Chapter 1

INTRODUCTION

1.1 Scope of the Book

The use of mathematics in modern physics can be rather daunting. In classical mechanics, the representation of the position of a particle by a euclidean three-vector can be directly related to our immediate impressions of the world around us. However, quantum theory is distinguished by a striking gap between the mathematical structure and the physical objects it seeks to represent: a gap that can become an unbridgeable chasm for students encountering the subject for the first few times. For this reason, it is worth being clear in advance what the main goals are in the material that follows.

A theoretical structure in modern physics typically has the following, quasi-axiomatic, features:

1. a specified domain of applicability that limits the class of physical systems to which the theory should be applied;
2. an identification of certain physical concepts that relate to the class of systems in this domain;
3. a specification of the general mathematical framework within which the theory is to be presented;
4. a collection of rules that relate the physical concepts to elements of the mathematical structure;

5. an overall conceptual scheme for analysing the meaning of fundamental terms employed in the statement of the rules;
6. a set of techniques for applying the rules to specific physical systems within the class admitted by the domain of applicability.

There are various criteria by which a structure of this type is judged. For example,

1. the domain of applicability should be as large as possible;
2. the theory should be empirically effective within that domain;
3. the techniques used in formulating the theory (some of which may involve new mathematics) should be internally consistent in a mathematical sense;
4. the physical and conceptual ideas used in elucidating the 'meaning' of the theory should be consistent in a genuine philosophical sense;
5. the overall formalism should be simple and elegant.

The original domain of applicability of quantum theory was the area of non-relativistic atomic and molecular physics, particularly systems of a finite number of elementary particles moving under the influence of electrostatic binding forces. The scope of quantum theory has been slowly developed since then to include relativistic atomic and subatomic processes, mainly within the context of quantum field theories. Schemes of this type have been very successful empirically, but certain problems arise (the 'ultra-violet' divergences) that still cause disquiet at a fundamental level. Attempts to extend quantum theory to include *general* relativity have encountered major mathematical and conceptual problems, and it is not yet clear what the outcome will be. In this book, the domain of applicability is limited to non-relativistic physics, although many of the ideas can be extended to include the relativistic case.

Many approaches to quantum theory take on a rather operationalist flavour by focussing on the concepts of 'observable' and 'measurement'. The emphasis is then placed on the instrinsically probabilistic nature of the predictions of the results of such measurements. This approach is also adopted in the first half of this book; later on we shall look at some aspects of what happens in less instrumentalist interpretations of the theory. In either case, the appropriate mathematical tools for quantum theory[1] are based on the theory of Hilbert spaces: finite- or infinite-dimensional vector spaces equipped with an inner product that mimics the overlap function of elementary wave mechanics.

The book begins with a short summary of some of the key ideas in wave mechanics, presented in a way that is adapted to the later discussion of the general quantum formalism. Chapters 2 and 3 then introduce the basic ideas of vector spaces and linear operators needed in the vector space approach to quantum theory. This material is kept to the minimum necessary for the task in hand: in particular, no attempt is made to deal rigorously with the mathematical problems that arise when infinite-dimensional spaces are treated properly.[2]

One of the main goals of the book is to explore some of the deep conceptual issues that arise in quantum theory. Of the many ways in which this topic can be approached, I have chosen to focus on the notion of a *physical property* and the extent to which it is, or is not, meaningful to talk about a quantum system 'possessing' such properties. For this reason, the exposition of quantum theory proper is preceded by a discussion in Chapter 4 of the analogous situation in classical physics: in particular, the way in which a certain philosophical view about the physical world is reflected in the mathematical framework used to describe it. In classical physics, the 'realist' and 'instrumentalist' views of science fit together seamlessly, whereas in quantum physics they differ sharply, especially in their attitudes towards the idea of physical properties. That such a distinction can arise at all is closely tied to the different mathematical structures employed in the formulations of classical and quantum physics.

The section on classical ideology is followed by two chapters that discuss in detail the general rules of quantum theory as they appear in the vector space formalism. The material covered includes vector states and mixed states, compatible operators, time development, unitary operators,

[1]The classic reference is von Neumann (1971).

[2]A more detailed exposition of the type of vector space theory used in quantum theory can be found in my book Isham (1989). A much more comprehensive treatment of the mathematics of quantum theory is contained in the excellent work by Reed & Simon (1972).

the derivation of the canonical commutation relations, and the generalised uncertainty relations. The main emphasis in these sections is on the basic mathematical framework, and therefore I have adopted a pragmatic, rather instrumentalist view of quantum theory that leaves open the question of what other types of interpretation are possible.

The interpretation of quantum theory is addressed in Chapter 8 where we turn to conceptual matters in a more formal way. The material is organised around four major topics: the meaning of probability, the role of measurement, reduction of the state vector, and quantum entanglement. These issues are of fundamental importance in any attempt to find a more realist interpretation of quantum theory: a key issue for anyone who, like myself, is interested in quantum cosmology. This challenge of realism is studied directly in the final chapter that deals with the status of 'properties' in quantum theory. The main topics discussed are the Kochen–Specker theorem, a short introduction to quantum logic, and the Bell inequalities.

The book concludes with a number of worked problems aimed at developing facility in the type of mathematical manipulations that are essential for any theoretical physicist who wants to use the vector space approach to quantum theory. Several worked problems are also included in the text proper as an aid to understanding various pieces of general formalism.

A quick note on references. The rather short bibliography reflects the origin of the book as lecture notes for an undergraduate course. For this reason, I have concentrated on citing papers and books that should be accessible to an advanced physics undergraduate and which, if dipped into, will genuinely enhance his or her understanding without needing a lifetime of study devoted to the task. As emphasised in the Preface, this is intended to be a short textbook for undergraduates—it is not meant to be a definitive review of modern quantum theory!

1.2 A Summary of Wave Mechanics

It is useful to begin by summarising some of the basic formalism of elementary wave mechanics. This will be presented, with the minimum of comment, in the form of four rules that will be generalised in Chapter 5 to apply to arbitrary quantum systems. For simplicity, I shall present the