

In the following chapters of the present book we will frequently take advantage of the more simple fluid approach in spite of the simplifying assumptions underlying any fluid theory. In fact, many of the nonlinear problems cannot be treated in the framework of the precise kinetic approximation. In these cases one is forced to return to the fluid description in order to treat the nonlinear effects at least in an approximate way.

1.4. Plasma Waves

The system of Vlasov-Maxwell equations or its hydrodynamic simplifications allow for the propagation of disturbances on the background of the plasma. Generally, these disturbances are nonlinear time-varying states the plasma can assume. But as long as their amplitudes are small when compared with the undisturbed field and particle variables, they can be treated in a linear approximation as small disturbances. This condition can be written as $|\delta A(\mathbf{x}, t)| \ll |A_0(\mathbf{x}, t)|$, where δA is the amplitude of the variation of some quantity $A(\mathbf{x}, t)$, and A_0 is its equilibrium undisturbed value which may also vary in time and space. In the linear approximation such disturbances of the plasma state represent propagating waves of frequency, $\omega(\mathbf{k})$, and wavenumber, \mathbf{k} . As usual, the phase and group velocities of these waves are defined as

$$\mathbf{v}_{ph} = \frac{\omega(\mathbf{k})}{k^2} \mathbf{k} \quad (1.24)$$

$$\mathbf{v}_{gr} = \frac{\partial \omega(\mathbf{k})}{\partial \mathbf{k}} = \nabla_{\mathbf{k}} \omega(\mathbf{k}) \quad (1.25)$$

The *phase velocity* is directed parallel to \mathbf{k} and gives direction and speed of the propagation of the wave front or phase

$$\phi(\mathbf{x}, t) = \mathbf{k} \cdot \mathbf{x} - \omega(\mathbf{k})t \quad (1.26)$$

while the *group velocity* can point into a direction different from the phase velocity. It gives the direction of the flow of energy and information contained in the wave. Both can be calculated from knowledge of the frequency. The latter is the solution of the wave dispersion relation in both the linear approximation and the full nonlinear theory.

In the linear approximation the dispersion relation is particularly simple to derive. Because of the linear approximation, the full set of Maxwell-Vlasov or Maxwell-hydrodynamic equations contains only linear disturbances. Thus the system can be reduced to a set of linear algebraic equations with vanishing determinant

$$D(\omega, \mathbf{k}) = 0 \quad (1.27)$$

the *dispersion relation*. The analytical form of the dispersion relation is obtained from the linearized wave equation (I.9.45)

$$\nabla^2 \delta \mathbf{E} - \nabla(\nabla \cdot \delta \mathbf{E}) - \epsilon_0 \mu_0 \frac{\partial^2 \delta \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial \delta \mathbf{j}}{\partial t} \quad (1.28)$$

The linear current density, $\delta\mathbf{j}$, on the right-hand side is expressed by the linear *Ohm's law* given in Eq. (I.9.46)

$$\delta\mathbf{j}(\mathbf{x}, t) = \int d^3x' \int_{-\infty}^t dt' \boldsymbol{\sigma}(\mathbf{x} - \mathbf{x}', t - t') \cdot \delta\mathbf{E} \quad (1.29)$$

with $\boldsymbol{\sigma}(\mathbf{x} - \mathbf{x}', t - t')$ the linear conductivity tensor. Fourier transformation of Eqs. (1.28) and (1.29) with respect to time and space gives as equation for the Fourier amplitude of the wave field

$$\left[\left(k^2 - \frac{\omega^2}{c^2} \right) \mathbf{I} - \mathbf{k}\mathbf{k} - i\omega\mu_0\boldsymbol{\sigma}(\omega, \mathbf{k}) \right] \cdot \delta\mathbf{E}(\omega, \mathbf{k}) = 0 \quad (1.30)$$

The linear conductivity, $\boldsymbol{\sigma}(\omega, \mathbf{k})$, is a function of frequency, ω , and wavenumber, \mathbf{k} . The fields and the conductivity satisfy the following symmetry relations

$$\begin{aligned} \delta\mathbf{E}(-\mathbf{k}, -\omega) &= \delta\mathbf{E}^*(\mathbf{k}, \omega) \\ \boldsymbol{\sigma}(-\mathbf{k}, -\omega) &= \boldsymbol{\sigma}^*(\omega, \mathbf{k}) \end{aligned} \quad (1.31)$$

