i^\dagger = 0$ for all i. Hence, for $t > 0$,

$$G^0(t) = -\frac{i}{2M\Omega_0} e^{-i\Omega_0 t} \langle 0 | B_i B_i^\dagger | 0 \rangle$$

$$= -\frac{i}{2M\Omega_0} e^{-i\Omega_0 t} \langle 0 | 1 + B_i^\dagger B_i | 0 \rangle = -\frac{i}{2M\Omega_0} e^{-i\Omega_0 t}. \tag{1.5.9}$$

Similarly, for $t < 0$,

$$G^0(t) = -\frac{i}{2M\Omega_0} e^{i\Omega_0 t}, \tag{1.5.10}$$

so that, for all t, we have the result

$$G^0(t) = -\frac{i}{2M\Omega_0} e^{-i\Omega_0 |t|}. \tag{1.5.11}$$

This undamped traveling wave, of frequency Ω_0, is just the Green's function representing the response of an undamped harmonic oscillator of natural frequency Ω_0 driven by a unit impulse at time $t = 0$.

For comparison we quote the expressions for the unperturbed retarded and advanced Green's functions, which may similarly be calculated from the definitions (1.4.8) and (1.4.9). We obtain the standing waves

$$G^{0,R}(t) = -\frac{\theta(t)}{M\Omega_0} \sin \Omega_0 t, \quad G^{0,A}(t) = \frac{\theta(-t)}{M\Omega_0} \sin \Omega_0 t. \tag{1.5.12}$$

Having obtained the unperturbed solution $G^0(t)\, \delta_{ij}$ of the equation of motion (1.4.14), we can now proceed to derive the complete solution. We could of course solve Eq. (1.4.14) directly in closed form, but as we are illustrating a general technique we prefer to use a more general approach in which the equation is solved by iteration, as a power series in the parameter λ.[1]

The Green's function $G_{ij}(t)$ is now to be obtained as the solution of the set of differential equations (1.4.14) which satisfies the boundary (or "initial") condition that $G_{ij}(t) = G^0(t)\,\delta_{ij}$ when $\lambda = 0$, where $G^0(t)$ is given by Eq. (1.5.11). This defines a boundary-value problem which can equally well be formulated in terms of an equivalent set of *integral equations*. These integral equations incorporate the boundary condition at $\lambda = 0$ and are thus particularly well adapted to solution by iteration in λ. To set up the integral equations we need only observe, as is clear from (1.5.1), that the function $G^0(t-t')$ is just the Green's function for the unperturbed operator $M(d^2/dt^2 + \Omega_0{}^2)$. The solution of the full equations (1.4.14) can thus be expressed as

$$G_{ij}(t) = G^0(t)\,\delta_{ij} + \lambda M \int_{\infty}^{\infty} dt'\, G^0(t-t') \sum_k D_{ik} G_{kj}(t'). \qquad (1.5.13)$$

It is easily checked directly that these integral equations are equivalent to the differential equations (1.4.14) together with the boundary condition at $\lambda = 0$.

The iterative solution of Eq. (1.5.13), obtained by successive approximation, is

$$G_{ij}(t) = G^0(t)\,\delta_{ij} + \lambda M \int_{-\infty}^{\infty} dt'\, G^0(t-t') D_{ij} G^0(t')$$

$$+ (\lambda M)^2 \sum_{i'} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt'\, dt''\, G^0(t-t') D_{ii'}\, G^0(t'-t'')$$

$$\times D_{i'j} G^0(t'') + \cdots \qquad (1.5.14)$$

[1] This is a more general method, in the sense that it will work in cases where no closed exact solution exists. On the other hand, the iteration series will converge only for sufficiently small values of λ, although the sum of the series defines a function which, by a process of analytic continuation, exists and is unique for general values of λ. In the present, exactly soluble, case the analytic properties of the solution are easily investigated. In more complicated problems, the justification of formal summations of perturbation series and their analytic continuation is not easily made rigorous, and the mathematical argument must usually be supported by physical reasoning.

The different terms in this series show how propagation develops through interaction between successively greater numbers of atoms, with excitation energy being handed on from site to site. To visualize this process, and to represent the terms in the series in a clear and simple way, we draw diagrams as shown in Fig. 1.1. These are Feynman diagrams

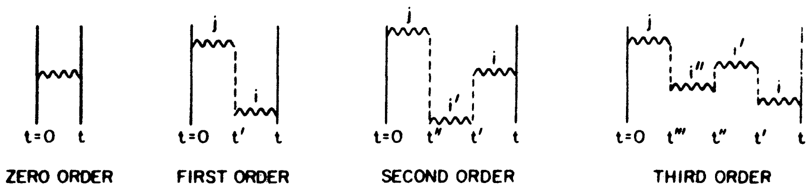

Fig. 1.1. Diagrammatic representation of the iteration solution for $G_{ij}(t)$.

of a specially simple kind, representing propagation (in time only) between times $t = 0$ and t (in general we have to deal with propagation in both time and space). We imagine (for $t > 0$) the time to be increasing from left to right. Each factor $G^0(t^\alpha - t^\beta)$ in any term of the perturbation series, which represents unperturbed propagation between successive scattering processes, is represented by a wavy line in a diagram connecting two times t^α and t^β and is labeled with the index of the atom carrying the excitation energy. At each time t^α at which a wavy line begins or ends, and excitation energy is handed on, there is an interaction factor D_{kl} represented by a dotted line in the diagram connecting two atoms k and l. The nth-order diagram contains n intermediate times between $t = 0$ and t, $(n + 1)$ wavy lines, n dotted lines and $(n - 1)$ intermediate atoms between i and j; a summation over all possible intermediate atoms and intermediate times is implied. Thus we can read off at once (reading from right to left) that the third-order diagram in Fig. 1.1 corresponds to the term

$$(\lambda M)^3 \sum_{i'i''} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt'\, dt''\, dt'''\, G^0(t - t')D_{ii'}G^0(t' - t'')D_{i'i''}$$

$$\times\, G^0(t'' - t''')D_{i''j}G^0(t''' - 0)$$

in the iteration series.