$$[\,\widehat{L}_y, \widehat{L}_z\,] = i\hbar\widehat{L}_x \tag{1.18}$$

$$[\,\widehat{L}_z, \widehat{L}_x\,] = i\hbar\widehat{L}_y \tag{1.19}$$

which, in turn, imply that

$$[\,\widehat{L}\cdot\widehat{L}, \widehat{L}_x\,] = [\,\widehat{L}\cdot\widehat{L}, \widehat{L}_y\,] = [\,\widehat{L}\cdot\widehat{L}, \widehat{L}_z\,] = 0, \tag{1.20}$$

where the total angular-momentum operator $\widehat{L}\cdot\widehat{L}$ is defined as the sum $\widehat{L}_x\widehat{L}_x + \widehat{L}_y\widehat{L}_y + \widehat{L}_z\widehat{L}_z$. The vanishing commutators of Eq. (1.20) mean that simultaneous eigenfunctions exist of $\widehat{L}\cdot\widehat{L}$ and any one[5] of $\widehat{L}_x$, $\widehat{L}_y$ or $\widehat{L}_z$. For various psychological reasons, the usual choice is $\widehat{L}_z$, with the z-axis being drawn pointing upwards.

One of the less gripping tasks in a course on wave-mechanics is to compute the explicit form of these simultaneous eigenfunctions, and to find their associated eigenvalues. The well-known result is that the eigenvalues of $\widehat{L}\cdot\widehat{L}$ have the form $\ell(\ell+1)\hbar^2$, where ℓ is an integer with $\ell \geq 0$. For any given value of ℓ, the associated eigenvalues of $\widehat{L}_z$ are of the form $m\hbar$, where the integer m ranges from $-\ell$ to $+\ell$ in steps of 1. The corresponding simultaneous eigenfunctions u_{lm} are the famous *associated Legendre polynomials.*

1.3 Beyond Introductory Wave Mechanics

The rules and postulates of wave mechanics have been used widely, and with considerable empirical success. However, a number of subtle and important issues wait to be uncovered. For example:

- What is the precise meaning of 'probability' as it arises in the context of quantum theory? And why does 'measurement' play such a prominent role? Can a measurement be regarded as just another type of physical interaction, or does it need to be considered as a fundamental concept in the very foundations of the theory? If the latter is true, how can this be reconciled with the fact that real measuring devices are composed of atoms, which certainly need to be described in quantum-mechanical terms?

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[5] It is only *one* of the operators $\widehat{L}_x$, $\widehat{L}_y$ or $\widehat{L}_z$ since a simultaneous eigenfunction of, for example, the pair of operators $\widehat{L}\cdot\widehat{L}$ and $\widehat{L}_z$ will generally *not* be a simultaneous eigenfunction of the pair $\widehat{L}\cdot\widehat{L}$ and $\widehat{L}_x$, or the pair $\widehat{L}\cdot\widehat{L}$ and $\widehat{L}_y$.

- Do the representations of $\widehat{x}$ and $\widehat{p}$ in Eq. (1.14–1.15) (and the commutation relation in Eq. (1.16)) need to be postulated, or can they be *derived* in some way from the general framework of the theory?

- Are there any other pairs of operators $\widehat{x}$ and $\widehat{p}$ that satisfy the commutation relation Eq. (1.16). If so, do the physical predictions depend on the choice made? In other words, what is more important: the general form of the commutation relation Eq. (1.16), or the specific representations Eq. (1.14–1.15) of the basic observables? Other operators satisfying Eq. (1.16) certainly exist. For example, there is an alternative version of wave mechanics in which states are represented by functions ϕ of momentum p (rather than position x), and the basic operators $\widehat{x}$ and $\widehat{p}$ are defined by

$$\begin{aligned}(\widehat{x}\phi)(p) &= i\hbar\frac{d\phi}{dp}(p) \qquad (1.21)\\ (\widehat{p}\phi)(p) &= p\phi(p) \qquad (1.22)\end{aligned}$$

  which, like the operators defined in Eq. (1.14–1.15), satisfy the canonical commutation relation Eq. (1.16). It is usually assumed that both forms of wave mechanics give the same physical answers. But why should this be so?