The dispersion relation follows from the condition that Eq. (1.30) should have nontrivial solutions

$$D(\omega, \mathbf{k}) = \text{Det} \left[\left(k^2 - \frac{\omega^2}{c^2} \right) \mathbf{I} - \mathbf{k}\mathbf{k} - i\omega\mu_0\boldsymbol{\sigma}(\omega, \mathbf{k}) \right] = 0 \quad (1.32)$$

It is convenient to introduce the *dielectric tensor* of the plasma

$$\boldsymbol{\epsilon}(\omega, \mathbf{k}) = \mathbf{I} + \frac{i}{\omega\epsilon_0} \boldsymbol{\sigma}(\omega, \mathbf{k}) \quad (1.33)$$

and to rewrite the dispersion relation into the shorter version

$$D(\omega, \mathbf{k}) = \text{Det} \left[\frac{k^2 c^2}{\omega^2} \left(\frac{\mathbf{k}\mathbf{k}}{k^2} - \mathbf{I} \right) + \boldsymbol{\epsilon}(\omega, \mathbf{k}) \right] = 0 \quad (1.34)$$

This dispersion relation is the basis of all linear plasma theory and is also used in nonlinear plasma theory. The dielectric tensor which appears in this relation must be calculated from the dynamical model of the plasma. Its most general analytical form derived from the linearized set of the Maxwell-Vlasov equations has been given in Eq. (I.10.94) of Chap. 10 of the companion volume, *Basic Space Plasma Physics*. For further reference we repeat this equation here

$$\begin{aligned} \boldsymbol{\epsilon}(\omega, \mathbf{k}) &= \left(1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \right) \mathbf{I} - \sum_s \sum_{l=-\infty}^{l=\infty} \frac{2\pi\omega_{ps}^2}{n_{0s}\omega^2} \\ &\int_0^{\infty} \int_{-\infty}^{\infty} v_{\perp} dv_{\perp} dv_{\parallel} \left(k_{\parallel} \frac{\partial f_{0s}}{\partial v_{\parallel}} + \frac{l\omega_{gs}}{v_{\perp}} \frac{\partial f_{0s}}{\partial v_{\perp}} \right) \frac{\mathbf{S}_{ls}(v_{\parallel}, v_{\perp})}{k_{\parallel} v_{\parallel} + l\omega_{gs} - \omega} \end{aligned} \quad (1.35)$$

The tensor appearing in the integrand, \mathbf{S}_{ls} , is of the form

$$\mathbf{S}_{ls}(v_{\parallel}, v_{\perp}) = \begin{bmatrix} \frac{l^2 \omega_{gs}^2}{k_{\perp}^2} J_l^2 & \frac{i l v_{\perp} \omega_{gs}}{k_{\perp}} J_l J_l' & \frac{l v_{\parallel} \omega_{gs}}{k_{\perp}} J_l^2 \\ -\frac{i l v_{\perp} \omega_{gs}}{k_{\perp}} J_l J_l' & v_{\perp}^2 J_l'^2 & -i v_{\parallel} v_{\perp} J_l J_l' \\ \frac{l v_{\parallel} \omega_{gs}}{k_{\perp}} J_l^2 & i v_{\parallel} v_{\perp} J_l J_l' & v_{\parallel}^2 J_l'^2 \end{bmatrix} \quad (1.36)$$

and the Bessel functions, $J_l, J_l' = dJ_l/d\xi_s$, depend on the argument $\xi_s = k_{\perp} v_{\perp} / \omega_{gs}$.

The determinant of the dispersion relation, $D(\omega, \mathbf{k})$, is a function of frequency, wavenumber and a set of plasma parameters. Its solution yields the frequency relation $\omega = \omega(\mathbf{k})$. Different versions of the dispersion determinant are derived in the companion volume, *Basic Space Plasma Physics*, and solved in several approximations. Generally spoken, in contrast to the vacuum where a continuum of electromagnetic waves can propagate, there is no continuum of plasma waves. Even in the linear approximation, neglecting all couplings, correlations and nonlinear interactions, plasmas are highly complicated dielectrics which possess only a few narrow windows where linear disturbances are allowed. These disturbances are the eigenmodes of the plasma. They appear as the *discrete spectrum* of eigenvalues of the basic linear system of equations governing the dynamics of a plasma as solutions of Eq. (1.27).

A further difference between wave propagation in vacuum and in a plasma is that the plasma allows for two types of waves, *transverse electromagnetic waves* and *longitudinal electrostatic waves*. The latter are nothing else but oscillations of the electrostatic potential and are not accompanied by magnetic fluctuations. Somehow they resemble sound waves in ordinary hydrodynamics, but there is a large zoo of electrostatic waves in a plasma most of which are not known in simple hydrodynamics.

The *electrostatic modes* are confined to the plasma, because oscillations of the electrostatic potential can be maintained only inside the plasma boundaries. Only two of the *electromagnetic modes* smoothly connect to the free-space electromagnetic wave and can leave the plasma, the *O-mode* and the high-frequency branch of the *X-mode*. The other low-frequency electromagnetic waves, the *Z-mode*, *whistlers* and *electromagnetic ion-cyclotron modes*, and the three magnetohydrodynamic wave modes, the *Alfvén wave*, and the *fast mode* and the *slow mode*, are all confined to the plasma. We have discussed the properties and propagation characteristics of these modes in Chaps. 9 and 10 of the companion volume, *Basic Space Plasma Physics*.

Waves propagating in a plasma can experience *reflection* and *resonance*. Reflection occurs when the wavenumber vanishes for finite frequency, $k \rightarrow 0$. Here the direction of the wave turns by an angle π , indicating that the wave is reflected from the particular point where its wavenumber vanishes. Resonance occurs where the wavenumber diverges at finite frequency, $k \rightarrow \infty$. At such a point the wavelength becomes very

short, and the interaction between the plasma particles becomes very strong. Here the wave may either dissipate its energy or extract energy from the plasma in order to grow.

As long as one looks only into the real solutions of the dispersion relation, no information can be obtained about the possible growth of a wave or its damping at the resonant point. However, as the possibility of resonances in a plasma shows, plasmas are active media. This is also realized when remembering that the charges and their motions themselves are sources of the fields. In order to investigate these processes one must include the possibility of complex solutions of the dispersion relation. The fluctuations of the fields can be excited or amplified or, in the opposite case, they can be absorbed in the plasma. The frequency becomes complex under these conditions

$$\omega(\mathbf{k}) \rightarrow \omega(\mathbf{k}) + i\gamma(\omega, \mathbf{k}) \quad (1.37)$$

Here $\gamma(\omega, \mathbf{k})$ is the growth or damping rate of the wave, which depends on the real part of the frequency and on the wavenumber. The wave grows for $\gamma > 0$, and it becomes damped for $\gamma < 0$.

In the companion volume we treated the damping rate, i.e., solutions with $\gamma < 0$. In the present volume, we will start with $\gamma > 0$ solutions. The next chapters are devoted to the discussion of these still linear effects leading to instability, before turning to nonlinear effects which arise when the amplitudes of the waves become so large that the linear assumption must be abandoned.

Introductory Texts

The literature listed below is a selection of introductory texts into plasma physics and space plasma physics which should be consulted before attempting to read this book.

- [1] W. Baumjohann and R. A. Treumann, *Basic Space Plasma Physics* (Imperial College Press, London, 1996).
- [2] F. F. Chen, *Introduction to Plasma Physics and Controlled Fusion, Vol. 1* (Plenum Press, New York, 1984).
- [3] N. A. Krall and A. M. Trivelpiece, *Principles of Plasma Physics* (McGraw-Hill, New York, 1973).
- [4] E. M. Lifshitz and L. P. Pitaevskii, *Physical Kinetics* (Pergamon Press, Oxford, 1981).
- [5] D. C. Montgomery and D. A. Tidman, *Plasma Kinetic Theory* (McGraw-Hill, New York, 1964).
- [6] D. R. Nicholson, *Introduction to Plasma Theory* (Wiley, New York, 1983).