- What is the meaning of the 'uncertainty relation' $\triangle x\triangle p \geq \frac{1}{2}\hbar$ associated with the commutation relation Eq. (1.16)? Does $\triangle x$ refer to 'the unavoidable disturbance in the act of making a measurement', as is often stated, or is there some other way of understanding this relation?

- Can the uncertainty relation be generalised to apply to any pair of operators whose commutator is non-zero? Examples are the angular momentum operators $\widehat{L}_x$, $\widehat{L}_y$ and $\widehat{L}_z$ that satisfy the cyclic commutation relations in Eq. (1.17–1.19).

Questions of this type open a Pandora's box of problems concerning the interpretation of quantum theory and the picture it gives of physical reality. Any serious attempt to tackle these problems inevitably encounters a number of profound philosophical issues that are still the subject of intense debate and controversy. One of the central goals of this course is to provide an introduction to some of these deep features of the quantum view of the world.

However, even at the technical level, the postulates above are deficient in several ways, not the least of which is that they apply only to a limited class of physical systems. It is straightforward to extend the wave-mechanical formalism to a particle moving in three dimensions, when the state is a function $\psi(\mathbf{x})$ of the particle's position vector $\mathbf{x}$, or even to a collection of N-particles moving in three dimensions, in which case the state is a function $\psi(\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_N})$ of the N position vectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_N}$. But there are many important physical systems whose quantum states cannot be described at all using only wave functions.

One example is relativistic quantum physics in which the number of particles can change as a result of interactions between them. For example, consider a scattering experiment in which two particles collide and turn into three particles. Ignoring internal and spin quantum numbers, the initial and final states could be described by wave functions $\psi(\mathbf{x_1}, \mathbf{x_2})$ and $\phi(\mathbf{y_1}, \mathbf{y_2}, \mathbf{y_3})$ respectively. However, it is by no means obvious what type of time-dependent Schrödinger equation could allow a function of two variables to evolve smoothly into a function of three variables.

Another famous example of a system that cannot be described using wave functions is electron spin. If one concentrates purely on its internal-spin properties, a state of an electron can be described by a column matrix[6] $\binom{a_1}{a_2}$, where the analogue of the normalisation condition Eq. (1.3) is that the complex numbers a_1 and a_2 satisfy $|a_1|^2 + |a_2|^2 = 1$. A pair of such states $\binom{a_1}{a_2}$ and $\binom{b_1}{b_2}$ can be superimposed with arbitrary complex coefficients α and β to give a new state $\alpha\binom{a_1}{a_2} + \beta\binom{b_1}{b_2}$ which is defined[7] to be the column matrix $\binom{\alpha a_1+\beta b_1}{\alpha a_2+\beta b_2}$ (the only restriction on α and β is that the normalisation condition must be preserved).

In this system, observables are represented by 2×2 complex hermitian[8] matrices that act on the states by matrix multiplication: an operation that is *linear* with respect to the superposition rule mentioned above. In

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[6] A column matrix is often called a column *vector*, reflecting the fact that, as we shall see later, it can be thought of as an element of a particular vector space.

[7] The significance of this definition will emerge in the next chapter.

[8] A square matrix A is said to be *hermitian* if $A = (A^T)^*$ where A^T denotes the transpose of A. In terms of matrix elements, $A_{ij} = A^*_{ji}$.

particular, the x, y and z components of the spin angular momentum are represented respectively by the matrices $S_x = \frac{\hbar}{2}\sigma_x$, $S_y = \frac{\hbar}{2}\sigma_y$ and $S_z = \frac{\hbar}{2}\sigma_z$, where σ_x, σ_y and σ_z are the well-known Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.23}$$

The spin operators $\widehat{S}_x$, $\widehat{S}_y$ and $\widehat{S}_z$ satisfy the same cyclic commutation relations Eq. (1.17–1.19) as the angular momentum operators $\widehat{L}_x$, $\widehat{L}_y$ and $\widehat{L}_z$. However, there is no way the spin variables can be represented as differential operators acting on wave functions.

Other examples of quantum ideas that cannot be described using wave functions are iso-spin, strangeness, charm *etc.*. The existence of systems of this type requires a significant generalisation of the quantum formalism, so as to be applicable to these more complex situations whilst reproducing the familiar results of elementary wave mechanics. The explication of such a formalism is one of the main goals of this course.